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# Staff Working Paper No. 596

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# The dynamic Black-Litterman approach to asset allocation

Richard D F Harris,[1] Evarist Stoja[2] and Linzhi Tan[3]

## Abstract

We generalise the Black-Litterman (BL) portfolio management framework to incorporate time-variation in the conditional distribution of returns in the asset allocation process. We evaluate the performance of the dynamic BL model using both standard performance ratios as well as other measures that are designed to capture tail risk in the presence of non-normally distributed asset returns. We find that dynamic BL model outperforms a range of different benchmarks. Moreover, we show that the choice of volatility model has a considerable impact on the performance of the dynamic BL model.

**Key words:** Black-Litterman model, multivariate conditional volatility, portfolio optimization, non-normality, tail risk.

**JEL classification:** C22, C53, G11.

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## 1. Introduction

The portfolio theory proposed by Markowitz (1952) is a cornerstone of modern finance. Markowitz argues that investors should balance risk and expected return to determine the optimal allocation of assets. However, when implemented in practice, the resulting portfolio suffers from various problems, including extreme weights (Green and Hollifield, 1992), corner solutions leading to highly concentrated portfolios (Grauer and Shen, 2000), sensitivity of the solution to the input parameters and large fluctuations in the weights over time (Best and Grauer, 1991). The main reason for this is estimation error in the inputs of the model, which has a disproportionate effect on the resulting portfolio weights, Kolm, Tütüncü, and Fabozzi (2014) summarise several methods that have been widely adopted in the literature to mitigate the impact of estimation error. These include Bayesian approaches and the Black Litterman model (Jorion, 1991; Black and Litterman, 1991), robust optimization methods (Tütüncü and Koenig, 2004; Stinstra and Hertog, 2008; Huang, Zhu, Fabozzi, and Fukushima, 2010), incorporating higher moments and tail-risk measures (Harvey, Liechty, and Müller, 2010), and imposing constraints on the portfolio weights (Jagannathan and Ma, 2003). In this paper, we focus on the Black-Litterman (1991, 1992, hereafter BL) model, which uses an equilibrium approach to estimate the expected returns of individual assets, and incorporates the investor's views by adjusting the equilibrium expected returns using a Bayesian approach. The BL model overcomes many of problems associated with the mean-variance model and, as a result, has become one of the most commonly used asset allocation approaches in practice (Bevan and Winklemann, 1998; Bertsimas, Gupta, and Paschalidis, 2012). In particular, the BL model provides stable and intuitively appealing mean-variance efficient portfolios based on the investor's subjective views, and eliminates the input sensitivity of mean-variance optimization (see also Herold and Maurer, 2003).

The assumptions that are implicit in the BL model include: (1) the expected return vector and the covariance matrix are constant over time; (2) the returns of individual assets are normally distributed; and (3) investors do not differentiate between positive and negative deviations from the mean. In reality, all three of these assumptions are questionable. For example, Bollerslev, Engle, and Wooldridge (1988) argue that investors have time-varying conditional expectations of returns. Moreover, it is well known that the variances and covariances of most financial time series are time-varying (Andersen, Bollerslev, and Diebold, 2004). In addition, there is overwhelming empirical evidence that asset returns are not normally distributed (Peiro, 1999; Ang and Chen, 2002), which would suggest that the standard deviation may not be a suitable measure of risk. As a result, alternative risk measures such as value at risk (VaR) and conditional value at risk (CVaR) have been developed (Artzner, Heath, Delbaen, and Eber, 1999; Rockafellar and Uryasev, 2002). Relatedly, when returns are not normally distributed, the Sharpe ratio is no longer an appropriate measure of portfolio performance. To overcome this problem, a number of other performance ratios have been proposed (see, for example, Farinelli and Tibiletti, 2008; Farinelli, Ferreira, Rossello, Thoeny, and Tibiletti, 2008; Farinelli, Ferreira, Rossello, Thoeny, and Tibiletti, 2009). For example, the standard deviation has been replaced by VaR and CVaR measures of tail risk (Biglova, Ortobelli, Rachev, and Stoyanov, 2004; Rachev, Jašić, Stoyanov, and Fabozzi, 2007; Giacometti, Bertocchi, Rachev, and Fabozzi, 2007). Finally, evidence suggests that investors have asymmetric attitudes towards upside and downside risks (Scott and Horvath, 1980). Mitton and Vorkink (2007) show that investors are willing to sacrifice portfolios with higher Sharpe ratios for those with higher skewness. Consequently, a number of researchers have incorporated skewness and kurtosis into portfolio selection (Harvey et al., 2010; Kerstens, Mounir, and Van de Woestyne, 2011), while others have incorporated measures of tail-risk (Goh, Lim, Sim, and Zhang, 2012).

A number of recent studies have attempted to extend the BL model to deal with non-normality in the distribution of asset returns (Giacometti et al., 2007; Martellini and Ziemann, 2007; Meucci, 2009), and in investors' views (Fabozzi, Focardi, and Kolm, 2006; Beach and Orlov, 2007; Palomba, 2008; Babameto and Harris, 2009; Chiarawongse, Kiatsupaibul, Tirapat, and Van Roy, 2012). However, none of these studies allows for time-variation in the conditional distribution of returns. Moreover, despite allowing for non-normality in returns, they evaluate the performance of the BL model using measures that are only valid under normality, namely the standard deviation and the Sharpe ratio.

We make the following contributions. First, we relax the assumption of a constant expected return vector and covariance matrix in the BL model. In particular, we generalize the BL model to incorporate time-variation in the conditional distribution of returns in the asset allocation process. Second, we relax the assumption of normally distributed returns and explicitly account for tail risk in the BL model. Moreover, we use alternative performance ratios that are appropriate for tail risk, namely the reward-to-VaR and reward-to-CVaR ratios, in order to construct the optimal dynamic BL portfolio. Third, we extend the method of Giacometti et al. (2007) to a dynamic setting and use the estimated time-varying VaR and CVaR to substitute for the time-varying standard deviation in the quadratic utility function.

Fourth, we estimate the tail risk-adjusted equilibrium returns and combine these with the investor's views to construct the dynamic BL portfolio.

We evaluate the out-of-sample performance of the following dynamic BL portfolios: the implied BL portfolio with reverse optimization; the SR-BL portfolio with maximal Sharpe ratio; the MVaR-BL portfolio with maximal reward-to-VaR ratio; the MCVaR-BL portfolio with maximal reward-to-CVaR ratio; and the risk-adjusted BL portfolio. The market portfolios are represented by the 10 FTSE sector indices for the US, UK, and Japan. To estimate the time-varying distribution of returns, we use various conditional volatility models (the Rolling Window, Exponentially Weighted Moving Average and Dynamic Conditional Correlation (DCC) models), distributional assumptions (the normal and t-distributions) and confidence levels (90%, 95%, and 99%). We report three main findings. First, the dynamic BL portfolios and risk-adjusted BL portfolios outperform the benchmark and the equally-weighted naive portfolios. Second, the dynamic BL portfolio based on the DCC model has the best out-of-sample performance. Third, the portfolios that account for tail risk outperform the portfolios that ignore tail risk.

The outline of the remainder of the paper is as follows. Section 2 describes the data used in the empirical analysis. Section 3 describes the dynamic BL asset allocation framework. Section 4 reports and discusses the empirical results. Section 5 summarizes our findings and offers some concluding remarks.

## 2. Data

We use monthly price indices and market values for the 10 FTSE industry sectors in the US, UK, and Japan obtained from DataStream for the period from December 1993 to August 2015, i.e. 261 observations. The FTSE sector indices are free from survivorship bias by construction[1]. The price indices and market values are measured in US Dollars. In addition, we obtained the US one-month Treasury Bill rate for the corresponding period from the Kenneth R. French Data Library. We use the price indices to compute simple returns and then subtract the Treasury Bill rate to give excess returns, which are used throughout the empirical

[1] FTSE uses the Industry Classification Benchmark (ICB) system to categorize over 70,000 companies and 75,000 securities worldwide, and enable the comparison of companies across four levels of classification (10 industries, 19 supersectors, 41 sectors and 114 subsectors) and national boundaries. Any significant change takes place in a company's structure as a result of a corporate event (such as a merger or demerger), its ICB classification may be reassessed and adjusted in the Subsector of ICB, and then further aggregated into the remaining levels of classification. The FTSE 10 Industries classification includes the widest range of categorized companies than any other levels of classification for the avoidance of survivorship bias.

analysis. We employ the market value of each FTSE sector index to compute the weight of each index in the benchmark portfolio during each month.

Table 1 reports the summary statistics of excess returns for each index from January 1994 to August 2015. The null hypothesis of normality is strongly rejected in all cases.

[Table 1]

Table 2 reports the time series properties of excess returns for each index from January 1994 to August 2015. In particular, it shows the first five autocorrelation coefficients and the value of the Ljung-Box test for serial correlation up to 10 lags, the ARCH test of Engle (1982) and the DCC test of Engle and Sheppard (2001). In only a few cases do excess returns display significant autocorrelations. This suggests that it is only for these series that we need to specify a conditional mean model to predict excess returns. However, taking the low power of the test and the possibility of non-linear dependence in excess returns into account, we use a momentum strategy for all 30 indices in order to capture predictability in excess returns as a basis for the investor's views in the BL model (see Fabozzi et. al, 2006). The results of the ARCH test suggest that there is significant volatility clustering for most of the excess return series. As shown in Table 2, the DCC test for all 30 excess return series fails to reject the null of a constant correlation in favour of a dynamic structure, with a p-value larger than 10%. We apply the DCC test to the 18 indices that exhibit a strong volatility clustering effect. The test suggests that these indices exhibit significant time-varying conditional correlations, with a p-value less than 1%, thus motivating the use of dynamic conditional covariance models. The non-rejection of the null hypothesis for the remaining 12 series may be due to the low power of the test. Overall, therefore, the results reported in Table 2 suggest that it is appropriate to use a conditional volatility model to estimate the covariance matrix in the dynamic asset allocation model.

[Table 2]

## 3. Theoretical Framework

We generalize the BL model in a dynamic asset allocation framework, and measure tail risk under the assumption of both normally distributed and t-distributed returns. The dynamic BL portfolios that we consider include both unconstrained as well as risk-adjusted portfolios. To construct the unconstrained BL portfolios, we calculate the optimal weights using the following methods: reverse optimization implied in the BL model (the implied BL portfolio); optimization with a maximal Sharpe ratio (the SR-BL portfolio); optimization with a maximal reward-to-VaR ratio (the MVaR-BL portfolio); and optimization with a maximal reward-to-CVaR ratio (the MCVaR-BL portfolio).

### 3.1. Dynamic BL Model

In the generalized dynamic BL model that we propose, we define the first and second moments of $N$ asset excess returns, conditional on the information set $Y$, as follows:

$$\mathbf{r}_t = \boldsymbol{\mu}_{BL,t} + \boldsymbol{\varepsilon}_t \qquad (1)$$

$$\boldsymbol{\varepsilon}_t \mid Y_{t-1} \sim G(0, \mathbf{V}_t)$$

$$\boldsymbol{\mu}_{BL,t} \sim G(\boldsymbol{\pi}_t, \tau\mathbf{H}_t)$$

where $\mathbf{r}_t = (r_{1t}, r_{2t}, \cdots, r_{Nt})'$ is the $N \times T$ excess returns vector, $\boldsymbol{\mu}_{BL,t}$ is the $N \times 1$ vector of expected excess returns in period $t$, $\boldsymbol{\varepsilon}_t$ is the $N \times 1$ error term vector, $G(.)$ is any location-scale family distribution, $\mathbf{V}_t$ is the $N \times N$ covariance matrix, $\boldsymbol{\pi}_t$ is the $N \times 1$ conditional equilibrium return vector of the market portfolio, and the scale parameter $\tau$ indicates the uncertainty of the CAPM prior. The smaller the value of $\tau$, the higher the confidence in the estimation of the implied equilibrium return. In Black and Litterman (1992) $\tau$ ranges between 0.01 and 0.05.

#### *3.1.1 Estimation of the Time-Varying Covariance Matrix*

In the dynamic asset allocation model, we use a time-varying conditional volatility model to estimate the covariance matrix. Indeed, Table 2 clearly shows that for the majority of the excess return series, the data exhibit volatility clustering and time-varying conditional correlations. The literature is replete with covariance matrix forecasting models. In order to narrow the scope of the research, we select three models in increasing order of sophistication. The Rolling Window (RW) model is the simplest but it also suffers from a number of limitations such as ghost features[2] (see Alexander, 1998). The Exponentially Weighted Moving Average (EWMA) model provides a more realistic approach to the weighting of past

[2] Using a historical averaging period of *T* days in order to forecast forward *T* days, if one extreme return is included in the averaging period, the volatility (or correlation) forecast will continue to keep artificially high for T days following that day, when actually nothing has happened in the markets afterwards.

observations, while preserving the simplicity of the RW model. This model puts more weight on recent observations relative to those in the distant past and remedies the drawback of ghost features in the Rolling Window model (see Alexander, 1998), and is better suited to capturing volatility persistence. Moreover, EWMA-based VaR forecasts have been shown to be superior to those based on GARCH models in many cases (see, for instance, Alexander and Leigh, 1997; Boudoukh, Richardson, and Whitelaw, 1997; Guermat and Harris, 2002). Finally, to capture time-varying conditional correlations in asset returns, we also employ the more sophisticated Dynamic Conditional Correlation (DCC) model (see, for instance, Engle and Sheppard, 2001; Engle, 2002; Kalotychou, Staikouras, and Zhao, 2014). An advantage of this model that it has a smaller number of parameters compared to traditional multivariate models such as the VEC and BEKK models, and can therefore be applied to problems involving a large number of assets. We rebalance the portfolio every month using the estimated conditional covariance matrix for that month.

*Rolling Window Covariance Matrix*

The Rolling Window (RW) covariance matrix is given by:

$$\mathbf{H}_t = \frac{1}{M-1}\sum_{j=1}^{M}(\mathbf{r}_{t-j} - \bar{\mathbf{r}})(\mathbf{r}_{t-j} - \bar{\mathbf{r}})' \qquad (2)$$

where $\bar{\mathbf{r}} = \frac{1}{M}\sum_{j=1}^{M}\mathbf{r}_{t-j}$ is the $N \times 1$ vector of sample mean excess returns within the window $t-M,...,t-1$.

*EWMA Covariance Matrix*

The EWMA covariance matrix is given by:

$$\mathbf{H}_t = \lambda\mathbf{H}_{t-1} + (1-\lambda)\mathbf{r}_{t-1}\mathbf{r}_{t-1}' \qquad (3)$$

where $\lambda$ is the decay factor, with $0 \le \lambda \le 1$, which determines how rapidly the weights on past observations decline and typically estimated to be between 0.92 and 0.96. In *RiskMetrics* (J.P. Morgan, 1994), the decay factor is set to 0.94.

*DCC Covariance Matrix*

The DCC covariance matrix is given by:

$$\mathbf{H}_t = \mathbf{D}_t \mathbf{R}_t \mathbf{D}_t \tag{4}$$

where $\mathbf{H}_t$ is the time-varying covariance matrix, $\mathbf{D}_t$ is the diagonal matrix of time-varying standard deviations from univariate GARCH models with $\sigma_{i,t}$ as the $i^{th}$ element at time $t$, i.e., $\mathbf{D}_t = diag\{\sigma_{i,t}\}$, $\mathbf{R}_t$ is the time-varying correlation matrix (see, Engle, 2002).

### *3.1.2. Conditional Equilibrium Returns*

Bollerslev et al. (1988) argue that investors have conditional expected returns that are time-varying. They employ a multivariate GARCH process within the CAPM framework to estimate expected returns. The idea is that the expected returns are proportional to the conditional non-diversifiable risk represented by the conditional covariance of each return with the market portfolio.

Following Bollerslev et al. (1988), let $\mathbf{r}_t$ be the $N \times 1$ vector of excess returns of all assets in the market at time $t$, let $\boldsymbol{\pi}_t$ be the $N \times 1$ conditional mean vector and let $\mathbf{H}_t$ be the $N \times N$ conditional covariance matrix of these returns given information available at time $t-1$. In addition, define $\mathbf{w}_{t-1}$ to be the $N \times 1$ vector of market capitalization weights at time $t-1$, and hence the excess return on the market portfolio is $r_{M,t} = \mathbf{w}_{t-1}'\mathbf{r}_t$. When the CAPM holds, the conditional mean vector $\boldsymbol{\pi}_t$ satisfies the following equation:

$$\boldsymbol{\pi}_t = \delta_t \mathbf{H}_t \mathbf{w}_{t-1} \tag{5}$$

where $\delta_t$ is the dynamic risk aversion coefficient. Brandt and Wang (2003) argue that the risk aversion coefficient is time-varying. We use a simple method to calculate the risk aversion coefficient as the value of the global market risk premium divided by the market variance (see Idzorek, 2004; Babameto and Harris, 2009).

*3.1.3. The Investor's Views*

An investor can possess views about some or all of the returns of the assets in a portfolio which may differ from the implied equilibrium returns. The uncertainty of the views is given by the error vector $\mathbf{e}_t$ with a mean of zero and covariance matrix $\mathbf{\Omega}$. The error terms are unknown and independent. The investor's views at time $t$ can thus be expressed as:

$$\mathbf{q}_t = \mathbf{P}_t \boldsymbol{\mu}_{BL,t} + \mathbf{e}_t \tag{6}$$

At time $t$, let $K \le N$ be the total number of the views, $\mathbf{P}_t$ be the $K \times N$ matrix of view portfolios and $\mathbf{q}_t$ be the $K \times 1$ vector of expected returns on the view portfolios.

Following Fabozzi et al. (2006), we utilize a momentum strategy to generate views. However, we extend their approach by substituting the constant standard deviation with a time-varying standard deviation to calculate the dynamic normalized returns. Further, there is evidence that the momentum effect is strongest at the six-month horizon (see Richard, 1997). Thus, we rank securities over the past six months and the momentum portfolios are formed at time $t$ and held for six months. Therefore, the normalized six-month return $Z_{i,t}$ is given by:

$$Z_{i,t} = \frac{p_{i,t-1} - p_{i,t-6}}{p_{i,t-6}\sigma_{i,t}} \tag{7}$$

where $p_{i,t-1}$ is the sector index price $i$ at time $t-1$; $p_{i,t-6}$ is the sector index price $i$ at time $t-6$; and $\sigma_{i,t}$ is the volatility of the sector index price $i$ at time $t$. The top half of the sector indices are allocated weights of $\omega_{i,t} = \frac{1}{\sigma_{i,t}c}$, while the bottom half of the sector indices are allocated weights of $\omega_{j,t} = -\frac{1}{\sigma_{j,t}c}$, where, $i, j = 1,...,30, i \ne j$. Then, the view matrix $\mathbf{P}_t$ in the BL model is a two-dimensional vector with the two quantities above as elements. The parameter $c$ is a constant whose role is to constrain the annual long-short portfolio volatility to a certain level (20% in this case). Note that the portfolio weights do not sum to zero with this non-zero-cost long-short portfolio. The expected return of this momentum portfolio is the expected view return $\mathbf{q}$. After constructing the momentum portfolio in each period $t$, we hold it for one month and observe its return $r_{m,t}$ over the holding period. For the same holding period, we also observe the realized return $r_{a,t}$ on the portfolio of the actual winners and losers. Then, the residual return is calculated as the difference between $r_{m,t}$ and $r_{a,t}$. The residual return series $\mathbf{e}_t$ is then obtained by rolling the evaluation window forward one month and repeating the process. The level of confidence in the views $\mathbf{\Omega}$ is equal to the variance of the series of residuals $\mathbf{e}_t$.

*3.1.4. Combining Conditional Equilibrium Returns and Views*

The next step in the estimation is to combine the conditional equilibrium returns with the views using a Bayesian approach. In the dynamic case, the $N \times 1$ vector of conditional expected returns $\boldsymbol{\mu}_{BL,t}$ at time $t$ is given by:

$$\boldsymbol{\mu}_{BL,t} = \boldsymbol{\pi}_t + \tau \mathbf{H}_t \mathbf{P}_t '(\mathbf{P}_t \mathbf{H}_t \mathbf{P}_t ' \tau + \mathbf{\Omega}_t)^{-1} (\mathbf{q}_t - \mathbf{P}_t \, \boldsymbol{\pi}_t) \tag{8}$$

and the estimated $N \times N$ covariance matrix $\mathbf{V}_t$ is given by:

$$\mathbf{V}_t = \mathbf{H}_t + ((\tau \mathbf{H}_t)^{-1} + \mathbf{P}_t ' \mathbf{\Omega}_t^{-1} \mathbf{P}_t)^{-1} \tag{9}$$

## 3.2. Unconstrained Dynamic BL Portfolio

We estimate the time-varying expected returns and covariance matrix from Equations (8) and (9). We use two methods to construct the unconstrained BL portfolio at each period $t$. For the first method, we use reverse optimization to compute the implied weights $\mathbf{w}^*_{BL,t}$ at time $t$, given by:

$$\mathbf{w}^*_{BL,t} = \frac{1}{\delta_t} \mathbf{V}_t^{-1} \boldsymbol{\mu}_{BL,t} \tag{10}$$

(see, for example, Idzorek, 2004). For the second method, we use mean-variance optimization, and maximize the Sharpe ratio:

$$\max \frac{\mathbf{w}_{BL,t}'\boldsymbol{\mu}_{BL,t}}{\sqrt{\mathbf{w}_{BL,t}'\mathbf{V}_t\mathbf{w}_{BL,t}}} \tag{11}$$

$$\text{subject to } \mathbf{w}_{BL,t}'\mathbf{1} = 1$$

where $\boldsymbol{\mu}_{BL,t}$ is the expected excess return of the BL portfolio in (8), $\sqrt{\mathbf{w}_{BL,t}'\mathbf{V}_t\mathbf{w}_{BL,t}}$ is the conditional portfolio standard deviation, $\mathbf{w}_{BL,t}$ is the $N \times 1$ vector of portfolio weights and $\mathbf{1}$ is an $N \times 1$ vector of ones. Thus, the vector of optimal weights for the SR-BL portfolio is given by:

$$\mathbf{w}_{BL,t}^{*} = \frac{\mathbf{V}_t^{-1}\boldsymbol{\mu}_{BL,t}}{\mathbf{1}'\mathbf{V}_t^{-1}\boldsymbol{\mu}_{BL,t}} \tag{12}$$

In both methods, the standard deviation is used to measure the portfolio risk. As noted above, this is only appropriate when returns are normally distributed, and so we consider a more general formulation of the maximal reward-to-risk portfolio, where risk is measured by the VaR or CVaR of the BL portfolio. In particular, the optimization problem with maximal reward-to-VaR (MVaR-BL) is:

$$\max \frac{\mathbf{w}_{BL,t}'\boldsymbol{\mu}_{BL,t}}{VaR_{\alpha,t}} \tag{13}$$

$$\text{subject to } \mathbf{w}_{BL,t}'\mathbf{1} = 1$$

where $VaR_{\alpha,t}$ is the expected maximum loss on the BL portfolio at time $t$ with a confidence level of $1-\alpha$. The VaR of the BL portfolio at time $t$ can be expressed as:

$$VaR_{\alpha,t} = \xi_\alpha \sqrt{\mathbf{w}_{BL,t}'\mathbf{V}_t\mathbf{w}_{BL,t}} - \mathbf{w}_{BL,t}'\boldsymbol{\mu}_{BL,t} \tag{14}$$

where $\xi_\alpha = -F^{-1}(1-\alpha)$ is the $1-\alpha\%$ quantile of the cumulative distribution $F(.)$ and $\alpha$ is equal to 1%, 5% and 10% (see, for example, Rockafellar and Uryasev, 2000).

Similarly, the optimization problem with maximal reward-to-CVaR (MCVaR-BL) is:

$$\max \frac{\mathbf{w}_{BL,t}'\boldsymbol{\mu}_{BL,t}}{CVaR_{\alpha,t}} \tag{15}$$

$$\text{subject to } \mathbf{w}_{BL,t}'\mathbf{1} = 1$$

where $CVaR_{\alpha,t}$ is the average loss exceeding the expected maximum loss $VaR_{\alpha,t}$ at time $t$ on the BL portfolio with a certain confidence level of $1-\alpha$. The CVaR of the BL portfolio at time $t$ is:

$$CVaR_{\alpha,t} = \zeta_{\alpha,t}\sqrt{\mathbf{w}_{BL,t}'\mathbf{V}_t\mathbf{w}_{BL,t}} - \mathbf{w}_{BL,t}'\boldsymbol{\mu}_{BL,t} \tag{16}$$

where $\zeta_{\alpha,t} = \frac{-\int_{-\infty}^{-F^{-1}(1-\alpha)} gf(g)dg}{1-\alpha}$ and $g = -\mathbf{w}_{BL,t}'\boldsymbol{\mu}_{BL,t} - VaR_{\alpha,t}$ (see, for example, Rockafellar and Uryasev, 2000).

### 3.3. The Dynamic Risk-adjusted BL Portfolio

Giacometti et al. (2007) improve the BL framework by relaxing the assumption of normally distributed returns. They also incorporate alternative risk measures into the calculation of equilibrium returns. However, they do not evaluate the performance of their method in a dynamic asset allocation framework, which we address here. The first step is to estimate the time-varying risk-adjusted equilibrium return. The optimization problem is given by:

$$\max(\mathbf{w}_t'\boldsymbol{\pi}_t - \frac{\delta_t}{2}\vartheta(\mathbf{w}_t'\mathbf{r}_t)) \tag{17}$$

where $\vartheta(\mathbf{w}_t'\mathbf{r}_t)$ indicates the measure of risk (i.e. variance, VaR or CVaR) of the portfolio return $\mathbf{w}_t'\mathbf{r}_t$ and the equilibrium returns at time $t$ are given by:

$$\boldsymbol{\pi}_t = \delta_t \mathbf{H}_t \mathbf{w}_t \tag{18}$$

$$\boldsymbol{\pi}_t = \frac{\delta_t}{2}(VaR_{\alpha,t} \frac{\mathbf{H}_t \mathbf{w}_t}{\sqrt{\mathbf{w}_t{}'\mathbf{H}_t \mathbf{w}_t}} - E(\mathbf{r})) \tag{19}$$

$$\boldsymbol{\pi}_t = \frac{\delta_t}{2}(CVaR_{\alpha,t} \frac{\mathbf{H}_t \mathbf{w}_t}{\sqrt{\mathbf{w}_t{}'\mathbf{H}_t \mathbf{w}_t}} - E(\mathbf{r})) \tag{20}$$

where $\mathbf{H}_t$ is the covariance matrix at time $t$ obtained from the conditional volatility models (2)-(4), under the normal or t-distribution. $VaR_{\alpha,t}$ and $CVaR_{\alpha,t}$ are the VaR and the CVaR at time $t$ for the corresponding distribution at the $1-\alpha$ confidence level and $E(\mathbf{r})$ is the expected return. Since we have estimated the risk-adjusted equilibrium return $\boldsymbol{\pi}_t$, we substitute $\boldsymbol{\pi}_t$ in Equation (8) with $\boldsymbol{\pi}_t$ in Equations (18) to (20). Then, we use Equation (10) to construct the implied risk-adjusted BL portfolio and use Equation (12) to construct the risk-adjusted SR-BL portfolio.

## 4. Empirical Results

In this section, we discuss the out-of-sample performance of the dynamic BL portfolios detailed above. The results of the in-sample analysis are available upon request. Given the weights obtained from the optimization of the unconstrained dynamic BL models and dynamic risk-adjusted BL models, we calculate buy-and-hold returns on the portfolio for a holding period of one month and repeat the calculation until the end of the sample, and thus obtain the time series of realized portfolio returns. We report the average, standard deviation, skewness and kurtosis of the portfolio returns in the out-of-sample period. We also use the average Herfindahl index (AHI) to measure diversification. The higher the value of the AHI, the less diversified the portfolio. To evaluate the performance of the portfolio, we use the information ratio, the reward-to-risk ratio (i.e. the Sharpe ratio) and the ratio of reward-to-downside risk, measured by VaR or CVaR computed from the empirical distribution. The information ratio measures the active return of the portfolio divided by the amount of portfolio risk relative to the benchmark. With these evaluation criteria, we compare the risk-adjusted BL portfolios to the benchmark, the naive 1/N portfolio, and the unconstrained BL portfolios. Finally, we investigate the impact of the choice of different distributions and different confidence levels on the performance of the dynamic BL portfolios.

Following Giacometti et al. (2007), we initially estimate each of the three volatility models using the first 110 observations (from January 1994 to February 2003) to generate a one-month-ahead out-of-sample forecast of the conditional covariance matrix for month 111 (March 2003). The estimation sample is then rolled forward by one month, the models are re-estimated and used to generate out-of-sample forecasts for month 112, and so on until the end of the sample. For each iteration, the starting parameter values for each model are set to the values estimated in the previous iteration. This procedure results in a total of 151 out-of-sample monthly forecasts. We then construct the momentum portfolio with a holding period of six months to use as the view vector in the BL model. Thus, the first month in which the BL portfolio is formed is August 2003 and so the total number of out-of-sample estimates is reduced to 145. We report the out-of-sample portfolio performance results in Table 3.

[Table 3]

Table 3 shows that all the unconstrained BL portfolios outperform both the benchmark portfolio and the 1/N portfolio, with better Sharpe ratios and reward-to-downside risk ratios. In addition, the unconstrained BL portfolios have a significantly lower values of the AHI (less than 1) than the traditional mean-variance portfolio (which has an AHI value of 12.636) implying higher diversification and more moderate weights.

### 4.1. The Benchmark Portfolio and the 1/N Portfolio

Panel A of Table 3 indicates that while the 1/N portfolio is more diversified, the benchmark portfolio performs better. In particular, the benchmark portfolio generates a 1.71% higher Sharpe ratio and a 0.95% higher reward-to-downside risk ratio relative to the 1/N portfolio. This result contradicts findings reported elsewhere in the literature regarding the performance of the 1/N portfolio relative to other, more sophisticated portfolio strategies (see, for example, DeMiguel et al., 2009).

### 4.2. The Dynamic Implied BL Portfolio and the SR-BL Portfolio

Next, we evaluate the performance of the two unconstrained portfolios, namely the implied BL portfolio and the SR-BL portfolio. We find that the implied BL portfolio constructed with the DCC model achieves the highest information ratio of 0.157 (see Panel B). In addition, this portfolio delivers the highest Sharpe, reward-to-VaR and reward-to-CVaR ratios, of 0.199, 0.078 and 0.074, respectively. These are considerably higher than the corresponding ratios for the DCC-based SR-BL portfolio (see Panel C). Moreover, the information ratio of the DCC-based SR-BL portfolio is 0.075, which is less than half of that of the implied BL portfolio. The DCC-based SR-BL portfolio has an AHI value of 0.097 and is thus less diversified than the implied DCC-based BL portfolio, which has an AHI value of 0.094.

We also examine the impact of the choice of the volatility model on the performance of the implied BL portfolio and the SR-BL portfolio. We find that the use of the DCC model leads to superior portfolio performance, regardless of the performance measure. Relative to the RW model, the EWMA model results in better performance for the SR-BL portfolio but worse performance for the implied BL portfolio.

### 4.3. The Dynamic MVaR-BL Portfolio

Panel D of Table 3 reports the results for the unconstrained MVaR-BL strategy. The findings on the impact of the volatility models discussed in section 4.2 are similar. In particular, we find that the DCC-based MVaR-BL portfolio is superior at each confidence level under the t-distribution. Further, the EWMA-based MVaR-BL portfolio is superior in terms of risk-adjusted performance but inferior in terms of active portfolio performance relative to the RW-based MVaR-BL portfolio. This finding appears robust to the choice of confidence level and distributional assumption. For example, Panel D1 shows that the DCC-MVaR-BL portfolio outperforms both the EWMA-MVaR-BL and the RW-MVaR-BL portfolios under the t-distribution at the 99% confidence level. Indeed, it achieves the highest information ratio (0.137), Sharpe ratio (0.200), reward-to-VaR ratio (0.080), and reward-to-CVaR ratio (0.067). However, with the highest AHI value (of 0.161), the outperformance comes at the cost of lower diversification. Similar conclusions can be drawn at lower confidence levels (Panels D2 and D3).

Next, we assess the impact of the confidence level and distributional assumption on portfolio performance. The EWMA-MVaR-BL and RW-MVaR-BL portfolios appear insensitive to the choice of confidence level and distributional assumption. However, the DCC-MVaR-BL portfolio performance is more sensitive, and performs better at the 99% confidence level under both the normal and t-distribution (see rows 1 and 4 of Panels D1 to D3). The DCC-MVaR-BL portfolio achieves a slightly better performance under the t-distribution than under the normal distribution at the 99% confidence level (Panel D1). However, the better performance comes at the cost of lower diversification. Similar conclusions can be drawn for the lower confidence levels (Panels D2 and D3).

### 4.4. The Dynamic MCVaR-BL Portfolio

The finding that the unconstrained MCVaR-BL portfolio with a DCC model is superior to the same portfolio constructed with other volatility models is robust to different confidence levels and distributional assumptions (Panel F). For example, the DCC-MCVaR-BL portfolio achieves higher Sharpe and reward-to-downside risk ratios than the EWMA-based MCVaR-BL portfolio under a t-distribution and at the 99% confidence level (Panel F1). The AHI value of 0.164 implies that the outperformance of the DCC-MCVaR-BL portfolio comes at the cost of lower diversification. Similar conclusions can be drawn under normality and lower confidence levels (Panels F1 to F3).

Turning to the impact of the confidence level and distributional assumption, we see that the performance of the DCC-MCVaR-BL portfolio is better under the t-distribution than under the normal distribution at the 99% confidence level, but seems insensitive to the choice of distribution at lower confidence levels. Further, the performance of the MCVaR-BL portfolios based on the EWMA and RW models does not appear to be sensitive to the choice of confidence level or distribution.

### 4.5. The Dynamic Risk-adjusted BL Portfolio

In this section, we discuss the performance of the risk-adjusted BL portfolios reported in Table 4.

[Table 4]

#### *4.5.1. The Variance-Adjusted BL Portfolio*

The variance-adjusted SR-BL portfolio performs much worse than the implied variance-adjusted BL portfolio, with performance measures that are lower by more than two thirds (Columns 8 to 11, Panels A and B of Table 4). This arises from the extreme negative skewness and high kurtosis of the variance-adjusted SR-BL portfolio, leading to higher tail risk.

#### *4.5.2. The VaR-Adjusted SR-BL Portfolio*

Panel C of Table 4 indicates that the distributional assumption has a significant impact on the performance of the VaR-adjusted SR-BL portfolio at each confidence level. The t-distribution is better suited to the VaR-adjusted SR-BL portfolio for both performance and diversification. At each confidence level, the Sharpe ratio and reward-to-downside risk ratios under the t-distribution are larger than under the normal distribution. Further, the AHI value under the t-distribution is at least 0.12 lower. Secondly, as the confidence level decreases, the risk-adjusted performance of the VaR-adjusted SR-BL portfolio improves under the normal distribution, while its performance deteriorates slightly under the t-distribution (Rows 1 and 2, Panels C1 to C3).

#### *4.5.3. The CVaR-Adjusted SR-BL Portfolio*

Panel D of Table 4 indicates that the confidence levels and distributional assumption have a moderate impact on the CVaR-adjusted SR-BL portfolio performance. As the confidence level is reduced from 99% to 90%, the Sharpe ratio falls from 0.183 to 0.165, while at the same time, the portfolio becomes more concentrated as suggested by the AHI, which increases from 0.425 to 0.431. A similar trend of deteriorating performance and diversification as the confidence level is reduced obtains under the t-distribution. Moreover, using the t-distribution rather than the normal distribution, both the diversification and risk-adjusted performance of the CVaR-adjusted BL portfolio performance improves. At the 99% confidence level under the t-distribution, the Sharpe ratio and the reward-to-downside risk ratios of the CVaR-adjusted BL portfolio are, respectively, 0.4% and 0.1% higher than under the normal distribution. The AHI under the t-distribution is about 4.4% lower than under the normal distribution. At lower confidence levels, the Sharpe ratio and the reward-to-downside risk ratios of the CVaR-adjusted BL portfolio are 1.4% higher on average than under the normal distribution.

### 4.6. Dynamic BL Portfolio Performance Ranking

In this section, we discuss the performance ranking of the dynamic BL portfolios reported in Table 5.

[Table 5]

We select the outperforming DCC-based dynamic BL portfolios in each strategy and report their rankings based on different performance measures. Based on the AHI, the 1/N and the benchmark portfolios have the best performance. However, these two portfolios are outperformed by most dynamic BL portfolios in terms of risk-adjusted performance and active performance. Table 5 shows that the best performing dynamic BL portfolios are the implied BL, the MVaR-BL, MCVaR-BL and the implied variance-adjusted BL portfolios. The implied BL portfolio offers the best active performance with information and reward-to-

CVaR ratios of 0.158 and 0.074 respectively. The MVaR-BL shows balanced performance. Overall, the MCVaR-BL portfolio under the t-distribution at the 99% confidence level performs better than any other portfolio. Although the implied variance-adjusted BL portfolio has the highest reward-to-VaR ratio (0.088) and second highest Sharpe ratio (0.206), its AHI value is nearly twice as large as that of the MCVaR-BL portfolio. All risk-adjusted BL portfolios have much higher AHI values, ranging from 0.307 to 0.541. Moreover, the CVaR-adjusted BL portfolio outperforms the VaR-adjusted BL portfolio. Occasionally, both the VaR- and the CVaR-adjusted BL portfolios perform better than the variance-adjusted BL portfolio, but they do not outperform the unconstrained DCC-BL portfolios.

Finally, our results highlight the importance of using an appropriate reward-to-risk ratio in portfolio performance evaluation (see also Alexander and Baptista, 2003). The performance ranking obtained by the Sharpe ratio under non-normality can be misleading. In this case, the reward-to-downside risk ratios should be used as complementary performance criteria.

### 4.6. Summary of the Findings

The out-of-sample analysis highlights the following important findings. First, the dynamic BL portfolios outperform both the 1/N and the benchmark portfolios. In addition, the dynamic BL portfolios are more diversified than the mean-variance portfolio. Further, the DCC-based dynamic BL portfolios outperform the EWMA and RW-based portfolios in most cases. Second, the implied BL portfolio outperforms the SR-BL portfolio when using the DCC model and the RW model. Third, both the MVaR-BL and MCVaR-BL portfolios outperform the SR-BL portfolios at high levels of confidence. Further, the MCVaR-BL portfolio outperforms the MVaR-BL portfolio and the implied BL portfolio, particularly under the t-distribution and at the 99% confidence level. Fourth, the MVaR-BL and the MCVaR-BL portfolios constructed with a DCC model perform better under the t-distribution than under the normal distribution. However, there is not a consistent ranking of the risk-adjusted portfolio performance and the active portfolio performance, which depends on the volatility model employed. Fifth, the implied variance-adjusted BL portfolio cannot beat DCC-based MCVaR-BL portfolio under the t-distribution and at the 99% confidence level, however, it outperforms most of the unconstrained BL portfolios. It also has the best risk-adjusted performance and active portfolio performance of the risk-adjusted BL portfolios.

## 5. Conclusion

In this paper, we extend the Black–Litterman methodology to a dynamic asset allocation framework. The main contribution of this paper is to relax the assumption of a constant conditional distribution of returns in the BL model. Moreover, this paper also improves the BL framework by using alternative risk measures (VaR and CVaR) under different distributional assumptions (normal and t-distributions). The dynamic unconstrained BL portfolio nests the implied BL portfolio formed by the implied reverse optimization of the BL model, the SR-BL portfolio with maximal Sharpe ratio, the MVaR-BL portfolio with maximal reward-to-VaR ratio, and the MCVaR-BL portfolio with maximal reward-to-CVaR ratio. We analyse four risk-adjusted BL portfolios, the implied variance-adjusted BL portfolio, variance-adjusted SR-BL portfolio, VaR-adjusted SR-BL portfolio and CVaR-adjusted SR-BL portfolio. We also use different performance measures, which lead to different rankings of these BL portfolios. Further, we examine the effect of the choice of volatility model, distributional assumption and the specified confidence level on portfolio diversification and portfolio performance.

The out-of-sample analysis suggests that the dynamic BL asset allocation framework performs well. The dynamic BL portfolios and risk-adjusted BL portfolios outperform the benchmark and the 1/N portfolio and lead to more diversified portfolios relative to the standard mean-variance portfolio. We find that the DCC model is a suitable volatility model choice when constructing a dynamic BL portfolio. Since the Sharpe ratio is not an appropriate performance ratio when asset returns are non-normal, alternative performance ratios (reward-to-VaR and reward-to-CVaR ratios) are used to construct unconstrained BL portfolios and evaluate portfolio performance. The alternative is to use the estimated risk-adjusted equilibrium returns in the dynamic BL asset allocation process, although this leads to much more concentrated portfolios relative to the unconstrained BL portfolios.

The dynamic BL asset allocation approach could be extended in several directions. First, it could be examined under the new performance measures of Biglova, Ortobelli, Rachev and Stoyanov (2004) and Rachev et al. (2007). In particular, it would be interesting to investigate the performance of the portfolios that optimize the Rachev ratio and Rachev generalised ratio. In addition, adding tracking error constraints to improve the active portfolio performance following the method of Palomba (2008) is another direction for future research. It would also be useful to investigate the performance of the dynamic BL portfolio in other markets, such as bonds, currencies, and commodities, and over a longer sample period.

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**Table 1 Summary Statistics for the FTSE Sector Indices Excess Returns**

Table 1 reports summary statistics for the monthly excess return series on 10 FTSE Sector Indices in UK, US and Japan countries for the period January 1994 to August 2015. The table also reports the statistics of the Jarque-Bera tests of the null hypothesis that the series are subject to normal distribution. All the statistics confirm the rejection of normality hypothesis at 5% significance level.

| | Mean *(1)* | Median *(2)* | Standard Deviation *(3)* | Skewness *(4)* | Kurtosis *(5)* | Min *(6)* | Max *(7)* | Jarque-Bera *(8)* | P-value *(9)* |
|---|---|---|---|---|---|---|---|---|---|
| UK BASIC MATS | 0.004 | 0.003 | 0.078 | -0.385 | 6.069 | -0.359 | 0.254 | 6.469 | 0.039 |
| UK CONSUMER GDS | 0.003 | 0.004 | 0.073 | 0.152 | 4.480 | -0.239 | 0.293 | 26.051 | 0.000 |
| UK CONSUMER SVS | -0.001 | 0.003 | 0.051 | -0.546 | 4.151 | -0.199 | 0.125 | 49.930 | 0.000 |
| UK FINANCIALS | 0.000 | 0.002 | 0.069 | -0.091 | 7.735 | -0.322 | 0.335 | 32.946 | 0.000 |
| UK HEALTH CARE | 0.002 | 0.005 | 0.045 | -0.071 | 3.588 | -0.144 | 0.162 | 63.269 | 0.000 |
| UK TECHNOLOGY | -0.001 | 0.004 | 0.068 | -0.751 | 5.151 | -0.257 | 0.155 | 32.250 | 0.000 |
| UK INDUSTRIALS | 0.005 | 0.005 | 0.061 | 0.064 | 3.688 | -0.171 | 0.178 | 58.102 | 0.000 |
| UK OIL & GAS | -0.005 | -0.002 | 0.118 | 0.018 | 3.277 | -0.307 | 0.332 | 80.364 | 0.000 |
| UK TELECOM | 0.000 | 0.003 | 0.066 | -0.331 | 3.357 | -0.207 | 0.169 | 80.399 | 0.000 |
| UK UTILITIES | 0.002 | 0.000 | 0.048 | 0.075 | 3.793 | -0.168 | 0.157 | 53.008 | 0.000 |
| USA BASIC MATS | 0.004 | 0.004 | 0.064 | -0.163 | 4.787 | -0.248 | 0.233 | 17.080 | 0.000 |
| USA CONSUMER GDS | 0.000 | 0.003 | 0.054 | -0.495 | 4.249 | -0.205 | 0.124 | 43.819 | 0.000 |
| USA CONSUMER SVS | 0.003 | 0.004 | 0.050 | -0.345 | 3.775 | -0.168 | 0.138 | 58.785 | 0.000 |
| USA FINANCIALS | 0.003 | 0.007 | 0.063 | -0.623 | 5.659 | -0.234 | 0.196 | 18.050 | 0.000 |
| USA HEALTH CARE | 0.004 | 0.009 | 0.041 | -0.568 | 3.726 | -0.126 | 0.103 | 70.018 | 0.000 |
| USA INDUSTRIALS | 0.004 | 0.009 | 0.054 | -0.441 | 4.824 | -0.201 | 0.174 | 23.405 | 0.000 |
| USA OIL & GAS | 0.006 | 0.004 | 0.054 | -0.082 | 4.009 | -0.184 | 0.177 | 43.217 | 0.000 |
| USA TECHNOLOGY | 0.008 | 0.014 | 0.083 | -0.301 | 3.495 | -0.276 | 0.223 | 71.890 | 0.000 |
| USA TELECOM | -0.001 | 0.004 | 0.047 | -0.384 | 3.459 | -0.136 | 0.130 | 76.348 | 0.000 |
| USA UTILITIES | -0.001 | 0.007 | 0.057 | 0.031 | 4.782 | -0.142 | 0.259 | 16.112 | 0.000 |
| JAPAN BASIC MATS | -0.001 | -0.004 | 0.071 | 0.298 | 3.766 | -0.214 | 0.227 | 57.899 | 0.000 |
| JAPAN CONSUMER GDS | 0.001 | 0.001 | 0.057 | 0.225 | 4.112 | -0.160 | 0.230 | 40.833 | 0.000 |
| JAPAN CONSUMER SVS | -0.003 | -0.008 | 0.052 | 0.384 | 3.303 | -0.144 | 0.156 | 85.214 | 0.000 |
| JAPAN FINANCIALS | -0.006 | -0.014 | 0.087 | 0.479 | 3.767 | -0.224 | 0.317 | 63.929 | 0.000 |

**Table 1 (continued)**

**Table 1 (continued)**

| | Mean<br>*(1)* | Median<br>*(2)* | Standard Deviation<br>*(3)* | Skewness<br>*(4)* | Kurtosis<br>*(5)* | Min<br>*(6)* | Max<br>*(7)* | Jarque-Bera<br>*(8)* | P-value<br>*(9)* |
|---|---|---|---|---|---|---|---|---|---|
| JAPAN HEALTH CARE | 0.000 | 0.000 | 0.051 | 0.430 | 4.726 | -0.152 | 0.218 | 25.586 | 0.000 |
| JAPAN INDUSTRIALS | 0.001 | 0.005 | 0.062 | -0.107 | 3.089 | -0.193 | 0.179 | 92.290 | 0.000 |
| JAPAN OIL & GAS | -0.001 | -0.005 | 0.086 | 0.095 | 3.763 | -0.271 | 0.271 | 54.617 | 0.000 |
| JAPAN TECHNOLOGY | 0.001 | -0.003 | 0.087 | 0.296 | 3.376 | -0.197 | 0.293 | 78.377 | 0.000 |
| JAPAN TELECOM | 0.000 | -0.003 | 0.079 | 0.855 | 6.032 | -0.230 | 0.375 | 31.672 | 0.000 |
| JAPAN UTILITIES | -0.002 | -0.006 | 0.052 | 0.428 | 4.372 | -0.117 | 0.231 | 36.644 | 0.000 |

## Table 2 Time Series Properties

Table 2 reports the test statistics for autocorrelation, autoregressive conditional heteroskedasticity (ARCH) and dynamic conditional correlation for the full sample from January 1994 to August 2015. The Ljung-Box-Q test statistic for autocorrelation of up to order 10 is asymptotically distributed as a central Chisquare with ten d.o.f. The ARCH (1) statistic is asymptotically distributed as a central Chi-square with one d.o.f. The DCC statistic is distributed as a central Chi-square with one d.o.f. *,** and *** denote significance at 10%, 5% and 1% levels respectively. In DCC test, 30 assets means the sample includes all assets, 18 assets means the sample includes assets selected with significant autocorrelation in the squared residuals with 1 lag.

| | ACF(1)<br>*(1)* | ACF(2)<br>*(2)* | ACF(3)<br>*(3)* | ACF(4)<br>*(4)* | ACF(5)<br>*(5)* | LB-Q(10)<br>*(6)* | ARCH(1)<br>*(7)* |
|---|---|---|---|---|---|---|---|
| UK BASIC MATS | 0.141* | 0.156*** | 0.076*** | 0.007*** | -0.130*** | 25.736*** | 35.848*** |
| UK CONSUMER GDS | -0.034 | -0.001 | 0.093 | -0.120 | -0.056 | 18.377* | 0.021 |
| UK CONSUMER SVS | 0.107 | -0.010 | 0.022 | 0.131 | -0.034 | 9.272 | 1.111 |
| UK FINANCIALS | 0.165*** | 0.029* | 0.090* | 0.070* | -0.033* | 17.040 | 11.760*** |
| UK HEALTH CARE | -0.075 | 0.059 | 0.009 | -0.094 | -0.005 | 5.758 | 1.319 |
| UK TECHNOLOGY | 0.070 | -0.009 | -0.001 | 0.032 | 0.028 | 5.662 | 0.268 |
| UK INDUSTRIALS | -0.119 | 0.024 | -0.021 | -0.018 | -0.027 | 8.585 | 7.661*** |
| UK OIL & GAS | 0.086 | 0.007 | 0.076 | 0.117 | -0.027 | 10.630 | 18.609*** |
| UK TELECOM | 0.093 | -0.018 | 0.189*** | -0.007* | 0.071* | 17.808 | 7.963*** |
| UK UTILITIES | 0.020 | 0.066 | 0.034 | 0.052 | -0.068 | 10.341 | 4.160* |
| USA BASIC MATS | 0.021 | 0.041 | 0.044 | 0.003 | -0.058 | 13.554 | 28.250*** |
| USA CONSUMER GDS | 0.027 | -0.146 | 0.008 | 0.008 | 0.006 | 11.018 | 1.458 |
| USA CONSUMER SVS | 0.063 | -0.113 | 0.070 | 0.010 | -0.024 | 6.323 | 7.070*** |
| USA FINANCIALS | 0.082 | -0.033 | 0.067 | 0.073 | 0.071 | 14.632 | 17.591*** |
| USA HEALTH CARE | 0.024 | -0.015 | 0.004 | -0.029 | 0.070 | 12.487 | 0.854 |
| USA INDUSTRIALS | 0.059 | -0.062 | 0.046 | 0.120 | 0.002 | 15.097 | 9.553*** |
| USA OIL & GAS | -0.035 | 0.052 | -0.065 | 0.063 | -0.055 | 6.950 | 3.471* |
| USA TECHNOLOGY | -0.027 | 0.006 | 0.117 | -0.068 | 0.012 | 8.017 | 44.773*** |
| USA TELECOM | 0.058 | -0.028 | 0.090 | 0.059 | 0.004 | 12.308 | 1.962 |
| USA UTILITIES | 0.005 | -0.049 | 0.085 | 0.013 | 0.081 | 13.521 | 13.540*** |
| JAPAN BASIC MATS | 0.083 | 0.005 | 0.095 | -0.027 | 0.070 | 11.657 | 19.241*** |
| JAPAN CONSUMER GDS | 0.063 | -0.028 | 0.158* | -0.006 | 0.025 | 15.853 | 1.574 |
| JAPAN CONSUMER SVS | 0.083 | -0.089 | 0.072 | -0.107 | 0.029 | 14.931 | 0.741 |
| JAPAN FINANCIALS | 0.072 | -0.041 | 0.079 | -0.029 | 0.031 | 9.811 | 0.043 |



**Table 2 (continued)**

**Table 2 (continued)**

| | ACF(1)<br>*(1)* | ACF(2)<br>*(2)* | ACF(3)<br>*(3)* | ACF(4)<br>*(4)* | ACF(5)<br>*(5)* | LB-Q(10)<br>*(6)* | ARCH(1)<br>*(7)* |
|---|---|---|---|---|---|---|---|
| JAPAN HEALTH CARE | 0.032 | -0.069 | -0.044 | -0.094 | 0.024 | 9.865 | 0.141 |
| JAPAN INDUSTRIALS | 0.161*** | 0.009** | 0.123*** | -0.024** | 0.012* | 14.524 | 7.407*** |
| JAPAN OIL & GAS | -0.021 | -0.105 | 0.108 | -0.076 | -0.054 | 15.681 | 9.266*** |
| JAPAN TECHNOLOGY | 0.124* | 0.114** | 0.176*** | 0.039*** | 0.068*** | 23.638** | 32.434*** |
| JAPAN TELECOM | 0.180*** | 0.038*** | 0.055** | 0.000*** | 0.152*** | 21.111* | 12.101*** |
| JAPAN UTILITIES | -0.068 | 0.067 | -0.111 | -0.041 | -0.046 | 8.596 | 0.764 |

| DCC test | statistic | p-Value |
|---|---|---|
| 30 Assets | 3.280 | 0.194 |
| 18 Assets | 11.911 | 0.003 |

**Table 3 Out-of-sample Unconstrained BL Portfolio Performance**

Table 3 reports realized unconstrained BL portfolio performance compared with the benchmark performance in the period from August 2003 to August 2015. Return is the average realized excess return, Sharpe Ratio (SR) is the average excess realized return divided by the standard deviation (SD). Average Herfindahl index (AHI) is used to measure diversification. Information Ratio (IR) is the average active return divided by the standard deviation of active return. Both VaR and CVaR are measured on the empirical distribution. $\mu / VaR$ and $\mu / CVaR$ evaluate the excess return per unit of tail risk. In the construction of the portfolio, both VaR and CVaR are estimated by the parametric method with assumption of normal distribution ('N') and t-distribution ('t') at confidence levels of 99%, 95% and 90%. The implied BL portfolio is constructed by reverse optimization of the utility function. The SR-BL portfolio is constructed by achieving maximal SR in the optimization problem. The MVaR-BL portfolio is constructed by achieving maximal return to VaR ratio in the optimization problem. The MCVaR-BL portfolio is constructed by achieving maximal return to CVaR ratio in the optimization problem. Note that the AHI of the static traditional mean-variance portfolio is equal to **12.636.**

| | | Return *(1)* | SD *(2)* | Skewness *(3)* | Kurtosis *(4)* | VaR *(5)* | CVaR *(6)* | AHI *(7)* | IR *(8)* | SR *(9)* | $\mu / VaR$ *(10)* | $\mu / CVaR$ *(11)* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Panel A. The benchmark portfolio and the 1/N portfolio* | | | | | | | | | | | | |
| Benchmark | | 0.005 | 0.039 | -0.855 | 5.487 | 0.107 | 0.149 | 0.075 | NaN | 0.135 | 0.049 | 0.036 |
| 1/N | | 0.005 | 0.038 | -0.903 | 5.441 | 0.126 | 0.150 | 0.033 | -0.070 | 0.118 | 0.036 | 0.030 |
| *Panel B. The implied BL portfolio* | | | | | | | | | | | | |
| Implied BL | DCC | 0.008 | 0.042 | -0.230 | 4.201 | 0.107 | 0.112 | 0.094 | 0.157 | 0.199 | 0.078 | 0.074 |
| | EWMA | 0.006 | 0.039 | -0.276 | 4.250 | 0.105 | 0.107 | 0.102 | 0.047 | 0.153 | 0.056 | 0.055 |
| | RW | 0.006 | 0.039 | -0.397 | 4.534 | 0.106 | 0.121 | 0.106 | 0.074 | 0.155 | 0.058 | 0.050 |
| *Panel C. The SR-BL portfolio* | | | | | | | | | | | | |
| SR-BL | DCC | 0.007 | 0.042 | -0.628 | 5.599 | 0.109 | 0.151 | 0.097 | 0.075 | 0.164 | 0.063 | 0.045 |
| | EWMA | 0.006 | 0.038 | -0.231 | 4.121 | 0.093 | 0.101 | 0.098 | 0.072 | 0.161 | 0.067 | 0.061 |
| | RW | 0.006 | 0.039 | -0.462 | 4.716 | 0.111 | 0.123 | 0.103 | 0.038 | 0.145 | 0.052 | 0.047 |

**Table 3 (continued)**

**Table 3 (continued)**

| | | Return (1) | SD (2) | Skewness (3) | Kurtosis (4) | VaR (5) | CVaR (6) | AHI (7) | IR (8) | SR (9) | $\mu / VaR$ (10) | $\mu / CVaR$ (11) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Panel D. The MVaR-BL portfolio* | | | | | | | | | | | | |
| *Panel D1. The MVaR-BL portfolio (99% Confidence Level)* | | | | | | | | | | | | |
| MVaR-BL N | DCC | 0.008 | 0.039 | -0.452 | 4.129 | 0.098 | 0.116 | 0.159 | 0.130 | 0.196 | 0.079 | 0.066 |
| | EWMA | 0.007 | 0.038 | -0.234 | 4.788 | 0.100 | 0.118 | 0.098 | 0.089 | 0.174 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.271 | 4.399 | 0.100 | 0.105 | 0.103 | 0.100 | 0.170 | 0.065 | 0.062 |
| MVaR-BL t | DCC | 0.008 | 0.039 | -0.439 | 4.101 | 0.098 | 0.116 | 0.161 | 0.137 | 0.200 | 0.080 | 0.067 |
| | EWMA | 0.007 | 0.038 | -0.237 | 4.778 | 0.100 | 0.118 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.274 | 4.390 | 0.100 | 0.105 | 0.103 | 0.099 | 0.170 | 0.065 | 0.062 |
| *Panel D2. The MVaR-BL portfolio (95% Confidence Level)* | | | | | | | | | | | | |
| MVaR-BL N | DCC | 0.007 | 0.040 | -0.686 | 4.694 | 0.129 | 0.132 | 0.146 | 0.099 | 0.180 | 0.056 | 0.055 |
| | EWMA | 0.007 | 0.038 | -0.235 | 4.793 | 0.100 | 0.118 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.271 | 4.401 | 0.100 | 0.105 | 0.103 | 0.100 | 0.170 | 0.065 | 0.062 |
| MVaR-BL t | DCC | 0.008 | 0.039 | -0.451 | 4.128 | 0.098 | 0.116 | 0.158 | 0.130 | 0.196 | 0.079 | 0.066 |
| | EWMA | 0.007 | 0.038 | -0.233 | 4.798 | 0.100 | 0.118 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.271 | 4.399 | 0.100 | 0.105 | 0.103 | 0.100 | 0.170 | 0.065 | 0.062 |
| *Panel D3. The MVaR-BL portfolio (90% Confidence Level)* | | | | | | | | | | | | |
| MVaR-BL N | DCC | 0.007 | 0.040 | -0.689 | 4.708 | 0.130 | 0.132 | 0.146 | 0.099 | 0.180 | 0.055 | 0.054 |
| | EWMA | 0.007 | 0.038 | -0.237 | 4.799 | 0.100 | 0.118 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.271 | 4.402 | 0.100 | 0.105 | 0.103 | 0.100 | 0.170 | 0.065 | 0.062 |
| MVaR-BL t | DCC | 0.008 | 0.039 | -0.443 | 4.126 | 0.098 | 0.116 | 0.156 | 0.131 | 0.196 | 0.079 | 0.066 |
| | EWMA | 0.007 | 0.038 | -0.236 | 4.805 | 0.100 | 0.118 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.271 | 4.401 | 0.100 | 0.105 | 0.103 | 0.100 | 0.170 | 0.065 | 0.062 |

**Table 3 (continued)**

**Table 3 (continued)**

| | | Return (1) | SD (2) | Skewness (3) | Kurtosis (4) | VaR (5) | CVaR (6) | AHI (7) | IR (8) | SR (9) | $\mu / VaR$ (10) | $\mu / CVaR$ (11) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Panel F. The MCVaR-BL portfolio* | | | | | | | | | | | | |
| *Panel F1. The MCVaR-BL portfolio (99% Confidence Level)* | | | | | | | | | | | | |
| MCVaR-BL N | DCC | 0.007 | 0.039 | -0.460 | 4.041 | 0.098 | 0.116 | 0.154 | 0.119 | 0.190 | 0.076 | 0.064 |
| | EWMA | 0.007 | 0.038 | -0.237 | 4.784 | 0.100 | 0.118 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.271 | 4.399 | 0.100 | 0.105 | 0.103 | 0.100 | 0.170 | 0.065 | 0.062 |
| MCVaR-BL t | DCC | 0.008 | 0.039 | -0.436 | 4.063 | 0.098 | 0.116 | 0.164 | 0.143 | 0.207 | 0.083 | 0.069 |
| | EWMA | 0.007 | 0.038 | -0.232 | 4.766 | 0.100 | 0.117 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.272 | 4.390 | 0.100 | 0.105 | 0.103 | 0.098 | 0.170 | 0.065 | 0.062 |
| *Panel F2. The MCVaR-BL portfolio (95% Confidence Level)* | | | | | | | | | | | | |
| MCVaR-BL N | DCC | 0.008 | 0.039 | -0.451 | 4.127 | 0.098 | 0.116 | 0.158 | 0.131 | 0.196 | 0.079 | 0.066 |
| | EWMA | 0.007 | 0.038 | -0.233 | 4.798 | 0.100 | 0.118 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.271 | 4.398 | 0.100 | 0.105 | 0.103 | 0.100 | 0.170 | 0.065 | 0.062 |
| MCVaR-BL t | DCC | 0.008 | 0.039 | -0.452 | 4.124 | 0.098 | 0.116 | 0.158 | 0.130 | 0.196 | 0.079 | 0.066 |
| | EWMA | 0.007 | 0.038 | -0.237 | 4.782 | 0.100 | 0.118 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.274 | 4.390 | 0.100 | 0.105 | 0.103 | 0.099 | 0.170 | 0.065 | 0.062 |
| *Panel F3. The MCVaR-BL portfolio (90% Confidence Level)* | | | | | | | | | | | | |
| MCVaR-BL N | DCC | 0.008 | 0.039 | -0.444 | 4.126 | 0.098 | 0.116 | 0.157 | 0.131 | 0.196 | 0.079 | 0.066 |
| | EWMA | 0.007 | 0.038 | -0.236 | 4.805 | 0.100 | 0.118 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.272 | 4.399 | 0.100 | 0.105 | 0.103 | 0.100 | 0.170 | 0.065 | 0.062 |
| MCVaR-BL t | DCC | 0.008 | 0.039 | -0.452 | 4.127 | 0.098 | 0.116 | 0.159 | 0.131 | 0.197 | 0.079 | 0.066 |
| | EWMA | 0.007 | 0.038 | -0.236 | 4.785 | 0.100 | 0.118 | 0.098 | 0.088 | 0.173 | 0.066 | 0.056 |
| | RW | 0.007 | 0.038 | -0.271 | 4.399 | 0.100 | 0.105 | 0.103 | 0.100 | 0.170 | 0.065 | 0.062 |

**Table 4 Out-of-sample Risk-Adjusted Unconstrained BL Portfolios Performance**

Table 4 reports realized risk-adjusted unconstrained BL portfolio performance in the period from August 2003 to August 2015. Return is the average realized excess return, Sharpe Ratio (SR) is the average excess realized return divided by the standard deviation (SD). Average Herfindahl index (AHI) is used to measure diversification. Information Ratio (IR) is the average active return divided by the standard deviation of active return. Both VaR and CVaR are measured on the empirical distribution. $\mu / VaR$ and $\mu / CVaR$ evaluate the excess return per unit of tail risk. In the construction of the portfolio, both VaR and CVaR are estimated by the parametric method with assumption of normal distribution ('N') and t-distribution ('t') at confidence levels of 99%, 95% and 90%. The implied variance-adjusted BL portfolio is constructed by reverse optimization of the utility function. The variance-adjusted SR-BL portfolio is constructed by achieving maximal SR in the optimization problem. The $\pi$ -VaR portfolio is constructed by achieving maximal VaR-adjusted return to SD ratio in the optimization problem. The $\pi$ -CVaR portfolio is constructed by achieving maximal CVaR-adjusted return to SD ratio in the optimization problem. Note that the AHI of the static traditional mean-variance portfolio is equal to **12.6362.**

| | Return *(1)* | SD *(2)* | Skewness *(3)* | Kurtosis *(4)* | VaR *(5)* | CVaR *(6)* | AHI *(7)* | IR *(8)* | SR *(9)* | $\mu / VaR$ *(10)* | $\mu / CVaR$ *(11)* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *Panel A. The implied variance-adjusted BL portfolio* | | | | | | | | | | | |
| Implied variance-adjusted BL | 0.009 | 0.043 | -0.363 | 4.278 | 0.100 | 0.136 | 0.307 | 0.094 | 0.206 | 0.088 | 0.065 |
| *Panel B. The variance-adjusted SR-BL portfolio* | | | | | | | | | | | |
| Variance-adjusted SR-BL | 0.007 | 0.050 | -2.015 | 15.205 | 0.143 | 0.264 | 0.324 | 0.036 | 0.136 | 0.047 | 0.025 |
| *Panel C. The $\pi$* -VaR portfolio | | | | | | | | | | | |
| *Panel C1. The $\pi$ -VaR portfolio (99% Confidence Level)* | | | | | | | | | | | |
| $\pi$ -VaR N | 0.008 | 0.063 | -1.624 | 13.727 | 0.319 | 0.335 | 0.541 | 0.046 | 0.126 | 0.025 | 0.024 |
| $\pi$ -VaR t | 0.008 | 0.045 | -0.658 | 6.454 | 0.128 | 0.178 | 0.381 | 0.080 | 0.180 | 0.063 | 0.045 |
| *Panel C2. The $\pi$ -VaR portfolio (95% Confidence Level)* | | | | | | | | | | | |
| $\pi$ -*VaR* N | 0.008 | 0.077 | -0.732 | 7.512 | 0.297 | 0.313 | 0.546 | 0.037 | 0.105 | 0.027 | 0.026 |
| $\pi$ -*VaR* t | 0.009 | 0.051 | -0.623 | 7.095 | 0.132 | 0.207 | 0.429 | 0.098 | 0.179 | 0.069 | 0.044 |
| *Panel C3. The $\pi$ -VaR portfolio (90% Confidence Level)* | | | | | | | | | | | |
| $\pi$ -*VaR* N | 0.008 | 0.059 | -0.787 | 6.805 | 0.221 | 0.233 | 0.886 | 0.043 | 0.131 | 0.035 | 0.033 |
| $\pi$ -*VaR* t | 0.007 | 0.046 | -1.881 | 15.289 | 0.137 | 0.244 | 0.440 | 0.049 | 0.162 | 0.054 | 0.030 |

**Table 4 (continued)**

**Table 4 (continued)**

| | Return<br>*(1)* | SD<br>*(2)* | Skewness<br>*(3)* | Kurtosis<br>*(4)* | VaR<br>*(5)* | CVaR<br>*(6)* | AHI<br>*(7)* | IR<br>*(8)* | SR<br>*(9)* | $\mu / VaR$<br>*(10)* | $\mu / CVaR$<br>*(11)* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *Panel D. The $\pi$ -CVaR portfolio* | | | | | | | | | | | |
| *Panel D1. The $\pi$ -CVaR portfolio (99% Confidence Level)* | | | | | | | | | | | |
| $\pi$ -CVaR N | 0.010 | 0.054 | -0.252 | 5.985 | 0.130 | 0.192 | 0.425 | 0.116 | 0.183 | 0.076 | 0.051 |
| $\pi$ -CVaR t | 0.009 | 0.048 | -0.372 | 5.491 | 0.127 | 0.170 | 0.382 | 0.104 | 0.187 | 0.070 | 0.052 |
| *Panel D2. The $\pi$ -CVaR portfolio (95% Confidence Level)* | | | | | | | | | | | |
| $\pi$ -CVaR N | 0.013 | 0.076 | 0.478 | 7.183 | 0.220 | 0.237 | 0.541 | 0.138 | 0.174 | 0.060 | 0.056 |
| $\pi$ -CVaR t | 0.009 | 0.051 | -0.337 | 5.812 | 0.129 | 0.184 | 0.407 | 0.110 | 0.185 | 0.073 | 0.051 |
| *Panel D3. The $\pi$ -CVaR portfolio (90% Confidence Level)* | | | | | | | | | | | |
| $\pi$ -CVaR N | 0.008 | 0.048 | -1.226 | 10.300 | 0.135 | 0.225 | 0.431 | 0.066 | 0.165 | 0.059 | 0.035 |
| $\pi$ -CVaR t | 0.010 | 0.056 | -0.182 | 6.060 | 0.138 | 0.198 | 0.436 | 0.120 | 0.182 | 0.074 | 0.051 |

**Table 5 Out-of-sample Dynamic BL Portfolios Performance Ranking**

Table 5 summarises out-of-sample performance evaluation ratios and report relevant rankings for the benchmark portfolio, 1/N portfolio, the DCC-based BL portfolios in the period from August 2003 to August 2015. Average Herfindahl index (AHI) is used to measure diversification. Information Ratio (IR) is the average active return divided by the standard deviation of active return. Both VaR and CVaR are measured on the empirical distribution. $\mu / VaR$ and $\mu / CVaR$ evaluate the excess return per unit of tail risk. The implied BL portfolio is constructed by reverse optimization of the utility function. The SR-BL portfolio is constructed by achieving maximal SR in the optimization problem. The MVaR-BL portfolio is constructed by achieving maximal return to VaR ratio in the optimization problem. The MCVaR-BL portfolio is constructed by achieving maximal return to CVaR ratio in the optimization problem. Based on Giacometti et al. (2007) method, the implied variance-adjusted BL portfolio is constructed by reverse optimization of the utility function. The variance-adjusted SR-BL portfolio is constructed by achieving maximal Variance-adjusted return to SD ratio in the optimization problem. The $\pi$ -VaR portfolio is constructed by achieving maximal VaR-adjusted return to SD ratio in the optimization problem. The $\pi$ -CVaR portfolio is constructed by achieving maximal CVaR-adjusted return to SD ratio in the optimization problem. Both VaR and CVaR are estimated by the parametric method with assumption of normal distribution ('N') and t-distribution ('t') at confidence levels of 99%. Note that the AHI of the static traditional mean-variance portfolio is equal to **12.636.**

| | SR | Ranking | $\mu / VaR$ | Ranking | $\mu / CVaR$ | Ranking | AHI | Ranking | IR | Ranking |
|---|---|---|---|---|---|---|---|---|---|---|
| Benchmark | 0.135 | 12 | 0.049 | 11 | 0.036 | 11 | 0.075 | **2** | NaN | NaN |
| 1/N | 0.118 | 14 | 0.036 | 13 | 0.030 | 12 | 0.033 | **1** | -0.070 | 13 |
| **Implied BL** | 0.199 | **4** | 0.078 | **5** | 0.074 | **1** | 0.094 | **5** | 0.157 | **1** |
| SR-BL | 0.164 | 10 | 0.063 | 10 | 0.045 | 10 | 0.097 | 6 | 0.075 | 10 |
| **MVaR 0.99 N** | 0.196 | **5** | 0.079 | **4** | 0.066 | **4** | 0.079 | **3** | 0.130 | **4** |
| **MVaR 0.99 t** | 0.200 | **3** | 0.080 | **3** | 0.067 | **3** | 0.080 | **4** | 0.137 | **3** |
| MCVaR 0.99 N | 0.190 | 6 | 0.076 | 6 | 0.064 | 6 | 0.154 | 7 | 0.119 | **5** |
| **MCVaR 0.99 t** | 0.207 | **1** | 0.083 | **2** | 0.069 | **2** | 0.164 | 8 | 0.143 | **2** |
| **Implied variance-adjusted BL** | 0.206 | **2** | 0.088 | **1** | 0.065 | **5** | 0.307 | 9 | 0.094 | 8 |
| Variance-adjusted SR-BL | 0.136 | 11 | 0.047 | 12 | 0.025 | 13 | 0.324 | 10 | 0.036 | 12 |
| VaR-adjusted SR-BL 0.99 N | 0.126 | 13 | 0.025 | 14 | 0.024 | 14 | 0.541 | 14 | 0.046 | 11 |
| VaR-adjusted SR-BL 0.99 t | 0.180 | 9 | 0.063 | 9 | 0.045 | 9 | 0.381 | 11 | 0.080 | 9 |
| CVaR-adjusted SR-BL 0.99 N | 0.183 | 8 | 0.076 | 7 | 0.051 | 8 | 0.425 | 13 | 0.116 | 6 |
| CVaR-adjusted SR-BL 0.99 t | 0.187 | 7 | 0.070 | 8 | 0.052 | 7 | 0.382 | 12 | 0.104 | 7 |