GAME-THEORETIC ANALYSIS OF BEHAVIOUR IN THE CONTEXT OF LONG-TERM RELATIONSHIPS

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Abstract

In this thesis an approach for modelling social interactions in a context of long-term relationships is developed in order to investigate apparently altruistic behaviour. The common model of social interactions is based on Prisoners' Dilemma game. Considering the interaction of the players not in isolation but in the context of conditions in which it takes place leads to the conclusion that cooperative behaviour may be rational or evolved. In this thesis this idea is taken further to consider different contexts of interaction.

Firstly, a three-player model is introduced in which the third player interacts with two other players engaged in a single interaction Prisoners' Dilemma. The existence of the third player in the interaction changes the payoffs in such a way that the two players are induced to cooperate.

The Iterated Prisoners' Dilemma is generalised by allowing additional states to exist in the game. This provides the possibility of introducing completely new types of strategies such as ''allocating tasks'' strategies. These strategies are relevant to the explanation of apparently altruistic behaviour since the observed behaviour for them is: one player cooperates while the other defects.

It is shown that ''allocating tasks with punishment'' and ''cooperating with punishment'' strategies can be Nash Equilibria. Populations which consist of different mixtures of ''allocating tasks (cooperating) without punishment'' and ''allocating tasks (cooperating) with punishment'' players can be the end points of the evolutionary process. There are ranges of parameters in the model for which the non-cooperative strategies considered are not Nash Equilibria, nor are they evolutionarily or asymptotically stable. Therefore, it can be concluded that cooperative populations can evolve under the influence of natural selection and it is possible to evolve to cooperative types of populations from populations initially composed of a majority of uncooperative individuals.

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Chapter 1

Introduction and Background

1.1 Introduction

Understanding of evolutionary mechanisms that can produce altruistic behaviour in animals has been regarded [1] as "the central theoretical problem in sociobiology" since the theory of evolution had been developed [2]. Gintis [3] gives the definition of the altruism as following: "an altruist is an agent who takes actions that improve the fitness or material well-being of other agents when more self-interested actions are available". Examples of altruistic behaviour among related and unrelated individuals can be observed in nature [4] and in many human societies. For instance, such behaviour has been observed in African lions [5], [6]. It has been suggested in [5] that female lions can be classified according to four discrete strategies: "unconditional cooperators" who always lead the response in the territorial defence, "unconditional laggards" who always lag behind, "conditional cooperators" who lag least when they are most needed, and "conditional laggards" who lag farthest when they are most needed. Although leaders recognize laggards and behave more cautiously in their presence, they continue to lead the response. For more examples see [7] where also further references can be found.

Since the pioneering work of Trivers [8] on reciprocal altruism the analysis of social interactions has become an important topic in Behavioural Ecology. In evolutionary models of interaction it is usually supposed that behaviours used by animals are determined by genes and, therefore, inherited and that welfare of the individuals is measured in terms of fitness (usually represented by the number of surviving offspring). If this point of view is adopted then by definition the fitness of an altruistic individual is less then others who choose "more self-interested actions", and therefore the result of evolutionary process must be extinction of the altruistic behaviour.

In order to explain the cooperative behaviour game-theoretic descriptions of social interactions are often used. Situations, in which the outcome of an interaction depends on the behaviour of all the individuals concerned, must be analysed using game theory. John von Neumann developed the mathematical foundations of Game Theory [9] and this theory was then applied to problems in Economics [10], [11]. In 1950 John Nash introduced the concept of the Nash Equilibrium [12] which made it possible to solve a wider range of games. Later, game theory was used to analyse problems in the biology of animal behaviour [13], [14], and the concept of an evolutionarily stable strategy was introduced [14]. Although evolutionarily stable strategies were introduced for biological reasons, evolutionary stability can be viewed as a generalisation of the Nash equilibrium concept to the case of models which may include evolution: for example, in economics a "learning evolution" (where the players switch to the most beneficial strategy) is used instead of Natural selection.

There is a standard game which has been used as a generic model of social interactions both by economists [15] and by biologists [16]. It is called Prisoners' Dilemma game. In the non-iterated (single interaction) form of this game, two prisoners decide independently whether or not to confess to a crime that they committed together. The police lack sufficient evidence to secure a conviction unless at least one of the prisoners confesses. If neither confesses, then they both will be convicted of a minor offence and sentenced to one month in prison. If one of them confesses, he will be released, while the other will be sentenced to nine months in prison. If both confess, they will both be convicted, but their sentences will be reduced to six months. We can represent this game by the following table

$Player_1 \setminus Player_2$	Do not confess	Confess
Do not confess	$-1, -1$	$-9, 0$
Confess	$0, -9$	$-6, -6$

. (1.1)

To solve this game let us find what the first player should do. If the second player does not confess, then the first player should confess because then he will be released. If the second player confesses, then the first player should also confess because then he will be sentenced to six months in prison rather than nine. So whatever the second player does, the first player is better off if he confesses. Similarly, the second player is better off confessing no matter what the first player does. So both prisoners should confess, i.e. [Confess, Confess] is the solution of the game. If prisoners do not confess it can be interpreted as cooperating with each other. Here if a prisoner confesses it can be interpreted as a defection since the other prisoner will end up worse off in this case.

In general a game is considered to be a Prisoners' Dilemma if it is described by a table

$Player_1 \setminus Player_2$	Cooperate	Defect
Cooperate	h_1, h_1	h_2, h_3
Defect	h_3, h_2	h_4, h_4

, (1.2)

where the payoffs h_j, $j = 1, 2, 3, 4$, satisfy the inequalities

$$h_2 < h_4 < h_1 < h_3. \tag{1.3}$$

The solution of this game can be obtained in the same way as it has just been done for the game (1.1). If the second player cooperates, then the first player should defect because that gives him payoff equal h_3 rather than h_1. If the second player defects, then the first player should also defect because that gets him payoff h_4 rather than h_2. So whatever the second player does, the first player is better off if he defects. Similarly, the second player is better off defecting no matter what the first player does. So both players should defect, i.e. [Defect, Defect] is the solution of the game. Nevertheless cooperative behaviour is observed in many human and animal societies.

There exist different approaches that try to shed light on the existence of the altruistic behaviour. One of them is to consider selection on groups [3]: "suppose there are many groups, and the altruists so enhance the fitness of the groups they are in, compared to the groups without altruists, that the former outcompete the latter, so that the average fitness of the altruist is higher than that of the selfish agent." The main idea here that the selection works not on individuals but on a type of behaviour. By benefiting the group which contains a big fraction of individuals who use the same (altruistic) behaviour, an altruistic individual therefore contributes to benefiting of its own type of behaviour. In this instance there exists Hamilton's Law [17] which describes such situation. This law is as follows. Suppose there are two types of agents: "selfish" and "altruist". By cooperating, an agent produces a fitness increment $b > 0$ for his partner, at personal cost $c < b$, and by defecting an agent produces zero at zero cost. Notice that this interaction is the Prisoners' dilemma and can be described by the following bi-matrix

$Player_1 \setminus Player_2$	Cooperate	Defect
Cooperate	$b-c, b-c$	$-c, b$
Defect	$b, -c$	$0, 0$

.

Suppose the associating agents pair off in each period, and each type is likely to meet its own type with probability $r \geq 0$, and a random member of the population with probability $1-r$. Then a small number of cooperators can invade a population of selfish actors if and only if $br \geq c$. In this approach the agent is transferred to the type of behavior rather than individual and "altruistic" acts occur because the agent, in fact, is following its own self-interest. Therefore, this approach states that altruistic behaviour is only "apparently altruistic" and not "really altruistic".

Another approach that is used to explain the evolution of cooperative behaviour involves a repeated game based on the Prisoners' Dilemma. Individuals interact repeatedly and play the same game (the Prisoners' Dilemma) at each interaction [16], [18]. Repeating the Prisoners' Dilemma Game provides the possibility for considering "punishing strategies". The basic idea is that " if you do not cooperate now, I will punish you in the future". Using this approach it is possible to conclude that cooperative behaviour may be rational or evolved [16], [18]. In this case the observed altruistic behaviour is also shown to be only "apparently altruistic" since the individual is expecting reciprocation many times in the future.

Although this approach has played an important role in the explanation of cooperative behaviour, it is insufficient. For example, in a population where all members are using a "punishing strategy", cooperative behaviour should be observed at all interactions. On the other hand, if a member of the population does not cooperate during an interaction, then punishment should be observed subsequently. Nevertheless, for example in African lions, non-cooperative behaviour has been observed but this has not been followed by any identifiable punishment behaviour [5], [6]. Therefore this approach is not sufficient to explain all social behaviour.

The Prisoners' Dilemma model of social interaction is one of the most studied of all game theoretic models. There are many approaches to investigating the possibility of cooperation in this model, most of which are based on computer simulations [19]-[25]. One approach consists of considering ever more complex strategies in an evolutionary model [19], [20]. One way of producing an elaborate strategy is to allow players to learn from experience [21]. The results of [21] show that mutual cooperation can be maintained when players have a primitive learning ability. It was shown that under proposed learning evolution some cooperative strategies can invade not only unconditional cooperation, Tit for Tat and Pavlov strategies but also noncooperative strategies. Another approach is to embed the prisoners' dilemma into a spatial context [22], [23], [24]. For example in [22] the version of the iterated prisoners' dilemma with only unconditionally cooperating and unconditionally defecting players interacting with the immediate neighbours was considered. It was shown [22] that such a model can generate chaotically changing spatial patterns, in which cooperators and defectors both persist indefinitely. In [23] it was found that the pattern generated by groups of cooperators exhibits the scale-invariance which is typical of self-organized criticality [26]. In [24] it is shown that a spatial context for the interaction encourages cooperative behaviour. Multi players games are also considered [25] with solutions been found by computer simulations. There is also a direction of research based on the idea of private information available to the players. That is, each player may have some information about themselves or state of the game which is not available to other players. An overview of the recent developments in this area which have revealed the possibility of cooperation under condition of private information can be found in [27].

Whatever approach is taken cooperative behaviour becomes understandable when the interaction is considered in a wider context (such as length of interaction, for example). In this thesis we will generalise the Prisoners' Dilemma to consider models with more complex interactions. In chapter 2 we will give an example showing the importance of considering an interaction in the context of other possible interactions. We will show that if there is the third player interacting with the two players engaged in Prisoners' Dilemma, then the payoffs of the two players can be changed in such a way that cooperation becomes a rational outcome. Existing models based on the Prisoners' Dilemma do not allow for the possibility that individuals may interact in more than one context (i.e. play more than one game). We will extend (chapters 5-7) the game theoretic approach to modelling social interactions to include behaviours beyond simple cooperation and defection in a single context.

To analyse games two main approaches are usually used in game theory. Classical game theory supposes that the analysed game is played once by rational players. In this approach a concept of Nash Equilibrium introduced by Nash in [12] is used in order to describe the solutions of the game, where

NE: a Nash Equilibrium is a profile of strategies such that each player's strategy is an optimal response to the other player's strategies.

We will discuss this concept in more detail in section 1.2. There the formal definitions are introduced and the Prisoners' Dilemma game is analysed as an example. We will show how to obtain the standard conclusion that cooperative behaviour is not rational in the Prisoners' Dilemma.

Another approach used to analysed games comes from evolutionary game theory which supposes that the game is played repeatedly by players who are randomly selected from a large population. Each player is programmed with some type of behaviour. It is assumed that some evolutionary selection process operates over time on the population distribution of behaviours. Under this approach the ideas of an Evolutionarily Stable Strategy (introduced by J. Maynard Smith and G. R. Price in [13]) and Replicator Dynamics (introduced by P. D. Taylor and L. B. Jonker in [28]) are used to investigate the game:

ESS: an Evolutionarily Stable Strategy is a strategy such that, if all the members of a population adopt it, then no mutant strategy could invade the population under the influence of the evolutionary selection process, and

RD: Standard Replicator Dynamics is a dynamical system that describes changes of a population state in a population whose members are playing a symmetric two-person game.

In this thesis when we refer to the Replicator Dynamics we will mean the standard Replicator Dynamics for which members of the population are only using pure strategies while playing a symmetric two-person game.

The main idea of evolutionary game theory is that the evolutionary selection process is operating as a force that changes the structure of population towards the optimisation of fitness. The notion of an Evolutionarily Stable Strategy describes stable results of the selection process. But the explicit dynamic analysis shows that populations may exist that are the end-points of an evolutionary process, but the corresponding strategy (that prescribes choosing each particular action of the two-person game with probabilities which are equal to fractions of individuals who use this action) is not an Evolutionarily Stable Strategy [29], [30]. We discuss these ideas in more detail in sections 1.3 and 1.4

In section 1.5 the structure of the thesis is outlined and the results obtained are described.

1.2 Basic Game theory.

In this section the definition of finite games in normal form and Nash equilibrium solutions for such games are discussed.

Let

- $I = \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, $n \geq 2$, be a set of players,
- $A^i = \{a_1^i, \dots, a_{m_i}^i\}$ be a set of actions for player i, and
- $\pi_i(a) = \pi_i(a^1, \dots, a^n)$ be the payoff for player i, specified for every possible combination of actions $a = \{a^1, \dots, a^n\}$ chosen by all the players, where $a^j \in A^j$ is the action chosen by player j.

Definition 1.1 *Sets I, A^i and payoffs $\pi_i(a)$, $i = 1, \dots, n$, define a game in normal form.*

To play the game, each player specifies a strategy: a rule which determines an action to be chosen by the player for every circumstance in which a decision must be made. Pure and mixed strategies are usually distinguished.

Definition 1.2 *For each player $i \in I$ let $\mathfrak{A}^i = \{\alpha_1^i, \dots, \alpha_{m_i}^i\}$ be a set of pure strategies, where strategy α_j^i, $j = 1, \dots, m_i$, specifies that player i chooses action a_j^i with probability 1. A vector $\alpha = \{\alpha^1, \dots, \alpha^n\}$, where $\alpha^i \in \mathfrak{A}^i$ is a pure strategy for player i, is called a pure-strategy profile.*

Definition 1.3 *A mixed (behavioural) strategy σ_i for player i, is a probability distribution over the set of actions A^i.*

A mixed strategy $\boldsymbol{\sigma}_i = \{\sigma_i^1, \dots, \sigma_i^{m_i}\}$ is a vector in $\mathbb{R}^{m_i}$. Its j^{th} coordinate σ_i^j is the probability assigned by $\boldsymbol{\sigma}_i$ to the action a_j^i. For a mixed-strategy profile $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\}$ payoffs $\pi_i(\boldsymbol{\sigma})$ for each player i are calculated as

$$\pi_i(\boldsymbol{\sigma}) = \pi_i(\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n) = \sum_{\substack{j_1 = 1, \dots, m_1; \\ \vdots \\ j_n = 1, \dots, m_n.}} \left(\prod_{k=1}^{n} \sigma_k^{j_k} \right) \pi_i(a_{j_1}^1, \dots, a_{j_n}^n).$$

Remark 1.1 *The pure strategy α_j^i that prescribes choosing action a_j^i with probability 1 can be considered as being equivalent to the mixed strategy $\boldsymbol{\sigma}_i = \left\{0, \dots, 0, \underset{j}{1}, 0, \dots, 0\right\}$.*

Remark 1.2 *Since $\boldsymbol{\sigma}_i$ is a probability distribution,* $\sum_{j=1}^{m_i} \sigma_i^j = 1$, $i \in I$.

Let us now define the notion of Nash equilibrium (introduced by Nash in [12]). Denote $\boldsymbol{\sigma}_{-i} = (\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_{i-1}, \boldsymbol{\sigma}_{i+1}, \dots, \boldsymbol{\sigma}_n)$ and $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) = (\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n)$.

Definition 1.4 *A best reply strategy $\widehat{\boldsymbol{\sigma}}_i$ of the i^{th} player to the strategies $\widetilde{\boldsymbol{\sigma}}_{-i}$ is a strategy such that* $\widehat{\boldsymbol{\sigma}}_i = \arg\max_{\sigma_i} \pi_i(\boldsymbol{\sigma}_i, \widetilde{\boldsymbol{\sigma}}_{-i})$.

Definition 1.5 *A Nash equilibrium for a game in normal form is a vector of mixed strategies* $\boldsymbol{\sigma}^* = \{\boldsymbol{\sigma}_1^*, \dots, \boldsymbol{\sigma}_n^*\}$ *such that* $\boldsymbol{\sigma}_i^* = \arg\max_{\sigma_i} \pi_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}^*)$, *for each player $i \in I$.*

The concept of Nash Equilibrium assumes that no player wishes to change his strategy (since it gives him the highest payoff) when the Nash Equilibrium strategies are played by all players. We will refer to this as to rationality of the players.

Example 1.1. Let us consider the Prisoners' Dilemma game defined by table (1.2) and conditions (1.3). The set of players for this game is $I = \{1, 2\}$. The sets of actions A^i, $i = 1, 2$, consist of two actions "Cooperate" and "Defect" for each player: $A^i = \{\text{Cooperate, Defect}\}$, $i = 1, 2$. Payoffs for players are defined as follows

$$\pi_1(C,C) = h_1, \quad \pi_1(C,D) = h_2, \quad \pi_1(D,C) = h_3, \quad \pi_1(D,D) = h_4,$$
$$\pi_2(C,C) = h_1, \quad \pi_2(C,D) = h_3, \quad \pi_2(D,C) = h_2, \quad \pi_2(D,D) = h_4.$$

The set of pure strategies for each player consists of two strategies

$$\alpha_1^i = \{\text{choose action "Cooperate" with probability } 1\}, \qquad i = 1, 2,$$

and

$$\alpha_2^i = \{\text{choose action "Defect" with probability } 1\}, \qquad i = 1, 2.$$

Mixed strategies for i^{th} player can be described by the formula $\boldsymbol{\sigma}_i = (\sigma_i, 1 - \sigma_i)$, where σ_i is the probability with which action "Cooperate" is chosen. Hence the payoffs $\pi_i(\boldsymbol{\sigma})$ for mixed strategy profile $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ are given by the formula

$$\begin{aligned}\pi_i(\boldsymbol{\sigma}) &= \sigma_1 \sigma_2 \pi_i(C,C) + \sigma_1 (1 - \sigma_2) \pi_i(C,D) \\ &\quad + (1 - \sigma_1) \sigma_2 \pi_i(D,C) + (1 - \sigma_1)(1 - \sigma_2) \pi_i(D,D), \qquad i = 1, 2.\end{aligned}$$

Therefore,

$$\begin{aligned}\pi_1(\boldsymbol{\sigma}) &= ((h_1 + h_4 - h_2 - h_3)\sigma_2 + h_2 - h_4)\sigma_1 + (1 - \sigma_2) h_4 + \sigma_2 h_3, \\ \pi_2(\boldsymbol{\sigma}) &= ((h_1 + h_4 - h_2 - h_3)\sigma_1 + h_2 - h_4)\sigma_2 + (1 - \sigma_1) h_4 + \sigma_1 h_3.\end{aligned}$$

To find Nash equilibrium solutions we need to solve the following system of equations

$$\begin{cases} \sigma_1^* = \arg\max_{\sigma_1} \pi_1(\sigma_1, \sigma_2^*) \\ \quad = \arg\max_{\sigma_1} \left((h_1 + h_4 - h_2 - h_3)\,\sigma_2^* + h_2 - h_4\right)\sigma_1 + (1 - \sigma_2^*)\,h_4 + \sigma_2^* h_3 \\ \sigma_2^* = \arg\max_{\sigma_2} \pi_2(\sigma_1^*, \sigma_2) \\ \quad = \arg\max_{\sigma_2} \left((h_1 + h_4 - h_2 - h_3)\,\sigma_1^* + h_2 - h_4\right)\sigma_2 + (1 - \sigma_1^*)\,h_4 + \sigma_1^* h_3 \end{cases} . \tag{1.4}$$

Let us notice that, since inequalities (1.3) hold,

$$(h_1 + h_4 - h_2 - h_3)\,x + h_2 - h_4 \neq 0 \qquad \forall x \in [0, 1]\,.$$

Therefore, since $\pi_1(\sigma_1, \sigma_2^*)$ and $\pi_2(\sigma_1^*, \sigma_2)$ are linear in σ_1 and σ_2, respectively, they take their maximum values on the interval $[0, 1]$ at the boundary points. Hence, any solutions for system (1.4) must have one of the following forms

$$\begin{cases} \sigma_1 = 1 \\ \sigma_2 = 1 \end{cases}, \quad \begin{cases} \sigma_1 = 1 \\ \sigma_2 = 0 \end{cases}, \quad \begin{cases} \sigma_1 = 0 \\ \sigma_2 = 1 \end{cases} \quad or \quad \begin{cases} \sigma_1 = 0 \\ \sigma_2 = 0 \end{cases}.$$

Substituting these values into system (1.4) and taking into account inequalities (1.3), we find that there is only one solution $\sigma_1 = 0$ and $\sigma_2 = 0$. This means that for the Prisoners' Dilemma game there exists only one Nash equilibrium solution, which is a symmetric strategy profile $\boldsymbol{\sigma}^* = \{\sigma_1^*, \sigma_2^*\}$, with $\boldsymbol{\sigma}_1^* = \boldsymbol{\sigma}_2^* = (0, 1)$. That is, both players choose "Defect" with probability 1.

From this example we can see that analysis of social interactions based on the single-interaction Prisoners' Dilemma game leads to the conclusion that cooperative behaviour is not rational. Also, since $h_4 < h_1$, rational behaviour leads to an outcome which is sub-optimal for both players. It is this dilemma which provides the interest in this simple game, and has led to it being used as a model for economic and biological interactions. In the next chapters I introduce more complex models as an attempt to overcome this dilemma. But first let us discuss the standard model based on the idea of iterating the Prisoners' Dilemma.

1.3 Repeated games.

Many existing models of social interactions which investigate the evolution of cooperative behaviour involve a repeated game based on the Prisoners' Dilemma [16], [18]. Let us consider again the Prisoner's Dilemma game defined by table (1.2) and conditions (1.3). Assume that the game is played an infinite number of times and that there is a constant discount factor β between each round of the game, so that the expected number of rounds in the game is $\frac{1}{1-\beta}$. The game obtained in this way is called the *Iterated Prisoners' Dilemma*. In Economics the discount factor β usually represents inflation over time. From the biological point of view β represents the probability that players participate in the next round of interaction and can be interpreted as surviving probability.

The Iterated Prisoners' Dilemma game is commonly used as an example of a model of social interaction for which cooperative behaviour may be evolved [16], [18]. The main difference of such models from the single interaction models is that when players are making a decision about which action to choose in each Prisoners' Dilemma game they can use information about actions chosen by both players in the past (for example, in the previous round). Therefore, players can "punish" their partner if he did not cooperate in the past. The most famous of such strategies is Tit for Tat [18], [31], [32], which begins by cooperating and thereafter mimics an opponent's play in the previous stage. Many numerical, analytical and computer simulation studies [15], [18], [31], [32] show that this strategy is a successful one. Later [16] it was discovered that, in simulated interactions of stochastic strategies with the memory of one previous state, the players using a Pavlov-type strategy (Win-Stay Lose-Switch) eventually dominate a population which was started in the completely random configuration.

Natural selection on strategies with memory of one previous state has also been studied in [20]. In this work a deterministic dynamical system based on differential reproductive success has been introduced. It has been shown that the Pavlov strategy is the only cooperative strategy which cannot be invaded by a similar strategy when the model is restricted to strategies with memory of one previous state with 1% noise level.

It is known [20] that an Iterated Prisoner's Dilemma interaction between players using strategies with memory of one previous state can be modeled by a Markov Process. Another closely related approach for analysis of repeated games is developed in [33]. In this work strategies are represented by way of a state space, where a player's choice of action depends on the player's current state. We will extend the idea of using competitive Markov decision processes (see [20], [34]) to include the possibility for analysis of more complex models of social interactions. We will use properties of such processes to find the Nash equilibrium solutions for these models.

1.4 Evolutionary Dynamics.

1.4.1 Evolutionarily Stable Strategies.

The concept of an Evolutionarily Stable Strategy [13], [14] is one of the main concepts in evolutionary game theory. In the approach proposed by J. Maynard Smith and G. R. Price it is supposed that there is a large population of individuals. The individuals in the population are "programmed" with certain types of behaviour. Then a small population share of individuals who are "programmed" to play some other pure or mixed strategy is introduced into the population. A strategy σ' is called an Evolutionarily Stable Strategy if, for each mutant strategy, there exists a positive invasion barrier such that if the population share of individuals playing the mutant strategy falls below this barrier, then the strategy σ' earns a higher payoff than the mutant strategy [35].

This definition can be formalised as follows [13], [14], [35].

Definition 1.6 *A strategy σ' is an Evolutionarily Stable Strategy if for every strategy $\sigma \neq \sigma'$ there exists some $\varepsilon_\sigma \in (0,1)$ such that for all $\varepsilon \in [0, \varepsilon_\sigma)$ the payoff to a player who uses strategy σ' in a population consisting of ε-fraction of σ players and $(1-\varepsilon)$-fraction of σ' players is greater than the payoff to a player who uses strategy σ in the same population. It means that inequality*

$$\pi(\sigma', \varepsilon\sigma + (1-\varepsilon)\,\sigma') > \pi(\sigma, \varepsilon\sigma + (1-\varepsilon)\,\sigma') \tag{1.5}$$

holds for all $\varepsilon \in [0, \varepsilon_\sigma)$.

If the case of pairwise contests is considered then there is an equivalent way of defining an Evolutionarily Stable Strategy which is given by the following proposition.

Proposition 1.1 *A strategy σ' is an Evolutionarily Stable Strategy if and only if it satisfies the first-order and the second-order best-reply conditions:*

1. $\pi(\sigma', \sigma') \geq \pi(\sigma, \sigma')$ *for any* σ;

2. *if* $\pi(\sigma', \sigma') = \pi(\sigma, \sigma')$ *then* $\pi(\sigma', \sigma) > \pi(\sigma, \sigma)$ *for any* $\sigma \neq \sigma'$.

Example 1.2. The defection strategy in the Prisoners' Dilemma is evolutionarily stable. In section 4.1.3 we will analyse the Iterated Prisoners' Dilemma and it will be shown that neither Tit for Tat nor unconditional cooperation (the strategy that prescribes cooperation at every round) are evolutionarily stable since, for example, we will have that

$$\pi(TFT,TFT) = \pi(TFT,C) = \pi(C,C) = \pi(C,C).$$

Here C stands for unconditional cooperation and TFT stands for Tit for Tat.

It is immediately apparent from Proposition 1.1 that in order to be Evolutionarily Stable a strategy must be a Nash Equilibrium of the corresponding two-person game. Therefore, the concept of Evolutionarily Stable Strategy generalises the concept of Nash Equilibrium, which assumes the rationality of playing individuals, to the case of evolving population. However, both the Evolutionarily Stable Strategy concept and the Nash Equilibrium concept describe only results of the evolutionary process and do not explain the dynamics leading to such an outcome. In the next section we discuss the concept of Replicator Dynamics which models the selection mechanism.

1.4.2 Replicator Dynamics.

Consider an infinitely large population of individuals who are programmed with certain types of behaviour (pure strategies) $i \in \{1, \dots, n\}$. Denote by x^i the proportion of the individuals in the population who adopt behaviour i. The Replicator Dynamics proposed by P. D. Taylor and L. B. Jonker [28] describes changes of a population state $X = (x_1, \dots, x_n)$ in a population whose members are playing a symmetric two-person game with the payoffs given by matrix A, with elements a_{ij}. In this work we will use two forms of the most commonly used version of the Replicator Dynamics: the continuous-time form and the discrete-time form. The continuous-time Replicator Dynamics is usually used for an analytical approach to the problem. On the other hand the discrete-time version of the Replicator Dynamics is more applicable when computational analysis and simulations are performed.

The Replicator Dynamics in continuous time is the following dynamical system.

$$\frac{dx_i}{dt} = x_i \left(\left\{ \sum_{j=1}^{n} a_{ij} x_j \right\} - XAX^T \right), \qquad i = 1, \dots, n. \tag{1.6}$$

Using relationship $x_n = 1 - x_1 - \dots - x_{n-1}$, which is preserved by the dynamical system, we can reduce the number of equations in system (1.6) to $n-1$. Denote $x = (x_1, \dots, x_{n-1})$.

Then

$$\frac{dx_i}{dt} = G_i = x_i \left(\left\{ \sum_{j=1}^{n-1} (a_{ij} - a_{in})\, x_j + a_{in} \right\} - \mu(x) \right), \qquad i = 1, \ldots n-1, \tag{1.7}$$

where

$$\mu(x) = (x_1, \cdots, x_{n-1}, 1 - x_1 - \ldots - x_{n-1})\, A\, (x_1, \cdots, x_{n-1}, 1 - x_1 - \ldots - x_{n-1})^T .$$

Since the x_i represent population shares (proportions), we consider the solutions of the system (1.7) that are restricted to the simplex

$$\Delta = \left\{ x : \bigcap_{i=1}^{n-1} \left(0 \le x_i \right) \cap \left(\sum_{i=1}^{n-1} x_i \le 1 \right) \right\}. \tag{1.8}$$

This system of differential equations, in general, has no explicit analytical solution. To analyse the behaviour of the solutions and describe the dynamics we apply methods of the qualitative theory of the dynamical systems (due to [36]-[40]) and the concepts of evolutionarily and asymptotic stability presented in [35], [41], [42]. In chapter 4 we discuss these methods in detail.

To obtain the discrete time analogue of the Replicator Dynamics (see [35]) let us make the following time substitution $\tau = \mu(x)\, t$, then the dynamical system (1.7) becomes

$$\frac{dx_i}{d\tau} = \frac{x_i}{\mu(x)} \left\{ \sum_{j=1}^{n-1} (a_{ij} - a_{in})\, x_j + a_{in} \right\} - x_i, \qquad i = 1, \ldots n-1.$$

By replacing $\frac{dx_i}{d\tau}$ by $\frac{\Delta x_i}{\Delta \tau}$, where $\Delta\tau = h$ is fixed and $\Delta x_i = x_i^{k+1} - x_i^k$ we obtain

$$\frac{\Delta x_i}{\Delta \tau} = \frac{x_i^{k+1} - x_i^k}{h} = \frac{x_i^k}{\mu(x^k)} \left\{ \sum_{j=1}^{n-1} (a_{ij} - a_{in})\, x_j^k + a_{in} \right\} - x_i^k, \qquad i = 1, \ldots n-1. \tag{1.9}$$

Choosing $h = 1$ in the above formula, the following discrete time analogue of the Replicator Dynamics (see [35]) is obtained

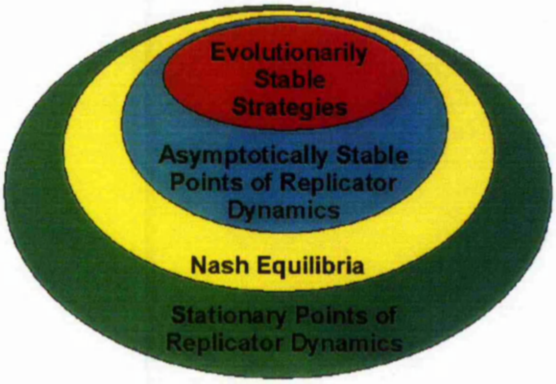
$$x_i^{k+1} = \frac{x_i^k}{\mu(x^k)} \left\{ \sum_{j=1}^{n-1} (a_{ij} - a_{in})\, x_j^k + a_{in} \right\}, \qquad i = 1, \ldots n-1. \tag{1.10}$$

This form of the Replicator Dynamics was used to perform computer simulations.

1.4.3 Pointwise and setwise evolutionarily stability.

The end points of the evolutionary process are stable points of the dynamical system (1.7). Such points can be considered as a generalisation of the Nash Equilibrium and evolutionarily stable strategy concepts. For every Nash Equilibrium strategy (prescribing to choose action i with probability σ_i) of the two-person game there exists a corresponding stationary point of Replicator Dynamics representing a population in which the proportion x^i of the individuals who adopt behaviour i is equal σ_i, that is $x^i = \sigma_i$. The proof of this fact can be found in [35]. It is also the case that for every evolutionarily stable strategy there exists an asymptotically stable point of the corresponding Replicator Dynamics system [35]. The relationship between different concepts of stability is illustrated by the diagram in figure 1.1.

Figure 1.1. Relationship between stability concepts.



If stationary points are not isolated then the asymptotically stable sets and evolutionarily stable sets are considered. Such sets often appear in the Replicator dynamics when the payoff bi-matrix for corresponding two-person game is non-generic.

The definition of an asymptotically stable set (in the sense of Lyapounov stability theory) is standard [35]. It generalises the concept of asymptotically stable point. The definition is as follows.

Definition 1.7 *A closed set $\mathfrak{S}$ is asymptotically stable if every neighbourhood B of $\mathfrak{S}$ contains a neighbourhood B^0 of $\mathfrak{S}$ such that for any $x \in B^0$ $\xi(t,x) \in B$ $\forall t > 0$ and there exists neighbourhood B^* of $\mathfrak{S}$ such that $\xi(t,x) \longrightarrow \mathfrak{S}$ for all $x \in B^*$.*

The concept of evolutionarily stable sets was introduced by Thomas [41]. It generalises the concept of evolutionarily stable strategy. There exist two version of this definition: the "strategy" version and the "population" version. The correspondence between the two versions can be established if instead of a strategy σ prescribing the choice of action i with probability σ_i we consider a corresponding stable point of Replicator Dynamics representing a population in which the proportion of the individuals who adopt behaviour i is equal σ_i. In this thesis we will use the "population" version of this definition which is formulated as follows [35].

Definition 1.8 *A set $\mathfrak{S}$ in simplex Δ is evolutionarily stable if it is non empty and closed, and each $x^0 \in \mathfrak{S}$ has a neighbourhood V_{x^0} such that $f_{x^0}(x) = \left(x^0 - x\right) A x^T > 0$ for all population states $x \in V_{x^0} \backslash \mathfrak{S}$. Here A is a payoff matrix for the corresponding two-player game.*

The concept of an asymptotically stable set is a general concept of the stability theory. The concept of evolutionarily stable set is introduced in a context of population evolution. As a result the latter one is more specific. It is true [35] that every evolutionarily stable set is also an asymptotically stable set. In general, for the Replicator Dynamics with a generic matrix A (which we investigate in this thesis), the converse statement is not true. But in the special case of Replicator Dynamics for which the corresponding two-player game is doubly symmetric (for which matrix A is symmetric) every asymptotically stable set is also an evolutionarily stable set.

1.5 Outline of the thesis.

In this thesis I generalise existing models of social interactions based on the Prisoners' Dilemma game in order to explore social and other behaviours in the context of long-term interaction and investigate the conditions for existence of cooperative behaviour. The thesis is structured as follows.

In chapter 2 I consider an example of an interaction which shows the importance of taking into account the conditions in which the interaction takes place. I introduce a third player into the model, who interacts with two other players engaged in a single interaction Prisoners' Dilemma. The two players interacting in the Prisoners' Dilemma Game represent two companies deciding whether to form an alliance. The third player is another company attacking the first two. The analysis of the model shows that, under the threat of an attack by another player, cooperation may be stabilised. I establish the relationship between the power of the attack, the level to which the attack affects the players and the possibility for forming an alliance.

The rest of the thesis is dedicated to the consideration of multi-state models which generalise the models of social interactions based on the Iterated Prisoners' Dilemma as follows. I will consider a repeated interaction between two individuals in which the game played at any particular time is randomly selected from a specified set of games. For example, individuals may interact either to hunt for food or to defend a jointly held territory. Such an interaction can also model two companies deciding whether to form an alliance for trading in a home country market and a foreign market. In addition to examining the behaviour of the individuals in these two contexts, I will take into account whether or not they wish to continue their long-term association. Chapters 3 and 4 contain explanations of the main techniques used in the analysis. Chapters 5, 6 and 7 contain the results obtained for the multi-state models.

In chapter 3 I explain the approach which allows us to check that a strategy is a Nash Equilibrium for a stochastic game with finite memory. I generalise the techniques of competitive Markov decision processes (see [34]) to be applicable for the analysis of general multi-state games. I illustrate this technique by considering the Iterated Prisoners' Dilemma game, and obtain Nash Equilibrium conditions for any one-stage-memory pure strategy.

In chapter 4 I show how the theory of qualitative analysis of differential equations can be used to obtain a qualitative picture of the evolutionarily dynamics for a game. I also introduce the concept of evolutionarily attractive sets. It will be shown in chapters 6 and 7 that evolutionarily attractive sets appear in the analysis of the multi-state games considered. I also describe the technique of singular coordinate transformations ("blowing up"), which can be used to determine the stability properties of non-hyperbolic fixed points. Such points may occur in the Replicator dynamics of stochastic and other non-generic games.

In chapter 5 I introduce a multi-state game model that allows for the possibility that individuals may interact in more than one context. I consider a repeated interaction between two individuals in which the game played at any particular time is randomly selected from a specified set of two context-games G_1 and G_2. In addition to examining the behaviour of the individuals in these two context-games, I will also allow them to decide whether or not they wish to continue their long-term association. Such a model will allow us to explore the consequences of a richer structure of interaction. For example, the possibility for introducing a new type of behaviour in such games arise: strategies based on the idea of division of labor or allocating tasks. That is one player cooperates in game G_1 and defects in game G_2 while the other player defects in G_1 and cooperates in G_2. I will show that under certain conditions on the parameters of the model such strategy is a Nash Equilibrium. This result is particularly important since the observed behavior for such strategies (in each of G_1 and G_2 separately) is: one player cooperates while the other defects. Therefore this Nash Equilibrium is relevant to the explanation of apparently altruistic behaviour among unrelated individuals. It is also possible to find interactions in which the strategies similar to unconditional defection strategy in the Iterated Prisoners' Dilemma game are not Nash Equilibria. This result shows that cooperative society may be able to evolve from initially uncooperative population which is not the case for the models based on the Iterated Prisoners' Dilemma. In chapter 7 I investigate this possibility for a particular choice of parameters of the model.

In chapter 6 I analyse the Replicator Dynamics obtained for the multi-state model with a generic set of parameters. I obtain conditions on the parameters such that cooperative or alternating tasks populations are the outcome of the selection process, but the pure non cooperative populations are not.

In chapter 7 I consider some interesting examples. In particular, I show that there are examples of interaction such that the only outcome of the selection process is either cooperative or alternating tasks behaviour. I also show that uncooperative population can be unstable and that cooperation may evolve from such populations. Another interesting result concerns the investigation of non reciprocal cooperation. In this instance I show the following. Let context-games G_1 and G_2 be modelled by Prisoners Dilemma Games with the same payoffs and let a Hawk-Dove Game be used to model the decision of whether or not to continue a long-term association. Assume that the probability of playing game G_1 is p and therefore the probability of playing game G_2 is $1-p$. Suppose that the probability p can change over time and can become either zero or one (then only one of the games G_1 or G_2 will be observed). The observed behaviour for the alternating tasks strategies (which is an outcome of the selection process for the set of parameters considered in the example) are then as follows: one player is always cooperating and another is defecting. Since the other game is never played in this situation such a behaviour gives an example of evolution of non-reciprocal cooperative (apparently altruistic) behavior among unrelated individuals.

In the final chapter the main results of the thesis are summarised.

Chapter 2

Beyond the Prisoners' Dilemma: a three-player game.

In this chapter I generalise the single interaction Prisoners' Dilemma game by introducing a third player who can choose the degree to which he interacts with the first and the second players. If there is no interaction between the third player and the two other players, the first and the second players are assumed to play a Prisoners' Dilemma type game. For example, the two players interacting in Prisoners' Dilemma game may represent two companies trying to ensure an alliance. Then the third player may represent another company attacking the alliance. The third player can also represent 'Nature', that is environmental or other conditions that have an impact on the game and do not depend on the behaviour of the first two players (for example, a tax regime by which a government ensures "nice" corporate behaviour). The main idea here is the same as in considering repeated interaction: the context of the interaction may change the conclusions drawn from the analysis. By considering the two players not in isolation but in the context of their interaction with the third player it becomes possible to obtain a cooperative Nash Equilibrium.

The idea of introducing the third player was suggested to me by Prof. L. Fletcher (in private communication) who proposed that symmetric attack forces the players to form an alliance. Besides the game-theoretic interest this game provides business application interest, since the problem of investigating the mechanisms allowing the formation and maintenance of an alliance in business is quite important [43].

In the next section a general model of the interaction described above is introduced and the effect which the existence of the third player has on the possibility of cooperation is investigated. For example, I establish the relationship between the power of the attack (favorable or unfavorable conditions in the 'Nature' case), the level to which the attack affects the players and the possibility for forming an alliance. Considering the problem from the two-player plus 'Nature' perspective, I look for Nash Equilibrium solutions that are relevant to the understanding of cooperative behaviour. When, taking the third player into account, I also show how to find an optimal strategy for attack.

2.1 Two-player perspective.

Let us start with considering two players who are playing the game with the payoffs determined by the following bi-matrix.

$P_1 \setminus P_2$	Cooperate	Defect
Cooperate	$l\ ,\ l$	$l-v-x\ ,\ l+v-x$
Defect	$l+v-x\ ,\ l-v-x$	$l-x-f\ ,\ l-x-f$

(2.1)

Here we suppose that $l \geq 0$, $v \geq 0$, $x \geq 0$ and $f \geq 0$.

If we consider an example of two companies trying to ensure an alliance then we can interpret the payoffs (2.1) as follows.

- If both players choose to cooperate we will interpret it as the companies agreeing to form an alliance. Here $2l$ represents the value of a market share available for the two companies. The players share the market equally and obtain payoff of l.

- If one player chooses to defect and another to cooperate, it will be taken to represent the defecting player is attacking the cooperating player who does not fight back. Here v represents the value of market share transferred as a result of the attack from player who cooperates to the player who defects. The parameter x represents the extra cost of running business outside the alliance compared with the cost of running business in alliance.

- If both players chose to defect then there is a fight and each player has an equal probability of obtaining the whole market. Here f represents the cost of fighting.

In order for the game (2.1) to be identified as a Prisoners' Dilemma conditions (1.3) on payoffs (2.1) must be satisfied. This means that the following inequalities must hold

$$l + v - x > l > l - x - f > l - v - x.$$

Hence we must have $v > x$ and $v > f$.

Now, let us assume that there is a third player who may choose to attack player one with level p and player two with level q. For convenience of analysis we normalise the levels of attack to be between zero and one: $0 \leq p \leq 1$ and $0 \leq q \leq 1$.

- If player three attacks with level p or q a market share proportional to the appropriate level of attack is lost by players one and two, respectively.
- We assume that there may be a cost of being attacked which is proportional to the level of attack and will be expressed as pc and qc. We assume that $c \geq 0$.
- We also assume that withstanding an attack in an alliance may be easier for a company than on its own, so that there maybe an extra cost py or qy of attack if companies are not in an alliance. Here $y \geq 0$.
- Finally, we suppose that if players are in an alliance they share the cost of the attack equally.

Taking into account the above assumptions, we obtain the following payoff bi-matrix for players one and two if player three attacks player one with level p and player two with level q.

$P_1 \setminus P_2$	C	D
C	$l(1-p)-\frac{1}{2}(pc+qc)$, $l(1-q)-\frac{1}{2}(pc+qc)$	$(l-v)(1-p)-x-p(c+y)$, $(l+v)(1-q)-x-q(c+y)$
D	$(l+v)(1-p)-x-p(c+y)$, $(l-v)(1-q)-x-q(c+y)$	$l(1-p)-x-p(c+y)-f$, $l(1-q)-x-q(c+y)-f$

(2.2)

We will now describe conditions for a particular strategy profile to be a Nash Equilibrium.

CC : If

$$\begin{cases} l\,(1-p) - \frac{1}{2}\,(pc+qc) \geq (l+v)\,(1-p) - x - p\,(c+y) \\ l\,(1-q) - \frac{1}{2}\,(pc+qc) \geq (l+v)\,(1-q) - x - q\,(c+y) \end{cases}$$

then the pair of strategies where both players choose cooperation with probability one is a Nash Equilibrium for the game (2.2). This conditions reduce to

$$p \geq \frac{2v - 2x + cq}{2v + 2y + c} \text{ and } q \geq \frac{2v - 2x + pc}{2v + 2y + c}. \tag{2.3}$$

This means that to obtain the cooperative Nash Equilibrium the levels of attack on both players should be quite high.

DD : If

$$\begin{cases} (l - v)(1 - p) - x - p(c + y) \leq l(1 - p) - x - f - p(c + y) \\ (l - v)(1 - q) - x - q(c + y) \leq l(1 - q) - x - f - q(c + y) \end{cases}$$

then the pair of strategies where both players choose defect with probability one is a Nash Equilibrium for the game (2.2). This conditions reduce to

$$p \leq 1 - \frac{f}{v} \text{ and } q \leq 1 - \frac{f}{v}. \tag{2.4}$$

Therefore, there is the non cooperative Nash Equilibrium if the levels of attack are not high enough.

CD : If

$$\begin{cases} (l - v)(1 - p) - x - p(c + y) \geq l(1 - p) - x - f - p(c + y) \\ l(1 - q) - \frac{1}{2}(pc + qc) \leq (l + v)(1 - q) - x - q(c + y) \end{cases}$$

then the pair of strategies where the first player chooses cooperation and the second chooses defection with probability one is a Nash Equilibrium for the game (2.2). This conditions reduce to

$$p \geq 1 - \frac{f}{v} \text{ and } q \leq \frac{2v - 2x + pc}{2v + 2y + c}. \tag{2.5}$$

Therefore, if attack on the first player is strong enough and attack on the second player is weak enough then there is a Nash Equilibrium there the first player cooperates and the second player defects.

DC : If

$$\begin{cases} l(1 - p) - \frac{1}{2}(pc + qc) \leq (l + v)(1 - p) - x - p(c + y) \\ (l - v)(1 - q) - x - q(c + y) \geq l(1 - q) - x - f - q(c + y) \end{cases}$$

then the pair of strategies where the first player chooses defection and the second chooses cooperation with probability one is a Nash Equilibrium for the game (2.2). This conditions reduce to

$$p \leq \frac{cq + 2v - 2x}{2v + c + 2y} \text{ and } q \geq 1 - \frac{f}{v}. \tag{2.6}$$

This Nash Equilibrium solution is "symmetric" to the CD Nash Equilibrium: if attack on the first player is weak enough and attack on the second player is strong enough then there is a Nash Equilibrium there the first player defects and the second player cooperates.

M : There could be also a mixed strategy Nash Equilibrium as we will show in the example below.

Example 2.1. As an example we consider the game with the following parameters

$$v = 5, \qquad x = 2, \qquad y = 2, \qquad f = 2, \qquad c = 3.$$

Then the payoff bi-matrix (2.2) is

$P_1 \setminus P_2$	C	D
C	$l(1-p)-\frac{3}{2}(p+q)$, $l(1-q)-\frac{3}{2}(p+q)$	$l(1-p)-7$, $l(1-q)+3-10q$
D	$l(1-p)+3-10p$, $l(1-q)-7$	$l(1-p)-4-5p$, $l(1-q)-4-5q$

(2.7)

Using formulae (2.3), (2.4) (2.5) and (2.6), we obtain the following conditions for the various Nash Equilibria.

CC : If $\frac{3q+6}{17} \leq p \leq 1$ and $\frac{3p+6}{17} \leq q \leq 1$ then the pair of strategies where both players choose cooperation with probability one is a Nash Equilibrium for the game (2.7).

DD : If $0 \leq p \leq \frac{3}{5}$ and $0 \leq q \leq \frac{3}{5}$ then the pair of strategies where both players choose defect with probability one is a Nash Equilibrium for the game (2.7).

CD : If $\frac{3}{5} \leq p \leq 1$ and $0 \leq q \leq \frac{3p+6}{17}$ then the pair of strategies where the first player chooses cooperation and the second chooses defection with probability one is a Nash Equilibrium for the game (2.7).

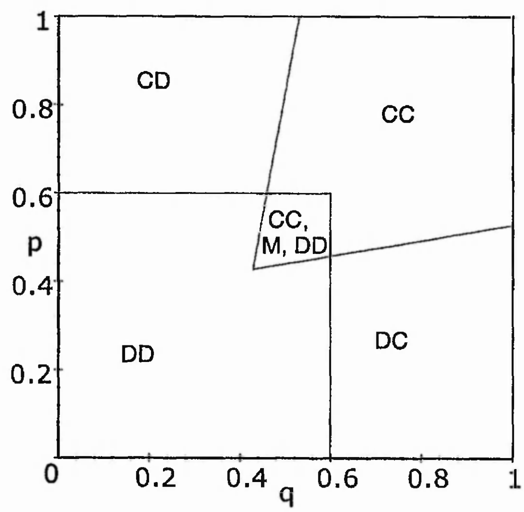
DC : If $0 \leq p \leq \frac{3q+6}{17}$ and $\frac{3}{5} \leq q \leq 1$ then the pair of strategies where the first player chooses defection and the second chooses cooperation with probability one is a Nash Equilibrium for the game (2.7).

M : We can also find that if $\frac{3q+6}{17} \leq p \leq \frac{3}{5}$ and $\frac{3p+6}{17} \leq q \leq \frac{3}{5}$ then there is a mixed strategy Nash Equilibrium which is as follows

$$f_1 = \left(\frac{6-10q}{7q-3p}, 1 - \frac{6-10q}{7q-3p}\right), \qquad f_2 = \left(\frac{6-10p}{7p-3q}, 1 - \frac{6-10p}{7p-3q}\right). \tag{2.8}$$

We illustrate conditions (2.3), (2.4) (2.5) and (2.6) on figure 2.1 below. Here $p = \frac{3q+6}{17}$ is shown is blue, $q = \frac{3p+6}{17}$ is shown is green, $p = \frac{3}{5}$ and $q = \frac{3}{5}$ are shown in red. For each combination of the parameters we have a unique Nash Equilibrium solution except for region where $\frac{3q+6}{17} \leq p \leq \frac{3}{5}$ and $\frac{3p+6}{17} \leq q \leq \frac{3}{5}$ in the middle of the square, for which there are three Nash Equilibria: ('Cooperate', 'Cooperate'), ('Defect', 'Defect') and the mixed Nash Equilibrium (2.8).

Figure 2.1. Nash Equilibria solutions for the game (2.7) depending on parameters p and q.



It can be shown that for the region where $\frac{3q+6}{17} \leq p \leq \frac{3}{5}$ and $\frac{3p+6}{17} \leq q \leq \frac{3}{5}$ ('Cooperate', 'Cooperate') is "Pareto efficient" Nash Equilibrium, that is both players obtain their highest payoffs if the ('Cooperate', 'Cooperate') equilibrium is played.

2.2 Third player perspective.

Now we will show how to determine the Nash Equilibrium levels of attack p and q.

In this model we assume that before the game (2.2) is played the third player chooses levels of attack p and q. This assumption is equivalent to the statement that the first two players are certain about the strength with which they are attacked. This approach simplifies the analysis of the game. The first and the second player make their choices at the game (2.2) which determine their own payoffs and also the payoff obtained by the third player. The payoffs to the third player will be given in the following matrix.

$P_1 \setminus P_2$	C	D
C	$lp + lq - (\sigma + \varepsilon)(p + q)$	$(l - v)p + (l + v)q - \sigma(p + q)$
D	$(l + v)p + (l - v)q - \sigma(p + q)$	$lp + lq - (\sigma - \epsilon)(p + q)$

(2.9)

Here σ represents the cost of the attack, ε represents the additional cost due to attacking an alliance and ϵ represents the decrement in the cost of the attack if it is made on the companies which are fighting with each other.

We now consider two examples showing how the optimal level of attack can be chosen.

Example 2.2. Let us choose the parameters of the game as follows.

$$v = 5, \quad x = 2, \quad y = 2, \quad f = 2, \quad c = 3, \quad l = 8, \quad \varepsilon = 1, \quad \epsilon = 1, \quad \sigma = 3.$$

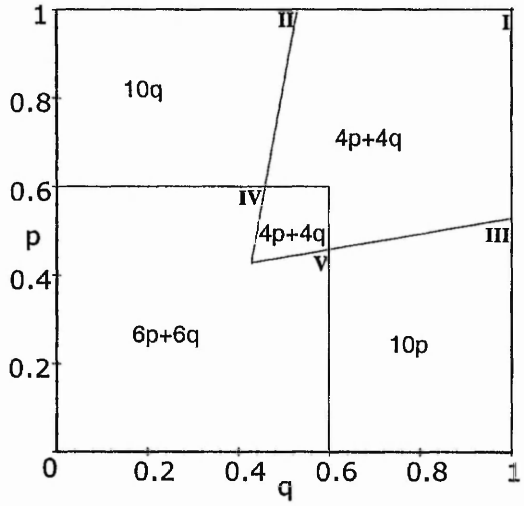
Then the third player payoff matrix (2.9) is

$P_1 \setminus P_2$	C	D
C	$4p + 4q$	$10q$
D	$10p$	$6p + 6q$

(2.10)

We will assume that the first and the second player use their Nash Equilibrium strategies and if the Nash Equilibrium is not unique for some range of the parameters p and q the Nash Equilibrium which gives the highest payoffs is played. Taking this remark into account we can draw a plot that represents the payoff obtained by the third player when he chooses different levels of attack (figure 2.2).

Figure 2.2. The payoff obtained by the third player depending on choice of attack levels $p$and q.



We have that

CC : If $\frac{3q+6}{17} \leq p \leq 1$ and $\frac{3p+6}{17} \leq q \leq 1$ then the first and the second players choose to cooperate in this region and the payoff to the third player is equal $4p + 4q$. This expression reaches its maximum of 8 at the point I $= \{p = 1, q = 1\}$.

CD : If $\frac{3}{5} \leq p \leq 1$ and $0 \leq q < \frac{3p+6}{17}$ then the first player chooses cooperation and the second chooses defection and therefore the payoff to the third player is equal $10q$, which reaches its maximum of $\frac{90}{17} \simeq 5.39$ at the point II $= \{p = 1, q = \frac{9}{17}\}$.

DC : In the same way if $0 \leq p < \frac{3q+6}{17}$ and $\frac{3}{5} \leq q \leq 1$ then the first player chooses defection and the second chooses cooperation and therefore the payoff to the third player is equal $10p$, which reaches its maximum of $\frac{90}{17} \simeq 5.39$ at the point III $= \{p = \frac{9}{17}, q = 1\}$.

DD : In the remaining area the first and the second players choose to defect and the payoff to the third player is equal $6p + 6q$. It reaches its maximum of $\frac{108}{17} \simeq 6.35$ at the points IV $= \{p = \frac{3}{5}, q = \frac{39}{85}\}$or V $= \{p = \frac{39}{85}, q = \frac{3}{5}\}$.

Comparing the values at the points I, II, III, IV and V we find that the optimal choice of attack levels is $p = 1$ and $q = 1$. Notice that such a choice by the third player forces the first and the second players to form (or maintain) an alliance, which means to cooperate.

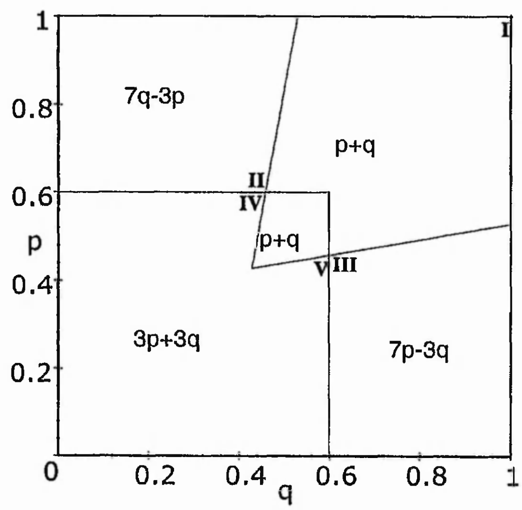
Example 2.3. Let us change the value of the parameter l which represents the value of a market share available for the first and the second players (which therefore can be taken over as a result of the attack). In this example we choose $l = 5$. Then the third player's payoff matrix (2.9) is

$P_1 \setminus P_2$	C	D
C	$p+q$	$7q-3p$
D	$7p-3q$	$3p+3q$

(2.11)

And arguing as in example 2.2 we obtain the following plot representing the payoff gained by the third player if he chooses different levels of attack (figure 2.3).

Figure 2.3. The payoff obtained by the third player depending on choice of attack levels $p$and q.



We have that

CC : If $\frac{3q+6}{17} \leq p \leq 1$ and $\frac{3p+6}{17} \leq q \leq 1$ then the first and the second players choose to cooperate in this region and the payoff to the third player is equal $p+q$. This expression reaches its maximum of 2 at the point I $= \{p = 1, q = 1\}$.

CD : If $\frac{3}{5} \leq p \leq 1$ and $0 \leq q < \frac{3p+6}{17}$ then the first player chooses cooperation and the second chooses defection and therefore the payoff to the third player is equal $7q - 3p$, which reaches its maximum of $\frac{24}{17} \simeq 1.4$ at the point II $= \left\{p = \frac{3}{5}, q = \frac{39}{85}\right\}$.

DC : In the same way if $0 \leq p < \frac{3q+6}{17}$ and $\frac{3}{5} \leq q \leq 1$ then the first player chooses defection and the second chooses cooperation and therefore the payoff to the third player is equal $7p - 3q$, which reaches its maximum of $\frac{24}{17} \simeq 1.4$ at the point III $= \{p = \frac{39}{85}, q = \frac{3}{5}\}$.

DD : In the remaining area the first and the second players choose to defect and the payoff to the third player is equal $3p + 3q$. It reaches its maximum of $\frac{54}{17} \simeq 3.2$ at the points IV $= \{p = \frac{3}{5}, q = \frac{39}{85}\}$ or V $= \{p = \frac{39}{85}, q = \frac{3}{5}\}$.

Comparing the values at the points I, II, III, IV and V we see that the third player obtains maximum payoff at the points IV $= \{p = \frac{3}{5}, q = \frac{39}{85}\}$ or V $= \{p = \frac{39}{85}, q = \frac{3}{5}\}$. Unfortunately for the third player, if he chooses the levels of attack to be exactly $\{p = \frac{3}{5}, q = \frac{39}{85}\}$ or $\{p = \frac{39}{85}, q = \frac{3}{5}\}$ the payoff bi-matrix (2.2) for the two firms is non-generic and there exist an infinite number of Nash Equilibria for this game. It includes the ('Cooperate', 'Cooperate') Nash Equilibrium which, if it is played by the first and the second players, gives them the highest payoffs. The third player obtains the payoff of $\frac{54}{17}$ only if the first and second players choose to defect. Therefore to prevent the first and the second players from swapping between Nash Equilibria the third player must not choose $\{p = \frac{3}{5}, q = \frac{39}{85}\}$ or $\{p = \frac{39}{85}, q = \frac{3}{5}\}$ but choose $\{p = \frac{3}{5} - \delta_1, q = \frac{39}{85} - \delta_2\}$ or $\{p = \frac{39}{85} - \delta_1, q = \frac{3}{5} - \delta_2\}$ such that these points are still in ('Defect', 'Defect') Nash Equilibrium region and δ_1 and δ_2 are very small. We can see that for this example the optimal choice of parameters p and q does not exist. The best strategy for the third player would be to break the alliance by using appropriate levels of attack. In this case a moderate, asymmetric attack will achieve higher payoff than a strong (symmetric or asymmetric) attack.

Remark 2.1 *For the example where parameters are as follows*

$$v = 5, \quad x = 2, \quad y = 2, \quad f = 2, \quad c = 3, \quad \varepsilon = 1, \quad \epsilon = 1,$$

it is possible to show that if the difference between value of the market l and the price of the attack δ is

- *less than -1 (note that in this case the actual cost of the attack is $\sigma - \epsilon$ so the difference between value of the market l and the actual cost of the attack is less than 0) then it is optimal not to attack, that is $p = 0$ and $q = 0$;*
- *between -1 and $\frac{13}{4}$ then the optimal choice of parameters p and q does not exist and the third player should follow the approach described in the example 2.3;*

- *greater than* $\frac{13}{4}$ *then the optimal choice of attack levels is* $p = 1$ *and* $q = 1$; *this choice of the third player forces the first and the second players to form (or maintain) an alliance.*

Remark 2.2 *For simplicity of the exposition we do not give the proof of this statement. The proof can be obtained using item-by-item examination of all possible cases. Similar conditions can be obtained for generic set of parameters.*

2.3 Summary.

In a single interaction Prisoners' Dilemma cooperative behaviour is not a Nash Equilibrium. However, the interacting players are considered in isolation from the environment they interact in or from other possible relationships in which they can be involved. The analysis of the game (2.2) shows that under the threat of an attack by another player (or under unfavorable conditions) the players may be induced to cooperate. It is clear that cooperative behaviour becomes a Nash Equilibrium of the game (2.2) because an attack on players changes their payoffs. But on the other hand, if in modelling the interaction we do not take into account the conditions in which the interaction takes place we would obtain the result that cooperation is not rational when, in fact, it is. Introducing the third player is only one of many possible ways to generalise the basic model of interaction. In the next chapters some other possibilities will be explored.

Another result of the above analysis concerns the third players' strategy. As we have seen from the examples, depending on the relationship between the value of the market and the costs of the attack, the third player may either wish to split the alliance or it may be better for the third player to engage in the strongest symmetric attack, which leads the first two players to form an alliance (for the conditions which produce each type of attacking behaviour see the remark 2.1 above). A strong attack, where the level of attack does not vary significantly between the target firms, will stabilise the possibility of forming an alliance. On the other hand, an asymmetric, weak or moderate attack does not provide an incentive for cooperation.

Chapter 3

Nash Equilibria and Markov Decision Processes.

In chapter 2 we have used the idea which came from the Iterated Prisoners' Dilemma model: considering an interaction in a wider context provides an opportunity for cooperation to be explained. In chapter 2 we introduced an additional player in the interaction. Another possibility is to introduce additional games in the model or allow players to discontinue their association. These models can be described as multi-state games. In chapters 5, 6, and 7 we will consider such models. But, before we can proceed with analysis of such models we need to discuss the techniques which we will use. In the analysis of multi-state games the techniques from the theory of Markov decision processes and stochastic games appear to be very useful.

Therefore, in this chapter, we will discussed the techniques and approaches that we will use later (in chapter 5) to find Nash Equilibria for the multi-state model we are interested in. To describe the approach we begin by recalling the definitions of competitive Markov Decision Processes and stochastic games with stationary strategies. After that we show how to enlarge the definition to deal with a special class of non-stationary strategies (finite memory strategies). The definitions of Nash equilibrium solutions for discounted stochastic games are also discussed. We show that Markov processes and dynamic programming can be used to analyse the discounted stochastic games and they are especially helpful if we wish to check that a particular strategy is a Nash Equilibrium for a game. In the last section of this chapter the definitions and techniques are illustrated using the Prisoners' Dilemma game. Using the approach proposed, we obtain the Nash Equilibrium conditions for any one-stage-memory pure strategy for the Iterated Prisoners' Dilemma game.

3.1 Markov decision processes and stochastic games.

In this section I recall (following [34]) the main definitions and facts about finite state competitive Markov decision processes and stochastic games that will be used later.

Let us consider a process that is observed at discrete time points $t = 0, 1, 2, \ldots$. At each time point t, the *state* of the process will be denoted S_t. Here S_t is a random variable that can take values from a finite set $\mathbf{S} = \{1, 2, \ldots, N\}$ which is called the *state space.*

The process is controlled by players P_1 and P_2 who independently choose actions $a^1 \in \mathbf{A}^1(s) = \left\{a_1^1(s), \ldots, a_{m_1(s)}^1(s)\right\}$ and $a^2 \in \mathbf{A}^2(s) = \left\{a_1^2(s), \ldots, a_{m_2(s)}^2(s)\right\}$ at time t if the process is in state $s : S_t = s$. The choice of $a^1 \in \mathbf{A}^1(s)$ and $a^2 \in \mathbf{A}^2(s)$ in state s results in immediate rewards $r^1(s, a^1, a^2)$ and $r^2(s, a^1, a^2)$, for the first and the second player respectively, and a probabilistic transition to a new state $s' \in \mathbf{S}$.

Definition 3.1 *A process is called Markov if for every $s, s' \in \mathbf{S}$ and $a^1 \in \mathbf{A}^1(s)$, $a^2 \in \mathbf{A}^2(s)$ the probability that $S_{t+1} = s'$ given that $S_t = s$ and the players choose actions a^1 and a^2 is independent of time and any previous states and actions. That is, there exist stationary transition probabilities $p\left(s'|s, a^1, a^2\right) := \mathbf{P}\left\{S_{t+1} = s'|S_t = s, A_t^1 = a^1, A_t^2 = a^2\right\}$ for all $t = 0, 1, 2, \cdots$. Here S_t is the state at time t, and A_t^1, A_t^2 denote the actions chosen by players P_1 and P_2 at time t, respectively.*

Definition 3.2 *The i^{th} player's strategy $\mathbf{f}_i = (\mathbf{f}_{i,1}, \ldots, \mathbf{f}_{i,s}, \ldots, \mathbf{f}_{i,N})$, $i = 1, 2$, in Markov decision process is a block row vector whose s^{th} block is a nonnegative row vector*

$$\mathbf{f}_{i,s} = \left(f_{i,s}(a_1^i(s)), f_{i,s}(a_2^i(s)), \ldots, f_{i,s}(a_{m_i(s)}^i(s))\right)$$

with entries that satisfy $\sum_{j=1}^{m_i(s)} f_{i,s}(a_j^i(s)) = 1$. These entries will be given the interpretation that $f_{i,s}(a_j^i(s))$ is probability that the i^{th} player chooses action $a_j^i(s) \in \mathbf{A}^i(s)$ in state $s \in \mathbf{S}$ whenever s is visited. A strategy $\mathbf{f}_i$ will be called pure if $f_{i,s}(a_j^i(s)) \in \{0, 1\}$ for all $a_j^i(s) \in \mathbf{A}^i(s)$, $s \in \mathbf{S}$. The property that the player's decisions in state s are invariant with respect to the time of visit to s is called the stationarity of the strategy, and such strategies are called stationary strategies.

Definition 3.3 *The strategies* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *define a probability transition matrix* $P(\mathbf{f}_1, \mathbf{f}_2) = (p(s'|s, \mathbf{f}_1, \mathbf{f}_2))_{s,s'=1}^{N}$ *with entries given by*

$$p(s'|s, \mathbf{f}_1, \mathbf{f}_2) = \sum_{j=1}^{m_1(s)} \sum_{k=1}^{m_2(s)} p\left(s'|s, a_j^1(s), a_k^2(s)\right) f_{1,s}(a_j^1(s)) f_{2,s}(a_k^2(s)).$$

We will consider the so-called "discounted" Markov decision models which are defined as follows.

Let $\{R_t^i\}_{t=0}^{\infty}, i = 1, 2$, denote the sequence of random rewards for the i^{th} player, with R_t^i being the reward for the i^{th} player at the time point t. Once an initial state s and strategies $\mathbf{f}_1$ and $\mathbf{f}_2$ are specified, then so is the probability distribution of R_t^i for every $t = 0, 1, 2, \ldots, i = 1, 2$. Thus the expectation of R_t^i is also well defined and will be denoted by $\mathbf{E}_{s,\mathbf{f}_1,\mathbf{f}_2}\left[R_t^i\right] := \mathbf{E}_{\mathbf{f}_1,\mathbf{f}_2}\left[R_t^i | S_0 = s\right]$.

The total discounted value of the strategies $\mathbf{f}_1$ and $\mathbf{f}_2$ from the initial state s for the i^{th} player will be defined by $v_\beta^i(s, \mathbf{f}_1, \mathbf{f}_2) := \sum_{t=0}^{\infty} \beta^t \mathbf{E}_{s,\mathbf{f}_1,\mathbf{f}_2}\left[R_t^i\right]$, where $\beta \in [0, 1)$ is called the *discount factor*. In order to calculate this value, define the *immediate expected reward vector* by

$$\mathbf{r}^i(\mathbf{f}_1, \mathbf{f}_2) = \left(r^i(1, \mathbf{f}_1, \mathbf{f}_2), r^i(2, \mathbf{f}_1, \mathbf{f}_2), \ldots, r^i(N, \mathbf{f}_1, \mathbf{f}_2)\right)^T$$

where, for each $s \in \mathbf{S}$,

$$r^i(s, \mathbf{f}_1, \mathbf{f}_2) := \sum_{j=1}^{m_1(s)} \sum_{k=1}^{m_2(s)} r\left(s, a_j^1(s), a_k^2(s)\right) f_{1,s}(a_j^1(s)) f_{2,S}(a_k^2(s)).$$

We can calculate now that for any $s \in \mathbf{S}$

$$\begin{aligned}
\mathbf{E}_{s,\mathbf{f}_1,\mathbf{f}_2}\left[R_0^i\right] &= r^i(s, \mathbf{f}_1, \mathbf{f}_2) = \left[\mathbf{r}^i(\mathbf{f}_1, \mathbf{f}_2)\right]_s \\
\mathbf{E}_{s,\mathbf{f}_1,\mathbf{f}_2}\left[R_1^i\right] &= \sum_{s'=1}^{N} p(s'|s, \mathbf{f}_1, \mathbf{f}_2) r^i(s, \mathbf{f}_1, \mathbf{f}_2) = \left[P(\mathbf{f}_1, \mathbf{f}_2) \mathbf{r}^i(\mathbf{f}_1, \mathbf{f}_2)\right]_s \\
\mathbf{E}_{s,\mathbf{f}_1,\mathbf{f}_2}\left[R_2^i\right] &= \sum_{s'=1}^{N} p_2(s'|s, \mathbf{f}_1, \mathbf{f}_2) r^i(s, \mathbf{f}_1, \mathbf{f}_2) = \left[P^2(\mathbf{f}_1, \mathbf{f}_2) \mathbf{r}^i(\mathbf{f}_1, \mathbf{f}_2)\right]_s \\
&\vdots \\
\mathbf{E}_{s,\mathbf{f}_1,\mathbf{f}_2}\left[R_t^i\right] &= \sum_{s'=1}^{N} p_t(s'|s, \mathbf{f}_1, \mathbf{f}_2) r^i(s, \mathbf{f}_1, \mathbf{f}_2) = \left[P^t(\mathbf{f}_1, \mathbf{f}_2) \mathbf{r}^i(\mathbf{f}_1, \mathbf{f}_2)\right]_s
\end{aligned}$$

where $[\mathbf{u}]_s$ denotes the s^{th} entry of a vector $[\mathbf{u}]$, and $p_t(s'|s, \mathbf{f}_1, \mathbf{f}_2)$ is the t-step transition probability from s to s' in the Markov chain defined by $\mathbf{f}_1$ and $\mathbf{f}_2$. It is well known from Markov chain theory that the t^{th} power of $P(\mathbf{f}_1, \mathbf{f}_2)$ contains all such t-step transition probabilities. That is, $P^t(\mathbf{f}_1, \mathbf{f}_2) = (p_t(s'|s, \mathbf{f}_1, \mathbf{f}_2))_{s,s'=1}^N$. Hence the *discounted value vector of the strategies* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *for the* i^{th} *player* $\mathbf{v}_\beta^i(\mathbf{f}_1, \mathbf{f}_2) := \left(v_\beta^i(1, \mathbf{f}_1, \mathbf{f}_2), \ldots, v_\beta^i(N, \mathbf{f}_1, \mathbf{f}_2)\right)$ can be calculated as $\mathbf{v}_\beta^i(\mathbf{f}_1, \mathbf{f}_2) = \sum_{t=0}^\infty \beta^t P^t(\mathbf{f}_1, \mathbf{f}_2)\,\mathbf{r}^i(\mathbf{f}_1, \mathbf{f}_2)$, where $P^0(\mathbf{f}_1, \mathbf{f}_2) := I_N$, the $N \times N$ identity matrix.

If β^{-1} is larger than any eigenvalue of the matrix $P(\mathbf{f}_1, \mathbf{f}_2)$, then the sum $\sum_{t=0}^\infty \beta^t P^t(\mathbf{f}_1, \mathbf{f}_2)$ is convergent and equal to the matrix

$$[I_N - \beta P(\mathbf{f}_1, \mathbf{f}_2)]^{-1} = I_N + \beta P(\mathbf{f}_1, \mathbf{f}_2) + \beta^2 P^2(\mathbf{f}_1, \mathbf{f}_2) + \cdots .$$

So, if the conditions on β and matrix $P(\mathbf{f}_1, \mathbf{f}_2)$ are satisfied we obtain the following compact matrix expression for the discounted value vector $\mathbf{v}_\beta^i(\mathbf{f}_1, \mathbf{f}_2)$ for the i^{th} player

$$\mathbf{v}_\beta^i(\mathbf{f}_1, \mathbf{f}_2) = [I_N - \beta P(\mathbf{f}_1, \mathbf{f}_2)]^{-1}\,\mathbf{r}^i(\mathbf{f}_1, \mathbf{f}_2). \tag{3.1}$$

Remark 3.1 *It is well known that if matrix* $P(\mathbf{f}_1, \mathbf{f}_2)$ *is a stochastic matrix (all rows sum to* 1*) then the maximal eigenvalue of this matrix is equal to* 1 *and, hence,* $\beta \in [0, 1)$ *satisfies the required condition.*

Definition 3.4 *A game that can be described as a discounted Markov decision process is called a discounted stochastic game.*

Let us now give the definition of a Nash equilibrium solution for such games.

Definition 3.5 *A pair of strategies* $(\mathbf{f}_1^*, \mathbf{f}_2^*)$ *is a Nash equilibrium of a discounted stochastic game if for all possible strategies* $\mathbf{f}_1$, $\mathbf{f}_2$ *and for all possible states* $s \in \mathbf{S}$ *the following condition holds*

$$\begin{cases} \mathbf{v}_\beta^1(S, \mathbf{f}_1^*, \mathbf{f}_2^*) \geq \mathbf{v}_\beta^1(S, \mathbf{f}_1, \mathbf{f}_2^*) \\ \mathbf{v}_\beta^2(S, \mathbf{f}_1^*, \mathbf{f}_2^*) \geq \mathbf{v}_\beta^2(S, \mathbf{f}_1^*, \mathbf{f}_2) \end{cases}. \tag{3.2}$$

3.2 Nash Equilibria for games with finite memory.

Suppose now that when players make a decision about which action to choose at a state of a discounted stochastic game they use information about actions chosen by both players at some previous state. That is an example of a model which we will call a *one-stage memory model* since the strategies that players use depend only on how the game was played at one particular stage in the past (usually this will be the previous stage). These are non-stationary strategies. However, the game can still be described by a Markov decision process with stationary strategies over an enlarged state space.

Remark 3.2 *It is clear that we can consider non-stationary strategies for discounted stochastic games (see, for example, [34] section 2.6). But for the convenience of the analysis it is useful to introduce a new Markov decision process for which the strategy we are considering becomes stationary over an enlarged state space. The idea of using the enlarged state space to model non-stationary strategies is well-know to game-theorists. For example such a Markov process was used in [20] for the Iterated Prisoner's Dilemma interaction between players using one-state memory strategies. Nevertheless, although the idea is very natural it seems that the general approach has not been described. Below we describe it for completeness of the discussion.*

Since we are considering a stochastic game there is a Markov process with the state space $\mathbf{S} = \{1, 2, \ldots, N\}$ which corresponds to this game. Let us denote the set of all possible memories (information states) which players can have during the game under some memory model as $\mathbf{I} = \{i_1, i_2, \ldots, i_k\}$. If we consider the one-stage memory model, then $i_j = (s', a_1, a_2)$, $j = 1, 2, \ldots, k$, is the information that actions a_1 and a_2 have been chosen at state s' by the first and the second players respectively.

To represent this model as finite state Markov decision processes with stationary strategies we need to enlarge the existing state space $\mathbf{S}$. If at some different time players use some specific information i_j to make their decision when they are at the state s, this situation will be represented by an additional state s_{i_j} of the enlarged state space $\bar{\mathbf{S}}$.

Definition 3.6 *Let $\mathbf{S}' \subset \mathbf{S}$ be a subset of all states s at which players use some information to make a choice of actions (the strategy is not stationary at these states). The enlarged state space $\bar{\mathbf{S}}$ consists of the following elements $\bar{\mathbf{S}} = \mathbf{S} \cup \left\{ s_{i_j} : s \in \mathbf{S}', i_j \in \mathbf{I} \right\}$.*

Example 3.1. In the Iterated Prisoners' Dilemma the number of states in $\mathbf{S}$ is one: $\mathbf{S} = \{1\} = \{PD\}$ (here PD stands for Prisoners' Dilemma). The number of possible combinations of actions chosen is four. Therefore, for the one-stage memory model of the Iterated Prisoners' Dilemma game

$$\mathbf{I} = \{i_1, i_2, i_3, i_4\} = \{(PD, C, C), (PD, C, D), (PD, D, C), (PD, D, D)\}.$$

The enlarged state space $\bar{\mathbf{S}}$ then is as follows

$$\bar{\mathbf{S}} = \left\{PD, PD_{(PD,C,C)}, PD_{(PD,C,D)}, PD_{(PD,D,C)}, PD_{(PD,D,D)}\right\}.$$

Here PD is the starting state when no previous interaction has occurred. We consider this example in more detail in the next section where we analyse all pure one-stage memory strategies for this model.

Using this idea, Markov processes with stationary strategies can be constructed for any stochastic game with a finite memory model. Here under finite memory we understand that the number of players and the number of states are finite for such a model and the information which the players use during the game depends only on finite number of previous rounds.

Definition 3.7 *Let $\bar{\mathbf{S}}$ be a state space of the Markov process with stationary strategies constructed for a stochastic game with finite memory. Space $\bar{\mathbf{S}}$ consists of two different types of states:*

- *the states at which the players have no particular information on the history of the game; these will be called empty memory states s^{em} (the set of all empty memory states will be denoted $\bar{\mathbf{S}}^{em}$), and*
- *the states at which the players have some information on the history of the game; these will be called information states.*

We will now introduce the definition of a Nash Equilibrium for a discounted stochastic game with finite memory. In this thesis we will use this definition when we solving such games.

Definition 3.8 *A pair of strategies $(\mathbf{f}_1^*, \mathbf{f}_2^*)$ is a Nash Equilibrium for a discounted stochastic game with finite memory if for all possible strategies $\mathbf{f}_1$, $\mathbf{f}_2$ for all* ***empty memory*** *states $s^{em} \in \bar{\mathbf{S}}^{em}$ the following condition holds*

$$\begin{cases} \mathbf{v}_\beta^1(s^{em}, \mathbf{f}_1^*, \mathbf{f}_2^*) \geq \mathbf{v}_\beta^1(s^{em}, \mathbf{f}_1, \mathbf{f}_2^*) \\ \mathbf{v}_\beta^2(s^{em}, \mathbf{f}_1^*, \mathbf{f}_2^*) \geq \mathbf{v}_\beta^2(s^{em}, \mathbf{f}_1^*, \mathbf{f}_2) \end{cases}. \tag{3.3}$$

Remark 3.3 *The main difference between this definition and the definition of a Nash equilibrium of memoryless discounted stochastic games is that in this case we consider only a subset of the state space (empty memory states). It seems that the difference between Definition 3.5* *and Definition 3.8 is similar to the difference between the concept of Nash Equilibrium and the concept of subgame perfect Nash Equilibrium, as Definition 3.8 does not require local best responses at every state* $s \in \bar{\mathbf{S}}$. *Such a definition is introduced in order to allow a wider class of strategies to be used in discounted stochastic games with memory. In section 3.3 we consider an example of the Iterated Prisoners' Dilemma game with memory of one previous state. It will be shown, for instance, that such a "good" strategy as Tit for Tat would be forbidden if Definition 3.8 were extended to* $\bar{\mathbf{S}} \backslash \bar{\mathbf{S}}^{em}$. *This fact is explained in more detail in section 3.3. For more discussion on this matter see also Remark 3.10 of section 3.3.*

If a particular game-model has many different states it can be difficult to find all Nash equilibria. For example, the number of only pure strategies for a model increases exponentially and equal 2^n if the number of states is n. From an application point of view it may be more useful not to attempt to solve the problem of finding all possible Nash Equilibria, but try to verify that some certain "interesting" strategies are Nash Equilibria. If we are able to find a few such "interesting" Nash Equilibria we then can analyse the Replicator Dynamics of these strategies and answer the question whether or not these strategies are likely to be the outcome of an evolutionary process. We now introduce the technique that allows us to check if a strategy f is a Nash Equilibrium for a game with finite memory. In our analysis we use the ideas and results of dynamic programming (which can be found, for example, in [44]).

Below we explain how to check that a pair of strategies $(\mathbf{f}_1^*, \mathbf{f}_2^*)$ is a Nash Equilibrium.

Firstly, let us consider the following suprema of the functions $v_\beta^i(s, \mathbf{f}_1, \mathbf{f}_2)$

$$\mathbf{V}_\beta^1(s, \mathbf{f}_2^*) = \sup_{\mathbf{f}_1} \mathbf{v}_\beta^1(s, \mathbf{f}_1, \mathbf{f}_2^*), \qquad \mathbf{V}_\beta^2(s, \mathbf{f}_1^*) = \sup_{\mathbf{f}_2} \mathbf{v}_\beta^2(s, \mathbf{f}_1^*, \mathbf{f}_2), \qquad s \in \bar{\mathbf{S}}, \tag{3.4}$$

The following two theorems give a system of equations on suprema $\mathbf{V}_\beta^1(s, \mathbf{f}_2^*)$ and $\mathbf{V}_\beta^2(s, \mathbf{f}_1^*)$ and guarantee that the solution is unique.

Theorem (i). (for proof see [44])

$$\begin{aligned}
&\mathbf{V}_\beta^1(s, \mathbf{f}_2^*) \\
&= \max_{a^1 \in \mathbf{A}^1(s)} \left\{ \sum_{a^2 \in \mathbf{A}^2(s)} r^1(s, a^1, a^2) f_2^*(s, a^2) + \beta \sum_{s' \in \bar{\mathbf{S}}} \sum_{a^2 \in \mathbf{A}^2(s)} p\left(s'|s, a^1, a^2\right) f_2^*(s, a^2) \mathbf{V}_\beta^1(s', \mathbf{f}_2^*) \right\}, \\
&\mathbf{V}_\beta^2(s, \mathbf{f}_1^*) \\
&= \max_{a^2 \in \mathbf{A}^2(s)} \left\{ \sum_{a^1 \in \mathbf{A}^1(s)} r^2(s, a^1, a^2) f_1^*(s, a^1) + \beta \sum_{s' \in \bar{\mathbf{S}}} \sum_{a^1 \in \mathbf{A}^1(s)} p\left(s'|s, a^1, a^2\right) f_1^*(s, a^1) \mathbf{V}_\beta^2(s', \mathbf{f}_1^*) \right\}, \\
&\text{for all } s \in \bar{\mathbf{S}}.
\end{aligned} \tag{3.5}$$

Theorem (ii). (for proof see [44]) $\mathbf{V}^1_\beta(s,\mathbf{f}^*_2)$ and $\mathbf{V}^2_\beta(s,\mathbf{f}^*_1)$ are the unique solutions for equations (3.5).

Suppose that, once these suprema are found, we can identify strategies $\widehat{\mathbf{f}}^1$ and $\widehat{\mathbf{f}}^2$ such that

$$\mathbf{v}^1_\beta(s,\widehat{\mathbf{f}}^1,\mathbf{f}^*_2) = \mathbf{V}^1_\beta(s,\mathbf{f}^*_2), \qquad \mathbf{v}^2_\beta(s,\mathbf{f}^*_1,\widehat{\mathbf{f}}^2) = \mathbf{V}^2_\beta(s,\mathbf{f}^*_1), \qquad s \in \bar{\mathbf{S}}.$$

This would mean that the actual values of the suprema are achieved as total values for some strategies. Then, which ever state we consider as the beginning of the game, $\widehat{\mathbf{f}}^1$ is the best reply for $\mathbf{f}^*_2$ and $\widehat{\mathbf{f}}^2$ is the best reply for $\mathbf{f}^*_1$. The following theorem guarantees that such strategies exist.

Theorem (iii). (for proof see [44]) The best reply strategies $\widehat{\mathbf{f}}^1$ and $\widehat{\mathbf{f}}^2$ exist and can be taken to be stationary and pure.

Then if the pair of strategies $(\mathbf{f}^*_1,\mathbf{f}^*_2)$ is a Nash Equilibrium the following equalities must be satisfied for all empty memory states $s^{em} \in \bar{\mathbf{S}}^{em}$

$$\mathbf{v}^1_\beta(s^{em},\mathbf{f}^*_1,\mathbf{f}^*_2) = \mathbf{v}^1_\beta(s^{em},\widehat{\mathbf{f}}^1,\mathbf{f}^*_2), \qquad \mathbf{v}^2_\beta(s^{em},\mathbf{f}^*_1,\mathbf{f}^*_2) = \mathbf{v}^2_\beta(s^{em},\mathbf{f}^*_1,\widehat{\mathbf{f}}^2).$$

Therefore, as a procedure for checking that a particular pair of strategies $(\mathbf{f}^*_1,\mathbf{f}^*_2)$ is a Nash Equilibrium I propose the following.

1. Find the total discounted value $\mathbf{v}^i_\beta\,(\mathbf{f}^*_1,\mathbf{f}^*_2)$ for the i^{th} player.

2. Solve the dynamic programming equations (3.5).

3. Check that for all empty memory states $s^{em} \in \bar{\mathbf{S}}^{em}$

$$\mathbf{v}^1_\beta(s^{em},\mathbf{f}^*_1,\mathbf{f}^*_2) = \mathbf{V}^1_\beta(s^{em},\mathbf{f}^*_2), \qquad \mathbf{v}^2_\beta(s^{em},\mathbf{f}^*_1,\mathbf{f}^*_2) = \mathbf{V}^2_\beta(s^{em},\mathbf{f}^*_1). \tag{3.6}$$

Remark 3.4 *Since Theorem (iii) guarantees that the best reply strategies* $\widehat{\mathbf{f}}^1$ *and* $\widehat{\mathbf{f}}^2$ *exist there is no necessity to identify these strategies exactly, as it is follows from the existence of the strategies* $\widehat{\mathbf{f}}^1$ *and* $\widehat{\mathbf{f}}^2$ *that suprema* $\mathbf{V}^1_\beta(s,\mathbf{f}^*_2)$ *and* $\mathbf{V}^2_\beta(s,\mathbf{f}^*_1)$ *are reached on the strategies* $\widehat{\mathbf{f}}^1$ *and* $\widehat{\mathbf{f}}^2$. *Therefore* $\mathbf{V}^1_\beta(s,\mathbf{f}^*_2)$ *and* $\mathbf{V}^2_\beta(s,\mathbf{f}^*_1)$ *give the maximum total value which it is possible to obtain playing against* $\mathbf{f}^*_2$ *and* $\mathbf{f}^*_1$ *respectively. Hence if equations (3.6) hold then Definition 3.8 is satisfied for strategies* $(\mathbf{f}^*_1,\mathbf{f}^*_2)$.

Remark 3.5 *Let us notice that using this approach we can verify that any strategy* f *(pure or mixed) is a Nash Equilibrium in a class of all possible strategies (pure or mixed) of the constructed process. When we say that a strategy* **f** *is a Nash Equilibrium it means that conditions of Definition 3.8 are satisfied for a pair of strategies* (**f**, **f**) *where strategy* **f** *is played by both players. For simplicity of the analysis in the problems studied in this thesis we only consider symmetric Nash Equilibria. Nevertheless the approach can easily be applied to non symmetric Nash Equilibria.*

3.3 Example: one-stage memory model for Iterated Prisoner's Dilemma game.

In this section we consider an example of the Iterated Prisoners' Dilemma game. We show that procedure described above can easily be used to verify that a certain strategy is a Nash equilibrium for this game. We will consider all one-stage memory pure strategies and obtain the Nash Equilibrium conditions for them.

Let us recall that in the Iterated Prisoners' Dilemma game

- there is one state, the Prisoner's Dilemma game defined by table (1.2) and conditions (1.3), which is played an infinite number of times and
- there is a constant discount factor β between each round of the game, so that the expected number of rounds in the game is $\frac{1}{1-\beta}$.

Suppose that when players make a decision about which action to choose in each Prisoners' Dilemma game they use information about the actions chosen by both players in the previous round.

It is known that an Iterated Prisoner's Dilemma interaction between players using such strategies can be modeled by a Markov Process [20]. In example 3.1 of the previous section we have obtained the state space of the Markov process with stationary strategies for the Iterated Prisoners' Dilemma

$$\bar{\mathbf{S}} = \left\{PD, PD_{(PD,C,C)}, PD_{(PD,C,D)}, PD_{(PD,D,C)}, PD_{(PD,D,D)}\right\}.$$

Here PD state accounts for first round condition when there is no history and $PD_{(PD,a_1,a_2)}$ is a state in which information that at the previous round actions a_1 and a_2 are chosen by the first and the second player, respectively. Since in this game the same single interaction game (Prisoners' Dilemma) is played at each state we can simplify the state description and use the following notation

$$\bar{\mathbf{S}} = \{PD, (C,C), (C,D), (D,C), (D,D)\}.$$

The sets of actions $\mathbf{A}^i(s)$, which can be chosen by the i^{th} player at state s, are $\mathbf{A}^i(s) = \{1,2\} = \{C,D\}$, $s = 0,1,\ldots,4$, $i = 1,2$. Immediate rewards $r^1(s,a^1,a^2)$ and $r^2(s,a^1,a^2)$ for the first and the second player, respectively, are the same for every state s and defined by table (1.2). The i^{th} player's strategy $\mathbf{f}_i = (\mathbf{f}_{i,0}, \mathbf{f}_{i,1}, \ldots, \mathbf{f}_{i,4})$, $\mathbf{f}_{i,s} = (f_{i,s}, 1 - f_{i,s})$, $s = 0,1,\ldots,4$, where $f_{i,s}$, $i = 1,2$, is the probability that action C is chosen at the state s by the i^{th} player. The strategies $\mathbf{f}_1$, $\mathbf{f}_2$ define a probability transition matrix as following

$$P(\mathbf{f}_1,\mathbf{f}_2) = \begin{pmatrix} 0 & f_{1,0}f_{2,0} & f_{1,0}(1-f_{2,0}) & (1-f_{1,0})f_{2,0} & (1-f_{1,0})(1-f_{2,0}) \\ 0 & f_{1,1}f_{2,1} & f_{1,1}(1-f_{2,1}) & (1-f_{1,1})f_{2,1} & (1-f_{1,1})(1-f_{2,1}) \\ 0 & f_{1,2}f_{2,2} & f_{1,2}(1-f_{2,2}) & (1-f_{1,2})f_{2,2} & (1-f_{1,2})(1-f_{2,2}) \\ 0 & f_{1,3}f_{2,3} & f_{1,3}(1-f_{2,3}) & (1-f_{1,3})f_{2,3} & (1-f_{1,3})(1-f_{2,3}) \\ 0 & f_{1,4}f_{2,4} & f_{1,4}(1-f_{2,4}) & (1-f_{1,4})f_{2,4} & (1-f_{1,4})(1-f_{2,4}) \end{pmatrix}$$

The vector $\mathbf{v}^i_\beta(\mathbf{f}_1,\mathbf{f}_2)$ can be calculated as $\mathbf{v}^i_\beta(\mathbf{f}_1,\mathbf{f}_2) = [I_5 - \beta P(\mathbf{f}_1,\mathbf{f}_2)]^{-1}\,\mathbf{r}^i(\mathbf{f}_1,\mathbf{f}_2)$.

Below we consider the pure strategies for the one-stage memory model for the Iterated Prisoners' Dilemma game (there are in total thirty two such strategies). We determine the necessary and sufficient conditions for the payoffs and discount factor β under which these strategies are Nash equilibria.

The state space descriptions of the strategies we consider are given in tables 3.1 and 3.2. Each strategy is given a number which is used to refer to it. The first column of tables 3.1 and 3.2 contains these numbers. For some strategies there are well known names (see, for example, [18], [32], [20]). These are also given in the first column of tables 3.1 and 3.2. The next five columns contain the probability f_s with which the strategy prescribes choosing action C at some particular state s. The last column contains the value of the strategy if it is played against itself (we denote $\mathbf{v}(\mathbf{f}_1,\mathbf{f}_2) = \mathbf{v}^1_\beta(0,\mathbf{f}_1,\mathbf{f}_2)$). Since $\{a_1,a_2\}$ is the information that at the previous round actions a_1 and a_2 are chosen by the first and the second player, respectively, the strategies for the first player only are given in tables 3.1 and 3.2. To obtain the same strategies used by the second player we need to swap the probabilities given for the states $s = 2 = (C,D)$ and $s = 3 = (D,C)$. Some strategies are invariant under such a transformation. Other strategies should be considered in pairs to obtain the complete strategy description. For example strategy description **6** from table 3.1 gives Tit for Tat strategy played by the first player, but strategy Tit for Tat played by the second player is described by line 4 from table 3.1. Therefore a pair of strategies where $(\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{6}, \mathbf{4})$ is symmetric where both players use strategy Tit for Tat. To calculate the expected value for Tit For Tat strategy played against itself we need to calculate the expected value for strategy described by line **6** played against strategy described by line **4**. Strategies 17-32 (see Table 3.2) are cautious versions of strategies 1-16: instead of starting the interaction with cooperation they prescribe choosing defection at the first round.

Table 3.1. Memory-one pure strategies (1-16) for the Iterated Prisoners' Dilemma.

Strategy	Probability of choosing action C at					Value of strategy
played by first player	*First Round*	*(C,C) state*	*(C,D) state*	*(D,C) state*	*(D,D) state*	played against itself
1=All C	1	1	1	1	1	v(1,1)=$\frac{h_1}{1-\beta}$
2	1	1	1	1	0	v(2,2)=$\frac{h_1}{1-\beta}$
3	1	1	1	0	1	v(3,5)=$\frac{h_1}{1-\beta}$
4=Stubborn	1	1	1	0	0	v(4,6)=$\frac{h_1}{1-\beta}$
5=Tweedledee	1	1	0	1	1	v(5,3)=$\frac{h_1}{1-\beta}$
6=Tit for Tat	1	1	0	1	0	v(6,4)=$\frac{h_1}{1-\beta}$
7=Pavlov	1	1	0	0	1	v(7,7)=$\frac{h_1}{1-\beta}$
8=Grim	1	1	0	0	0	v(8,8)=$\frac{h_1}{1-\beta}$
9	1	0	1	1	1	v(9,9)=$\frac{h_1+\beta h_4}{1-\beta^2}$
10	1	0	1	1	0	v(10,10)=$h_1 + \frac{\beta h_4}{1-\beta}$
11=Bully	1	0	1	0	1	v(11,13)=$\frac{h_1+\beta h_4}{1-\beta^2}$
12	1	0	1	0	0	v(12,14)=$h_1 + \frac{\beta h_4}{1-\beta}$
13=Fickle	1	0	0	1	1	v(13,11)=$\frac{h_1+\beta h_4}{1-\beta^2}$
14	1	0	0	1	0	v(14,12)=$h_1 + \frac{\beta h_4}{1-\beta}$
15	1	0	0	0	1	v(15,15)=$\frac{h_1+\beta h_4}{1-\beta^2}$
16	1	0	0	0	0	v(16,16)=$h_1 + \frac{\beta h_4}{1-\beta}$

Table 3.2. Memory-one pure strategies (17-32) for the Iterated Prisoners' Dilemma.

Strategy played by first player	Probability of choosing action C at *First Round*	*(C,C) state*	*(C,D) state*	*(D,C) state*	*(D,D) state*	Value of strategy played against itself
17=Cautious All C	0	1	1	1	1	v(17,17)=$h_4 + \frac{\beta h_1}{1-\beta}$
18	0	1	1	1	0	v(18,18)=$\frac{h_4}{1-\beta}$
19	0	1	1	0	1	v(19,21)=$h_4 + \frac{\beta h_1}{1-\beta}$
20	0	1	1	0	0	v(20,22)=$\frac{h_4}{1-\beta}$
21=Cautious Tweedledee	0	1	0	1	1	v(21,19)=$h_4 + \frac{\beta h_1}{1-\beta}$
22=Cautious Tit for Tat	0	1	0	1	0	v(22,20)=$\frac{h_4}{1-\beta}$
23=Cautious Pavlov	0	1	0	0	1	v(23,23)=$h_4 + \frac{\beta h_1}{1-\beta}$
24=Cautious Grim	0	1	0	0	0	v(24,24)=$\frac{h_4}{1-\beta}$
25	0	0	1	1	1	v(25,25)=$\frac{h_4+\beta h_1}{1-\beta^2}$
26	0	0	1	1	0	v(26,26)=$\frac{h_4}{1-\beta}$
27	0	0	1	0	1	v(27,29)=$\frac{h_4+\beta h_1}{1-\beta^2}$
28	0	0	1	0	0	v(28,30)=$\frac{h_4}{1-\beta}$
29	0	0	0	1	1	v(29,27)=$\frac{h_4+\beta h_1}{1-\beta^2}$
30	0	0	0	1	0	v(30,28)=$\frac{h_4}{1-\beta}$
31	0	0	0	0	1	v(31,31)=$\frac{h_4+\beta h_1}{1-\beta^2}$
32=All D	0	0	0	0	0	v(32,32)=$\frac{h_4}{1-\beta}$

Remark 3.6 *Some of the strategies we considered above are well studied (see, for example, [18], [20], [33], [45] and section 1.3 for more detail). The results of our analysis extend the existing picture as we analyse all pure strategies for the one-stage memory model for any possible combination of payoffs and discount factor β.*

Remark 3.7 *As has been noted already the approach works for mixed strategies as well; but, since there is an infinite number of such strategies, we do not consider such strategies in this* *work.*

Remark 3.8 *One of the main advantages of the approach which we use here is that it is algorithmic: it is possible to use computer programs (such as "Mathematica" or "Maple") to obtain the sufficient and necessary Nash Equilibrium conditions for all pure or (some specific mixed) finite memory strategies for a multi-state game.*

To obtain Nash Equilibrium conditions for strategy $\mathbf{f}$ we solve equations (3.5). Since we will only check the symmetric Nash Equilibria $(\mathbf{f},\mathbf{f})$ and the state game (Prisoners' Dilemma) is symmetric we only need to solve the first equation for $\mathbf{V}^1_\beta(s,\mathbf{f})$. For the Iterated Prisoners' Dilemma game the equation (3.5) reproduced below

$$\begin{aligned}&\mathbf{V}^1_\beta(s,\mathbf{f})\\&= \max_{a^1\in\mathbf{A}^1(s)}\left\{\sum_{a^2\in\mathbf{A}^2(s)} r^1(s,a^1,a^2)f(s,a^2)+\beta\sum_{s'\in\bar{\mathbf{S}}}\sum_{a^2\in\mathbf{A}^2(s)} p\left(s'|s,a^1,a^2\right)f(s,a^2)\mathbf{V}^1_\beta(s',\mathbf{f})\right\},\end{aligned}$$

has the following form

$$\mathbf{V}^1_\beta(s,\mathbf{f}) = \max\left\{u_{f_s}(C),u_{f_s}(D)\right\}, \tag{3.7}$$

where

$$\begin{aligned}u_{f_s}(C) &= f_sh_1+(1-f_s)\,h_2+\beta\left(f_s\mathbf{V}^1_\beta(1,\mathbf{f})+(1-f_s)\,\mathbf{V}^1_\beta(2,\mathbf{f})\right),\\u_{f_s}(D) &= f_sh_3+(1-f_s)\,h_4+\beta\left(f_s\mathbf{V}^1_\beta(3,\mathbf{f})+(1-f_s)\,\mathbf{V}^1_\beta(4,\mathbf{f})\right)\end{aligned}$$

and $s=0,1,2,3,4$.

It follows from equation (3.7) that if, for some strategy $\mathbf{f}$, the value $\mathbf{V}^1_\beta(s,\mathbf{f})$ is equal to $u_{f_s}(C)$ it means that by choosing action C at state s while playing against $\mathbf{f}$ the player would earn the highest payoff. In the same way if $\mathbf{V}^1_\beta(s,\mathbf{f})$ is equal $u_{f_s}(D)$ players should choose action D at state s while playing against $\mathbf{f}$ to optimise their payoff.

We will now demonstrate how to establish the range of parameters for which strategy Tit for Tat described by the pair $(\mathbf{6},\mathbf{4})$ of entries of table 3.1 and strategy Cautious Tit for Tat described by the pair $(\mathbf{22},\mathbf{20})$ of entries of table 3.2 are the Nash Equilibria. To analyse the Tit for Tat strategy we need to solve the equation (3.7) for this strategy played by the second player. As was explained above, if the Tit for Tat strategy is played by the second player it will be represented by line 4. The following lemma gives the solution for the equation (3.7) in this case.

Lemma 3.1 *If* $\mathbf{f} = 4$ *(Tit for Tat played by second player), then* $\mathbf{V}^1_\beta(s, \mathbf{f}) = a$, *for* $s = 0, 1, 2$, *and* $\mathbf{V}^1_\beta(s, \mathbf{f}) = b$, *for* $s = 3, 4$, *where*

$a = u_1(C) = \frac{h_1}{1-\beta}$ *and* $b = u_0(C) = h_2 + \frac{\beta h_1}{1-\beta}$	*if*	$\beta \geq \frac{h_3-h_1}{h_1-h_2}$ *and* $\beta \geq \frac{h_4-h_2}{h_1-h_2}$;	
$a = u_1(C) = \frac{h_1}{1-\beta}$ *and* $b = u_0(D) = \frac{h_4}{1-\beta}$	*if*	$\beta \geq \frac{h_3-h_1}{h_3-h_4}$ *and* $\beta \leq \frac{h_4-h_2}{h_1-h_2}$;	(3.8)
$a = u_1(D) = \frac{h_3+\beta h_2}{1-\beta^2}$ *and* $b = u_0(C) = \frac{h_2+\beta h_3}{1-\beta^2}$	*if*	$\beta \leq \frac{h_3-h_1}{h_1-h_2}$ *and* $\beta \geq \frac{h_4-h_2}{h_3-h_4}$;	
$a = u_1(D) = h_3 + \frac{\beta h_4}{1-\beta}$ *and* $b = u_0(D) = \frac{h_4}{1-\beta}$	*if*	$\beta \leq \frac{h_3-h_1}{h_3-h_4}$ *and* $\beta \leq \frac{h_4-h_2}{h_3-h_4}$.	

Proof. Since $f_0 = f_1 = f_2 = 1$ and $f_3 = f_4 = 0$ we have that

$$\begin{aligned}
\mathbf{V}^1_\beta(0, \mathbf{f}) &= \mathbf{V}^1_\beta(1, \mathbf{f}) = \mathbf{V}^1_\beta(2, \mathbf{f}) = \max\left\{h_1 + \beta \mathbf{V}^1_\beta(1, \mathbf{f}), h_3 + \beta \mathbf{V}^1_\beta(3, \mathbf{f})\right\}, \\
\mathbf{V}^1_\beta(3, \mathbf{f}) &= \mathbf{V}^1_\beta(4, \mathbf{f}) = \max\left\{h_2 + \beta \mathbf{V}^1_\beta(2, \mathbf{f}), h_4 + \beta \mathbf{V}^1_\beta(4, \mathbf{f})\right\}.
\end{aligned}$$

Denoting $\mathbf{V}^1_\beta(0, \mathbf{f}) = \mathbf{V}^1_\beta(1, \mathbf{f}) = \mathbf{V}^1_\beta(2, \mathbf{f}) = a$ and $\mathbf{V}^1_\beta(3, \mathbf{f}) = \mathbf{V}^1_\beta(4, \mathbf{f}) = b$ we obtain the following system

$$\begin{cases} a = \max\{h_1 + \beta a, h_3 + \beta b\} \\ b = \max\{h_2 + \beta a, h_4 + \beta b\} \end{cases}.$$

To solve this system we should consider four different cases.

1. $a = h_1 + \beta a$ and $b = h_2 + \beta a$. This can be solved to give $a = \frac{h_1}{1-\beta}$ and $b = \frac{h_2 + (h_1 - h_2)\beta}{1-\beta}$. This means that

$$\begin{aligned}
&h_1 + \beta \tfrac{h_1}{1-\beta} \geq h_3 + \beta \tfrac{h_2+(h_1-h_2)\beta}{1-\beta} \text{ and } h_2 + \beta \tfrac{h_1}{1-\beta} \geq h_4 + \beta \tfrac{h_2+(h_1-h_2)\beta}{1-\beta} \\
&\text{or } \beta \geq \tfrac{h_3-h_1}{h_1-h_2} \text{ and } \beta \geq \tfrac{h_4-h_2}{h_1-h_2}.
\end{aligned}$$

2. $a = h_1 + \beta a$ and $b = h_4 + \beta b$. This can be solved to give $a = \frac{h_1}{1-\beta}$ and $b = \frac{h_4}{1-\beta}$. This means that

$$\begin{aligned}
&h_1 + \beta \tfrac{h_1}{1-\beta} \geq h_3 + \beta \tfrac{h_4}{1-\beta} \text{ and } h_2 + \beta \tfrac{h_1}{1-\beta} \leq h_4 + \beta \tfrac{h_4}{1-\beta} \\
&\text{or } \beta \geq \tfrac{h_3-h_1}{h_3-h_4} \text{ and } \beta \leq \tfrac{h_4-h_2}{h_1-h_2}.
\end{aligned}$$

3. $a = h_3 + \beta b$ and $b = h_2 + \beta a$. This can be solved to give $a = \frac{h_3+\beta h_2}{1-\beta^2}$ and $b = \frac{h_2+\beta h_3}{1-\beta^2}$. This means that

$$\begin{aligned}
&h_1 + \beta \tfrac{h_3+\beta h_2}{1-\beta^2} \leq h_3 + \beta \tfrac{h_2+\beta h_3}{1-\beta^2} \text{ and } h_2 + \beta \tfrac{h_3+\beta h_2}{1-\beta^2} \geq h_4 + \beta \tfrac{h_2+\beta h_3}{1-\beta^2} \\
&\text{or } \beta \leq \tfrac{h_3-h_1}{h_1-h_2} \text{ and } \beta \geq \tfrac{h_4-h_2}{h_3-h_4}.
\end{aligned}$$

4. $a = h_3 + \beta b$ and $b = h_4 + \beta b$. This can be solved to give $a = \frac{h_3+(h_4-h_3)\beta}{1-\beta}$ and $b = \frac{h_4}{1-\beta}$. This means that

$$h_1 + \beta\frac{h_3+(h_4-h_3)\beta}{1-\beta} \leq h_3 + \beta\frac{h_4}{1-\beta} \text{ and } h_2 + \beta\frac{h_3+(h_4-h_3)\beta}{1-\beta} \leq h_4 + \beta\frac{h_4}{1-\beta}$$
$$\text{or } \beta \leq \frac{h_3-h_1}{h_3-h_4} \text{ and } \beta \leq \frac{h_4-h_2}{h_3-h_4}.$$

Therefore we obtain the statement of the lemma.

Remark 3.9 *As we can see from the proof of the lemma that if at two different states a strategy prescribes cooperation with the same probability, then the best actions to choose in reply are the same at these states. For example since for strategy Tit for Tat* $a = u_1(C) = \frac{h_1}{1-\beta}$ *and* $b = u_0(D) = \frac{h_4}{1-\beta}$ *if* $\beta \geq \frac{h_3-h_1}{h_3-h_4}$ *and* $\beta \leq \frac{h_4-h_2}{h_1-h_2}$, *the action C is the best reply at any state to which strategy Tit for Tat prescribes cooperation with probability 1 and the action D is the best reply at any state to which strategy Tit for Tat prescribes cooperation with probability 0. This is due to homogeneous nature of the Iterated Prisoners' Dilemma Game. If we consider the game at any time moment and forget about the past history of the game the rest of the game is exactly the same and does not depend on the time moment.*

Remark 3.10 *Taking Remark 3.9 into account, we can conclude that the standard stochastic game approach would be restricted if we wish to analyse the so-called "punishing strategies" for which the state description is not symmetric under permutation of players (for example Tit for Tat). Since such strategies take into account the history of the game it may fail to be optimal by the following reason. Due to its nature such strategies try to ensure cooperation by punishing the non-cooperative behaviour of the opponent: if an opponent defected in the past the strategy would prescribe choosing defection in spite of the opponent's present action (which could be to cooperate). Therefore such strategy is likely to prescribe different replies to the same action of the opponent at different states and hence cannot be optimal under the standard stochastic game approach. We have introduced Definition 3.8 in order to avoid this problem. It is clear that under this definition some Nash Equilibrium strategies are not optimal from every state of the process, but this is due to the nature of "punishment": sometimes a player punishing an opponent punishes himself as well. The approach that we have proposed here gives us the possibility of analysing "punishing strategies" since its only requires a strategy to be optimal at the empty-memory states. This approach can be justified if we assume that the players are not rational but programed with some types of behaviours. For example such an interpretation of players was used in [46].*

Proposition 3.2 *Strategy Tit for Tat (strategy pair* $(\mathbf{6}, \mathbf{4})$*) is a Nash equilibrium if and only if*

$$\beta \geq \frac{h_3 - h_1}{h_1 - h_2} \text{ and } \beta \geq \frac{h_4 - h_2}{h_1 - h_2} \quad \text{or} \quad \beta \geq \frac{h_3 - h_1}{h_3 - h_4} \text{ and } \beta \leq \frac{h_4 - h_2}{h_1 - h_2}. \tag{3.9}$$

Proof. The proof of this proposition is obvious since if conditions (3.9) are satisfied then the expected value for the Tit For Tat strategy against itself $\mathbf{v}(\mathbf{6}, \mathbf{4}) = \mathbf{v}^1_\beta(0, \mathbf{6}, \mathbf{4}) = \frac{h_1}{1-\beta}$ is equal to the $\mathbf{V}^1_\beta(s, \mathbf{4})$ which gives the maximum achievable value for the Tit For Tat strategy. If conditions (3.9) are not satisfied then

- if $\beta \leq \frac{h_3-h_1}{h_1-h_2}$ and $\beta \geq \frac{h_4-h_2}{h_3-h_4}$ strategy **29** obtains the maximum payoff $\frac{h_3+\beta h_2}{1-\beta^2}$ playing against Tit For Tat, and therefore Tit For Tat played against Tit For Tat is not a Nash equilibrium;
- in the same way if $\beta \leq \frac{h_3-h_1}{h_3-h_4}$ and $\beta \leq \frac{h_4-h_2}{h_3-h_4}$ strategy **32** (All D) obtains the maximum payoff $\frac{h_4}{1-\beta}$ playing against Tit For Tat.

Proposition 3.3 *Strategy pair* $(\mathbf{22}, \mathbf{20})$ *(Cautious Tit For Tat) is a Nash equilibrium if and only if*

$$\beta \geq \frac{h_3 - h_1}{h_3 - h_4} \text{ and } \beta \leq \frac{h_4 - h_2}{h_1 - h_2} \quad \text{or} \quad \beta \leq \frac{h_3 - h_1}{h_3 - h_4} \text{ and } \beta \leq \frac{h_4 - h_2}{h_3 - h_4}$$

Proof. The proof of this proposition is similar to the proof of Proposition 3.2. We would need to solve the equation (3.7) for strategy **20**. The solution is given again by (3.8) but in this case $\mathbf{V}^1_\beta(0, \mathbf{20}) = b$.

Using the ideas described we can solve equations (3.7) for each of the strategies given in tables 3.1 and 3.2. The results are presented in tables 3.3 and 3.4. The first column of these tables contains numbers of strategies. The strategies are given in pairs (for example strategies **4** and **20**), because solutions of equation (3.7) are given by the same expressions for strategies in a pair. In the same way as for strategies **4** and **20** the values of $\mathbf{V}^1_\beta(s, \mathbf{f})$, $s = 1, 2, 3, 4$, are equal for such strategies. The next five columns contain the values of $\mathbf{V}^1_\beta(s, \mathbf{f})$, $s = 0, 1, 2, 3, 4$, for each strategy. In the same way as has been done for strategy **4** we find for each strategy $\mathbf{f}$ the values $\mathbf{V}^1_\beta(s, \mathbf{f}) = a$ for any s such that $f_s = 1$ and $\mathbf{V}^1_\beta(s, \mathbf{f}) = b$ for any s such that $f_s = 0$. The last column of tables 3.3 and 3.4 contains the expressions for a and b for different parameter values.

Table 3.3. Solutions of the dynamic programming equation (3.7) for strategies 1-8 and 17-24.

Strategy	$\mathbf{V}^1_\beta(s,\mathbf{f})$ for different states s :					Formulae for a and b depending on parameter values
f	0	1	2	3	4	
1	a	a	a	a	a	$a{=}u_1(0){=}\frac{h_3}{1-\beta}$, $b{=}u_1(0){=}h_4{+}\frac{\beta h_3}{1-\beta}$.
17	b	a	a	a	a	
2	a	a	a	a	b	$a{=}u_1(0){=}\frac{h_3}{1-\beta}$. If $\beta{\geq}\frac{h_4-h_2}{h_3-h_2}$ then $b{=}u_0(1){=}h_2{+}\frac{\beta h_3}{1-\beta}$. If $\beta{\leq}\frac{h_4-h_2}{h_3-h_2}$ then $b{=}u_0(0){=}\frac{h_4}{1-\beta}$.
18	b	a	a	a	b	
3	a	a	a	b	a	If $\beta{\geq}\frac{h_3-h_1}{h_1-h_4}$ then $a{=}u_1(1){=}\frac{h_1}{1-\beta}$ and $b{=}u_0(0){=}h_4{+}\beta\frac{h_1}{1-\beta}$. If $\beta{\leq}\frac{h_3-h_1}{h_1-h_4}$ then $a{=}u_1(0){=}\frac{h_3+\beta h_4}{1-\beta^2}$ and $b{=}u_0(0){=}\frac{h_4+\beta h_3}{1-\beta^2}$.
19	b	a	a	b	a	
4	a	a	a	b	b	If $\beta{\geq}\frac{h_3-h_1}{h_1-h_2}$ and $\beta{\geq}\frac{h_4-h_2}{h_1-h_2}$ then $a{=}u_1(1){=}\frac{h_1}{1-\beta}$ and $b{=}u_0(1){=}h_2{+}\frac{\beta h_1}{1-\beta}$. If $\beta{\geq}\frac{h_3-h_1}{h_3-h_4}$ and $\beta{\leq}\frac{h_4-h_2}{h_1-h_2}$ then $a{=}u_1(1){=}\frac{h_1}{1-\beta}$ and $b{=}u_0(0){=}\frac{h_4}{1-\beta}$. If $\beta{\leq}\frac{h_3-h_1}{h_1-h_2}$ and $\beta{\geq}\frac{h_4-h_2}{h_3-h_4}$ then $a{=}u_1(0){=}\frac{h_3+\beta h_2}{1-\beta^2}$ and $b{=}u_0(1){=}\frac{h_2+\beta h_3}{1-\beta^2}$. If $\beta{\leq}\frac{h_3-h_1}{h_3-h_4}$ and $\beta{\leq}\frac{h_4-h_2}{h_3-h_4}$ then $a{=}u_1(0){=}h_3{+}\frac{\beta h_4}{1-\beta}$ and $b{=}u_0(0){=}\frac{h_4}{1-\beta}$.
20	b	a	a	b	b	
5	a	a	b	a	a	$a{=}u_1(0){=}\frac{h_3}{1-\beta}$, $b{=}u_0(0){=}h_4{+}\frac{\beta h_3}{1-\beta}$.
21	b	a	b	a	a	
6	a	a	b	a	b	$a{=}u_1(0){=}\frac{h_3}{1-\beta}$, $b{=}u_0(0){=}\frac{h_4}{1-\beta}$.
22	b	a	b	a	b	
7	a	a	b	b	a	If $\beta{\geq}\frac{h_3-h_1}{h_1-h_4}$ then $a{=}u_1(1){=}\frac{h_1}{1-\beta}$ and $b{=}u_0(0){=}h_4{+}\frac{\beta h_1}{1-\beta}$. If $\beta{\leq}\frac{h_3-h_1}{h_1-h_4}$ then $a{=}u_1(0){=}\frac{h_3+\beta h_4}{1-\beta^2}$ and $b{=}u_0(0){=}\frac{h_4+\beta h_3}{1-\beta^2}$.
23	b	a	b	b	a	
8	a	a	b	b	b	$b{=}u_0(0){=}\frac{h_4}{1-\beta}$. If $\beta{\geq}\frac{h_3-h_1}{h_3-h_4}$ then $a{=}u_1(1){=}\frac{h_1}{1-\beta}$. If $\beta{\leq}\frac{h_3-h_1}{h_3-h_4}$ then $a{=}u_1(0){=}h_3{+}\frac{\beta h_4}{1-\beta}$.
24	b	a	b	b	b	

Table 3.4. Solutions of the dynamic programming equation (3.7) for strategies 9-16 and 25-32.

Strategy	$V^1_\beta(s, f)$ for different states s :					Formulae for a and b depending on parameter values
f	0	1	2	3	4	
9 25	a b	b b	a a	a a	a a	$a=u_1(0)=\frac{h_3}{1-\beta}$, $b=u_0(0)=h_4+\beta\frac{h_3}{1-\beta}$.
10 26	a b	b b	a a	a a	b b	$a=u_1(0)=\frac{h_3}{1-\beta}$. If $\beta\geq\frac{h_4-h_2}{h_3-h_2}$ then $b=u_0(1)=h_2+\beta\frac{h_3}{1-\beta}$. If $\beta\leq\frac{h_4-h_2}{h_3-h_2}$ then $b=u_0(0)=\frac{h_4}{1-\beta}$.
11 27	a b	b b	a a	b b	a a	$a=u_1(0)=\frac{h_3+\beta h_4}{1-\beta^2}$, $b=u_0(0)=\frac{h_4+\beta h_3}{1-\beta^2}$.
12 28	a b	b b	a a	b b	b b	If $\beta\geq\frac{h_4-h_2}{h_3-h_4}$ then $a=u_1(0)=\frac{h_3+\beta h_2}{1-\beta^2}$ and $b=u_0(1)=\frac{h_2+\beta h_3}{1-\beta^2}$. If $\beta\leq\frac{h_4-h_2}{h_3-h_4}$ then $a=u_1(0)=h_3+\beta\frac{h_4}{1-\beta}$ and $b=u_0(0)=\frac{h_4}{1-\beta}$.
13 29	a b	b b	b b	a a	a a	$a=u_1(0)=\frac{h_3}{1-\beta}$, $b=u_0(0)=h_4+\beta\frac{h_3}{1-\beta}$.
14 30	a b	b b	b b	a a	b b	$a=u_1(0)=\frac{h_3}{1-\beta}$, $b=u_0(0)=\frac{h_4}{1-\beta}$.
15 31	a b	b b	b b	b b	a a	$a=u_1(0)=\frac{h_3+\beta h_4}{1-\beta^2}$, $b=u_0(0)=\frac{h_4+\beta h_3}{1-\beta^2}$.
16 32	a b	b b	b b	b b	b b	$a=u_1(0)=h_3+\beta\frac{h_4}{1-\beta}$, $b=u_0(0)=\frac{h_4}{1-\beta}$.

Using the results obtained we can find the conditions required for a certain strategy to be a Nash Equilibrium. Considering each strategy in turn, we find that the strategies given in the first column of table 3.5 are Nash Equilibria if parameters of the model are in a specific range. The conditions on the parameters are given in the second column of table 3.5.

Table 3.5. Pure memory-one Nash Equilibria for the Iterated Prisoners' Dilemma game.

Strategy	Conditions under which a strategy is a Nash equilibrium
5, 7, 21, 23	$\beta \geq F$
6	$\beta \geq A$ and $\beta \geq B$ or $\beta \geq C$ and $\beta \leq B$
8	$\beta \geq C$
18, 26	$\beta \leq E$
22	$\beta \geq F$ and $\beta \leq B$ or $\beta \leq C$ and $\beta \leq D$
30	$\beta \leq D$
20, 24, 28, 32	$0 < \beta < 1$

Where

$$A = \frac{h_3 - h_1}{h_1 - h_2},\ B = \frac{h_4 - h_2}{h_1 - h_2},\ C = \frac{h_3 - h_1}{h_3 - h_4},\ D = \frac{h_4 - h_2}{h_3 - h_4},\ E = \frac{h_4 - h_2}{h_3 - h_2},\ F = \frac{h_3 - h_1}{h_1 - h_4}.$$

Notice that the inequalities (1.3) on the Prisoners' Dilemma payoffs imply that all A, B, C, D, E and F are greater that zero, B, C and E are less than one, $C < F$ and $E < B$.

The following plots represent the above result graphically showing the range of the discount factor β for which a strategy is a Nash Equilibrium and allow the comparison of different strategies. Three different cases are represented.

1. If $h_4 - h_2 < h_3 - h_1$ then $D < B < C < A$. In this case the results are shown in figure 3.1. Here for example strategy **8** (Grim) is a Nash equilibrium if $\beta \geq C$ and strategies **20** (Cautious Stubborn) , **24** (Cautious Grim), **28** and **32** (All C) are Nash Equilibria for any value of β $(0 < \beta < 1)$.

2. If $h_4 - h_2 = h_3 - h_1$ then $D = B = C = A$. In this case the results are shown in figure 3.2.

3. If $h_4 - h_2 > h_3 - h_1$ then $A < C < B < D$. In this case three plots are produced (see figure 3.3) because the precise relationships between F and B or D and between E and A or C are not relevant.

Figure 3.1 Pure memory one Nash Equilibria for the Iterated Prisoners' Dilemma game if $h_4 - h_2 < h_3 - h_1$.

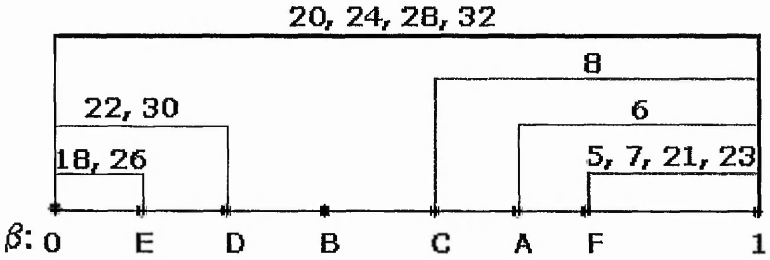


Figure 3.2 Pure memory one Nash Equilibria for the Iterated Prisoners' Dilemma game if $h_4 - h_2 = h_3 - h_1$.

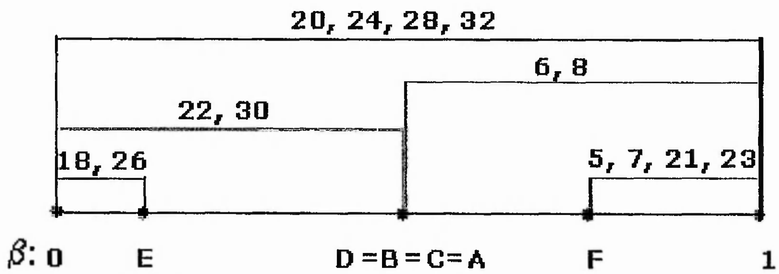
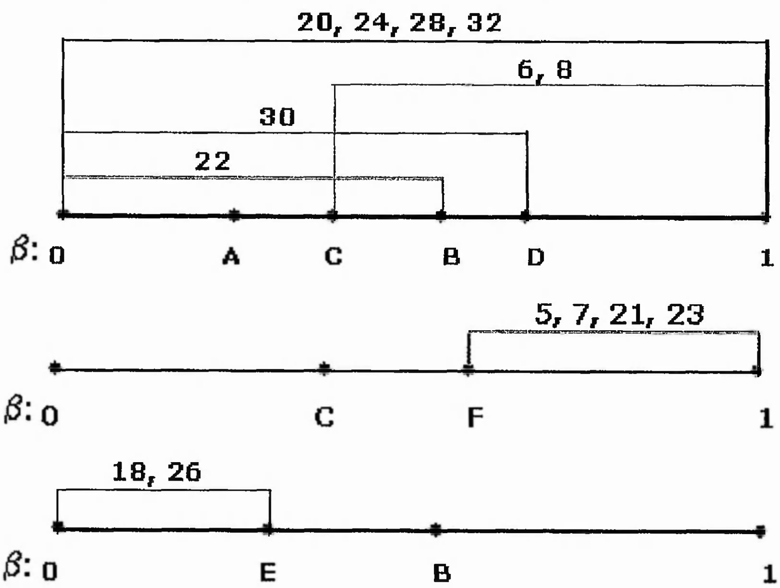


Figure 3.3 Pure memory one Nash Equilibria for the Iterated Prisoners' Dilemma game if $h_4 - h_2 > h_3 - h_1$.



Figures 3.1-3.3 give a complete set of one-stage memory symmetric pure Nash Equilibria for the Iterated Prisoners' Dilemma game. Notice that the strategies **5** (Tweedledee), **6** (Tit for Tat), **7** (Pavlov) and **8** (Grim) are cooperative strategies. We mean here that if both players use these strategies then cooperative behaviour is observed at each round of interaction. Strategies **21** (Cautious Tweedledee) and **23** (Cautious Pavlov) are the cautious versions of strategies **5** and **7** respectively. They can also be considered as cooperative since if they are used by both players the cooperative behaviour is observed at each round except for the first one. All other strategies which appear in table 3.5 are non-cooperative in the sense that if they are played against themselves then defection is observed at each round of interaction.

Example 3.2. For the standard set of parameters

$Player_1 \setminus Player_2$	Cooperate	Defect
Cooperate	3, 3	0, 5
Defect	5, 0	1, 1

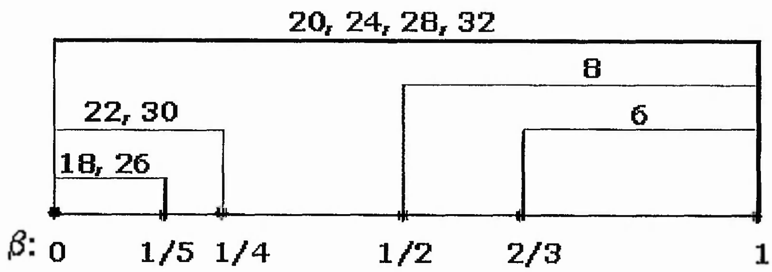
$$h_1 = 3, \qquad h_2 = 0, \qquad h_3 = 5, \qquad h_4 = 1,$$

we have that

$$A = \frac{2}{3}, \quad B = \frac{1}{3}, \quad C = \frac{1}{2}, \quad D = \frac{1}{4}, \quad E = \frac{1}{5}, \quad F = 1.$$

The Nash equilibrium strategies in this case are shown in figure 3.4 below.

Figure 3.4 Pure memory one Nash Equilibria for the Iterated Prisoners' Dilemma game if $h_1 = 3, h_2 = 0, h_3 = 5, h_4 = 1$.



Notice that, since $F = 1$, strategies **5** (Tweedledee), **7** (Pavlov) **21** (Cautious Tweedledee) and **23** (Cautious Pavlov) are not Nash Equilibria for any value of the discount factor β. The only cooperative Nash Equilibria in this case are **6** (Tit for Tat), for $\beta \geq \frac{2}{3}$, and **8** (Grim), for $\beta \geq \frac{1}{2}$.

3.4 Summary.

In this chapter we have introduced the approach which allows us to check that a strategy is a Nash Equilibrium for stochastic games with finite memory. We use this approach in chapter 5 to analyse a special multi-state game which we introduce. The main point of this approach consists in following. Once we have constructed the appropriate Markov process and obtained the state space representation of some strategy we are interested in analysing we can find the necessary and sufficient Nash Equilibrium conditions. These condition then guarantee that the strategy is a Nash Equilibrium in a class of all (pure and mixed) strategies which are allowed by the state space representation of the process. Unfortunately, the approach is restricted in the sense that it does not solve the problem of finding all Nash Equilibria for a model. In particular, mixed strategy Nash Equilibria cannot be found.

Chapter 4

Stability concepts and Evolutionary Dynamics.

In the previous chapter we have described how it is possible to verify that a certain strategy is a Nash Equilibrium. Assuming that we are able to find a few Nash Equilibrium strategies, the question arises: which one of the strategies will be adopted by players. From this point of view the Nash Equilibrium concept does not allow us to compare different strategies. If a game has multiple Nash Equilibria it is not possible to say which one will be played. The concepts of an Evolutionarily Stable Strategy and Replicator Dynamics considered in this chapter are aimed at answering the question: 'which strategies or types of behavior are likely to be an outcome of an evolutionary process'? We have already discussed the general definitions of these concepts in chapter 1. In this chapter we explain in more detail the techniques which we will use in order to analyse the Replicator Dynamics for the multi-state game we are considering in chapters 5, 6 and 7.

In the first section of this chapter we discuss the standard techniques of the theory of qualitative analysis of differential equations [36]- [40] which mostly consists in analysing the linearised system of equations. This approach allows us to produce a qualitative picture of the evolutionary dynamics and determine the strategies which can be the end points of an evolutionary process. We illustrate this technique considering the example of the Iterated Prisoners' Dilemma with three strategies. The results obtained for this example lead us to the conclusion that the standard concepts of asymptotically stable and evolutionarily stable sets (see section 1.4.3 for definitions) are not sufficient. In the example of the Iterated Prisoners'

Dilemma there appears to be a set of strategies such that all solution trajectories starting in a neighbourhood of this set terminate at this set, but the set does not satisfy any standard definition of stability. This motivates us to introduce a new kind of stable sets which we call evolutionarily attractive sets. This is done in the second section where the new definition is introduced. In the second section we also discuss the relationship between the standard and new concepts and consider examples which illustrate the new definition.

In the third section we present the so-called "blowing up" technique. When repeated games are represented in normal form the payoff bi-matrix is commonly non-generic. As a consequence, some of the fixed points in the standard replicator dynamics are not hyperbolic and their stability properties cannot be analysed by the standard approach of local linearisation. In such cases stability properties may be determined by Lyapounov's method. However, there is no procedure which guarantees that a Lyapounov function will be found and the problem of finding a suitable function by trial and error becomes increasingly intractable as the number of pure strategies being considered increases. In this section we illustrate a technique, known either as "blowing up" or "the sigma process", which can be used to determine the stability properties of non-hyperbolic fixed points in the replicator dynamics. This method was introduced by Bogoyavlensky and Novikov [47] and independently by McGehee [48] and Mather and McGehee [49]. Although this technique has been around for many years, it appears that it is not known to game theorists. We will illustrate its use by considering a replicator dynamics system which is similar to one that arises from dynamic models of social interactions based on multi-state games (see section 7.1 where the results of the similar analysis for the multi-state game are given).

Let us recall that the standard Replicator Dynamics [28] (see chapter 1), which assumes "pairwise contest" interaction and which we will consider in this thesis, describes changes of state $X = \{x_1, \dots, x_n\}$ in a population whose members are playing a symmetric two-person game with the payoffs given by matrix A, with elements a_{ij}. Here x_i is the proportion of the individuals in the population who adopt behaviour i. The Replicator Dynamics (4.1) (see formula (1.6) of chapter 1 which we reproduce here for convenience) in continuous time is the following dynamical system.

$$\dot{x}_i = \frac{dx_i}{dt} = x_i \left(\left\{ \sum_{j=1}^{n} a_{ij} x_j \right\} - XAX^T \right), \qquad i = 1, \dots, n. \tag{4.1}$$

Using relationship $x_n = 1 - \sum_{i=1}^{n} x_i$, the number of equations is reduced to $n-1$. Denoting $x = (x_1, \ldots, x_{n-1})$, we obtain system (4.2) (the system of equations labeled (1.7) in chapter 1)

$$\frac{dx_i}{dt} = G_i = x_i \left(\left\{ \sum_{j=1}^{n-1} (a_{ij} - a_{in}) x_j + a_{in} \right\} - \mu(x) \right), \qquad i = 1, \ldots n-1, \tag{4.2}$$

where

$$\mu(x) = (x_1, \cdots, x_{n-1}, 1 - x_1 - \ldots - x_{n-1}) A (x_1, \cdots, x_{n-1}, 1 - x_1 - \ldots - x_{n-1})^T.$$

We consider the solutions of the system (4.2) that are restricted to the simplex (4.3)

$$\Delta = \left\{ x : \bigcap_{i=1}^{n-1} \left(0 \leq x_i \right) \cap \left(\sum_{i=1}^{n-1} x_i \leq 1 \right) \right\}. \tag{4.3}$$

To analyse the behaviour of the solutions and describe the dynamics we apply methods of the qualitative theory of the dynamical systems (due to [36]-[40]) and the concepts of evolutionary stability.

4.1 Basic qualitative analysis.

4.1.1 Vertices.

To begin with the qualitative analysis of the Replicator Dynamics we obtain conditions on the asymptotic stability for the vertices of the simplex Δ. In order to do this let us analyse the linearised system and calculate eigenvectors and eigenvalues of the Jacobian $J_G|_x = \left\{ \frac{\partial G_i}{\partial x_k} \right\}\Big|_x$ at the vertices.

Proposition 4.1

(i) If $x = \Big\{ \underbrace{0, 0, \ldots, 0, 0}_{n-1} \Big\}$ *then eigenvectors* e_i *and corresponding eigenvalues* λ_i *of the Jacobian* $J_G|_x = \left\{ \frac{\partial G_i}{\partial x_k} \right\}\Big|_x$ *at the point* x *are as follows.*

$$e_i = \Big(\underbrace{0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0}_{n-1} \Big) \quad \text{and} \quad \lambda_i = a_{in} - a_{nn}, \quad i = 1, \ldots n-1.$$

(ii) If $x = \left\{ \underbrace{0, \ldots, 0, \underset{m}{1}, 0, \ldots, 0}_{n-1} \right\}$ *then eigenvectors* e_i *and corresponding eigenvalues* λ_i *of the Jacobian* $J_G|_x = \left\{ \frac{\partial G_i}{\partial x_k} \right\} \Big|_x$ *at the point* x *are as follows.*

$$e_i = \left(\underbrace{0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0, \underset{m}{-1}, 0, \ldots, 0}_{n-1} \right) \quad \text{and} \quad \lambda_i = a_{im} - a_{mm}, \quad i \neq m;$$

$$e_m = \left(\underbrace{0, \ldots, 0, \underset{m}{1}, 0, \ldots, 0}_{n-1} \right) \quad \text{and} \quad \lambda_m = a_{nm} - a_{mm}.$$

Proof. Firstly, let us obtain a formula for $\mu(x)$.

$$\begin{aligned} \mu(x) &= (x_1, \cdots, x_{n-1}, 1 - x_1 - \ldots - x_{n-1}) A (x_1, \cdots, x_{n-1}, 1 - x_1 - \ldots - x_{n-1})^T \\ &= \sum_{m=1}^{n-1} \sum_{l=1}^{n-1} (a_{ml} + a_{nn} - a_{mn} - a_{nl}) x_m x_l + \sum_{m=1}^{n-1} (a_{mn} + a_{nm} - 2a_{nn}) x_m + a_{nn} \end{aligned}$$

Then elements of the Jacobian are calculated as follows.

$$\begin{aligned} \frac{\partial G_i}{\partial x_k} &= \frac{\partial}{\partial x_k} \left(x_i \left(\left\{ \sum_{j=1}^{n-1} (a_{ij} - a_{in}) x_j + a_{in} \right\} - \mu(x) \right) \right) \\ &= \delta_{ik} \left(a_{in} + \sum_{j=1}^{n-1} (a_{ij} - a_{in}) x_j - \mu(x) \right) + x_i (a_{ik} - a_{in}) \\ &\quad - x_i \left((a_{nk} + a_{kn} - 2a_{nn}) + \sum_{l=1}^{n-1} (a_{lk} + a_{kl} - a_{ln} - a_{nl} - a_{nk} - a_{kn} + 2a_{nn}) x_l \right) \end{aligned}$$

Let $x = \left\{ \underbrace{0, 0, \ldots, 0, 0}_{n-1} \right\}$, then $\mu(x) = a_{nn}$ and $J_G|_x = \left\{ \frac{\partial G_i}{\partial x_k} \right\} \Big|_x = \{ \delta_{ik} (a_{in} - a_{nn}) \}$.

Therefore we obtain statement (i) of the proposition.

Now consider the other vertices. If $x = \left\{ \underbrace{0, \ldots, 0, \underset{m}{1}, 0, \ldots, 0}_{n-1} \right\}$, then $\mu(x) = a_{mm}$ and

$$J_G|_x = \left\{ \frac{\partial G_i}{\partial x_k} \right\} \Big|_x = \delta_{ik} (a_{im} - a_{mm}) + \delta_{im} (a_{nm} - a_{km}).$$

For example, if $x = \left\{ \underbrace{1, 0, \dots, 0}_{n-1} \right\}$, then $m = 1$ and we have that

$$J_G|_x = \left\{ \frac{\partial G_i}{\partial x_k} \right\} \Big|_x = \delta_{ik} \left(a_{i1} - a_{11} \right) + \delta_{i1} \left(a_{n1} - a_{k1} \right)$$

$$= \begin{bmatrix} a_{n1} - a_{11} & a_{n1} - a_{21} & \cdots & a_{n1} - a_{(n-2)1} & a_{n1} - a_{(n-1)1} \\ 0 & a_{21} - a_{11} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{(n-2)1} - a_{11} & 0 \\ 0 & 0 & \cdots & 0 & a_{(n-1)1} - a_{11} \end{bmatrix}.$$

Calculating eigenvectors and eigenvalues of the Jacobian $J_G|_x$ directly we find that

$$e_1 = \left(\underbrace{1, 0, \dots, 0}_{n-1} \right) \quad \text{and} \quad \lambda_1 = a_{n1} - a_{11},$$

$$e_i = \left(\underbrace{-1, 0, \dots, 0, \underset{i}{1}, 0, \dots, 0}_{n-1} \right) \quad \text{and} \quad \lambda_i = a_{i1} - a_{11}, \qquad i \neq 1.$$

In the same way we can calculate eigenvectors and eigenvalues for all other vertices and obtain statement (ii) of the proposition.

Remark 4.1 *Using the results of Proposition 4.1 we can easily check whether or not a vertex (which represents a monomorphic population in which all members use the corresponding strategy) is asymptotically stable. If all eigenvalues for the vertex are less than zero, then it is asymptotically stable. Therefore Proposition 4.1 gives the analytical explanation of the fact that in order to be asymptotically stable the strategy must earn a higher payoff against itself than all other strategies (compare the formulae for eigenvalues and the payoff matrix A).*

4.1.2 Stationary points and sets.

All stationary points for the systems (4.1) and (4.2) are described by the solutions of the system of equations

$$x_i \left(\sum_{j=1}^{n} a_{ij} x_j - XAX^T \right) = 0, \qquad i = 1, \dots, n. \tag{4.4}$$

To find all solutions of the system (4.4) it is necessary to enumerate all possible combinations of m indices from the set $\{1, \ldots, n\}$ where m consecutively takes values from 1 to n. If a combination $\{j_1, \ldots, j_m\}$ is chosen, then we suppose that coordinates $x_{j_1}, \ldots, x_{j_m}$ take non-zero values and all remaining coordinates x_i are zero. In this case system (4.4) is equivalent to the system

$$\begin{cases} \sum_{k=1}^{m} (a_{j_1 j_k} - a_{j_l j_k}) x_{j_k} = 0, \qquad l = 2, \ldots, m; \\ \sum_{k=1}^{m} x_{j_k} = 1. \end{cases} \tag{4.5}$$

If $m = 1$ then the solutions are the vertices of the simplex Δ.

Consider now the case when $m \in \{2, \ldots, n\}$ corresponding to polymorphic population. Denote by

$$A_{j_1, \ldots, j_m} = \begin{bmatrix} (a_{j_1 j_1} - a_{j_2 j_1}) & \ldots & (a_{j_1 j_m} - a_{j_2 j_m}) \\ \vdots & & \vdots \\ (a_{j_1 j_1} - a_{j_m j_1}) & \ldots & (a_{j_1 j_m} - a_{j_m j_m}) \\ 1 & \ldots & 1 \end{bmatrix}$$

the matrix of the system (4.5). If matrix $A_{j_1, \ldots, j_m}$ is non-singular then the system has a unique solution which can be obtained by Cramer's rule as follows.

$$x_{j_k} = \frac{\begin{Vmatrix} (a_{j_1 j_1} - a_{j_2 j_1}) & \cdots & (a_{j_1 j_{k-1}} - a_{j_2 j_{k-1}}) & 0 & (a_{j_1 j_{k+1}} - a_{j_2 j_{k+1}}) & \cdots & (a_{j_1 j_m} - a_{j_2 j_m}) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (a_{j_1 j_1} - a_{j_m j_1}) & \cdots & (a_{j_1 j_{k-1}} - a_{j_m j_{k-1}}) & 0 & (a_{j_1 j_{k+1}} - a_{j_m j_{k+1}}) & \cdots & (a_{j_1 j_m} - a_{j_m j_m}) \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{Vmatrix}}{\begin{Vmatrix} (a_{j_1 j_1} - a_{j_2 j_1}) & \cdots & (a_{j_1 j_k} - a_{j_2 j_k}) & \cdots & (a_{j_1 j_m} - a_{j_2 j_m}) \\ \vdots & & \vdots & & \vdots \\ (a_{j_1 j_1} - a_{j_m j_1}) & \cdots & (a_{j_1 j_k} - a_{j_m j_k}) & \cdots & (a_{j_1 j_m} - a_{j_m j_m}) \\ 1 & \cdots & 1 & \cdots & 1 \end{Vmatrix}}$$

If this solution is in the domain (4.3) then it corresponds to a stationary point of the dynamical system (4.2).

If the matrix of system (4.5) is singular then there are two possible cases.

1. The ranks of matrix $A_{j_1,\dots,j_m}$ and matrix

$$\overline{A}_{j_1,\dots,j_m} = \begin{bmatrix} (a_{j_1j_1} - a_{j_2j_1}) & \dots & (a_{j_1j_m} - a_{j_2j_m}) & 0 \\ \vdots & & \vdots & \vdots \\ (a_{j_1j_1} - a_{j_mj_1}) & \dots & (a_{j_1j_m} - a_{j_mj_m}) & 0 \\ 1 & \dots & 1 & 1 \end{bmatrix}$$

   are different. Then the equations of system (4.5) are incompatible and there are no solutions for this system.

2. The ranks of matrix $A_{j_1,\dots,j_m}$ and matrix $\overline{A}_{j_1,\dots,j_m}$ are equal. Then the system (4.5) has infinitely many solutions. If some of these solutions belong to the domain (4.3) then these solutions constitute a stationary set of the system (4.2).

It is often quite hard to obtain the global picture of the dynamics, but once the stationary points of the system are found it is possible to analyse the dynamics in neighbourhoods of such points. The following theorem is commonly used for the investigation of solution behaviour in the neighbourhood of a stationary point (see for example Guckenheimer and Holmes [50]).

Theorem (Centre Manifold Theorem for Flows). *Let G be a C^r vector field on R^n vanishing at the origin ($G(0) = 0$) and let $J_G|_0 = \left\{\frac{\partial G_i}{\partial x_k}\right\}\Big|_0$. Divide the spectrum of $J_G|_0$ into three parts, L_s, L_c, L_u with*

$$\operatorname{Re}\lambda \begin{cases} < 0 & \text{if} \quad \lambda \in L_s, \\ = 0 & \text{if} \quad \lambda \in L_c, \\ > 0 & \text{if} \quad \lambda \in L_u. \end{cases}$$

Let the generalized eigenspaces of L_s, L_c and L_u be E^s, E^c and E^u, respectively. Then there exist stable and unstable invariant manifolds W^s and W^u tangent to E^s and E^u at 0 and a centre manifold W^c tangent to E^c at 0. The manifolds W^s, W^u and W^c are all invariant for the flow of G. The stable and unstable manifolds are unique, but W^c need not be.

This theorem provides us with an approach for finding the stable and unstable manifolds (separatrixes) at each stable point. Then we can use the following observation to analyse the global dynamics. Let there be a sequence of stationary points x^i such that all outgoing separatrixes of the point x^1 approach the point x^2, and so on, generating a sequence of separatrixes $\dots \longrightarrow x^1 \longrightarrow x^2 \longrightarrow x^3 \longrightarrow x^4 \longrightarrow \dots$ Then, since the solutions depend continuously on the initial conditions, the trajectories that start sufficiently close to one of these separatrixes of stationary points move along this sequence of separatrixes for any finite moment of time t.

4.1.3 Example: Qualitative Analysis of the Evolutionary Dynamics of the Iterated Prisoners' Dilemma game.

In this section we demonstrate how the ideas discussed above can be used.

The Iterated Prisoners' Dilemma [18], [32] is the most common model used for the study of the evolution of cooperative behaviour in a population of selfish individuals. As such, it is probably the most intensively studied of all games. In the general Iterated Prisoners' Dilemma the number of available strategies is infinite, so I will consider a restricted set of strategies. In this section I consider a class of Iterated Prisoners' Dilemma games in which players are restricted to using one of the three strategies: unconditional defection (All D strategy from section 3.3), unconditional cooperation (All C strategy from section 3.3) and a strategy which attempts to ensure cooperation by invoking a punishment option in the event of an opponent's defection. The most famous of such strategies is Tit for Tat (see Tit for Tat from section 3.3) [18], [32], which begins by cooperating and thereafter mimics an opponent's play in the previous stage. This is the strategy which will be considered in this paper. The evolutionary dynamics for this class of games is well-known, having been obtained by simulation [16]. It is also possible to integrate the corresponding Replicator Dynamics to find exact solution trajectories [51]. Here the Iterated Prisoners' Dilemma is given as an example of how the qualitative approach given above can be used. Partial results have already been obtained by this method [15], but I give a complete analysis.

It is obvious that Tit for Tat is not an evolutionarily stable strategy in the sense of ([13], [14]), since in a population of Tit for Tat players unconditional cooperators gain the same payoff. In the replicator dynamics, this translates to the statement that the corresponding fixed point is not asymptotically stable. In the next section we show that there exists a set of populations of cooperative individuals which may be considered as potential outcomes of the evolutionary dynamics. These populations are composed of various mixtures of unconditional cooperators and Tit for Tat players. Although neither any particular population in the set nor the set as a whole is asymptotically stable, we show that the failure of asymptotic stability for the set is "mild" (this concept will be made more precise in section 4.2).

Let us consider the Iterated Prisoners' Dilemma game as described in section 3.3. That is the payoffs for the players in each stage game are symmetric and given by the bi-matrix (1.2) reproduced below.

		Player 2	
		C	D
Player 1	C	h_1, h_1	h_2, h_3
	D	h_3, h_2	h_4, h_4

In order for this repeated game to be identified as an Iterated Prisoners' Dilemma, the payoffs given in the table must satisfy the inequalities [52]

$$h_3 > h_1 > h_4 > h_2 \qquad \textit{and} \qquad 2h_1 > h_2 + h_3. \tag{4.6}$$

We set up the evolutionary dynamics by considering an infinitely large population of individuals who adopt one of three strategies.

1. Unconditional cooperation at every stage (which we denote σ_C).
2. Unconditional defection at every stage (which we denote σ_D).
3. Tit for Tat (which we denote σ_T).

Using a discounting factor $\beta \in [0, 1)$ we can calculate the payoffs in the repeated game, $\pi(\sigma, \sigma')$ for adopting strategy σ against an opponent who adopts strategy σ'. The payoffs can be summarised in the matrix

$$\begin{aligned} A &= \begin{bmatrix} \pi(\sigma_C, \sigma_C) & \pi(\sigma_C, \sigma_T) & \pi(\sigma_C, \sigma_D) \\ \pi(\sigma_T, \sigma_C) & \pi(\sigma_T, \sigma_T) & \pi(\sigma_T, \sigma_D) \\ \pi(\sigma_D, \sigma_C) & \pi(\sigma_D, \sigma_T) & \pi(\sigma_D, \sigma_D) \end{bmatrix} \\ &= \frac{1}{1-\beta} \begin{bmatrix} h_1 & h_1 & h_2 \\ h_1 & h_1 & h_2 + (h_4 - h_2)\beta \\ h_3 & h_3 + (h_4 - h_3)\beta & h_4 \end{bmatrix}. \end{aligned} \tag{4.7}$$

Remark 4.2 *Tit For Tat could be replaced by the strategy Grim (see table 3.1 chapter 3) as the payoff matrix A is exactly the same as calculated above.*

The strategy pair $[\sigma_D, \sigma_D]$ is a Nash equilibrium of this game for all values of h_i which satisfy inequalities (4.6). If the condition

$$\beta(1 - \alpha_0) \geq \frac{h_3 - h_1}{h_3 - h_4}. \tag{4.8}$$

is satisfied for some $\alpha_0 \in [0,1)$, there is also a continuum of symmetric Nash equilibria of the form $[\sigma_\alpha, \sigma_\alpha]$ with $\sigma_\alpha = (1-\alpha)\sigma_T + \alpha\sigma_C$ for all $\alpha \in [0, \alpha_0]$. Since the Nash equilibrium $[\sigma_D, \sigma_D]$ is strict, σ_D is an evolutionarily stable strategy in the sense of [13]. However, none of the strategies σ_α are evolutionarily stable since, for example,

$$\pi(\sigma_\alpha, \sigma_\alpha) = \pi(\sigma_C, \sigma_\alpha) = \pi(\sigma_\alpha, \sigma_C) = \pi(\sigma_C, \sigma_C).$$

Let x_1, x_2 and x_3 be the proportions of individuals who adopt σ_C, σ_T and σ_D respectively. The state of the population then is described by a vector $X = (x_1, x_2, x_3)$. Using the identity $x_3 = 1 - x_1 - x_2$ we reduce to two equations describing the evolution of a point $x = (x_1, x_2)$ in the domain

$$\Delta = \{(x_1, x_2) : (x_1 \geq 0) \cap (x_2 \geq 0) \cap (x_1 + x_2 \leq 1)\}. \tag{4.9}$$

The Replicator Dynamics are

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \frac{1}{1-\beta} \begin{pmatrix} G_1(x_1, x_2) \\ G_2(x_1, x_2) \end{pmatrix} \tag{4.10}$$

where

$$G_1(x_1, x_2) = x_1(x_1 + x_2 - 1)(\Lambda x_1 + (\Lambda - \Omega)x_2 + \Theta),$$
$$G_2(x_1, x_2) = x_2(x_1 + x_2 - 1)(\Lambda x_1 + (\Lambda - \Omega)x_2 + \Theta(1-\beta)).$$

Here we have used the following notation

$$\Lambda = h_3 + h_2 - h_1 - h_4, \quad \Omega = (h_3 + h_2 - 2h_4)\beta, \quad \Theta = h_4 - h_2.$$

For later use, we also define

$$\Psi = \Omega - \Lambda + \Theta(\beta - 1) \tag{4.11}$$

and note that conditions (4.6) and (4.8) imply the inequalities

$$\Theta > 0, \quad \Psi > 0, \quad \Omega - \beta\Lambda > 0, \quad \Omega - \Lambda > 0, \quad \Theta + \Lambda > 0. \tag{4.12}$$

Remark 4.3 *In [51] it was shown that this system is integrable. To integrate system (4.10) let us introduce the following coordinate substitution.*

$$\begin{cases} k = \frac{x_2}{x_1} \\ l = \frac{1-x_1-x_2}{x_1} \end{cases}, \quad i.e. \quad \begin{cases} x_1 = \frac{1}{1+k+l} \\ x_2 = \frac{k}{1+k+l} \end{cases}. \tag{4.13}$$

Then

$$\begin{cases} \dot{k} = \frac{\dot{x}_2 x_1 - x_2 \dot{x}_1}{x_1^2} = \frac{\beta\Theta k l}{(1-\beta)(1+k+l)} \\ \dot{l} = \frac{-\dot{x}_1 + x_2\dot{x}_1 - x_1\dot{x}_2}{x_1^2} = \frac{l(-\Psi k + \Theta l + \Theta + \Lambda)}{(1-\beta)(1+k+l)} \end{cases}.$$

Solution trajectories can be found by integrating

$$\frac{dl}{dk} = \frac{-\Psi k + \Theta l + \Theta + \Lambda}{\beta\Theta k}.$$

This can be done analytically to obtain

$$l\,(k) = k^{\frac{1}{\beta}} C + \left(\frac{(\Omega - \Lambda)}{\Theta\,(1-\beta)} - 1\right) k - \left(1 + \frac{\Lambda}{\Theta}\right),$$

where C is a constant that depends on the initial conditions. Finally, substituting the expressions for k and l (4.13) into the above formula, we find that the solution trajectories are described by the following expression.

$$\left(\frac{x_2}{x_1}\right)^{\frac{1}{\beta}} C + \frac{(\Omega - \Lambda)}{\Theta\,(1-\beta)} \frac{x_2}{x_1} - \frac{\Lambda}{\Theta} - \frac{1}{x_1} = 0 \qquad (4.14)$$

To obtain all solutions of the system (4.10) we should add the solutions

$$\begin{cases} x_1 = 0 \\ \forall x_2 \in [0,1] \end{cases}, \quad \textit{and} \quad \begin{cases} x_1 = \alpha \\ x_2 = 1 - \alpha \end{cases}, \qquad \alpha \in [0,1],$$

which were lost when the coordinate substitution (4.13) was performed.

Although the solution for the system (4.10) can be given by exact formula it is not possible to express the solution as a function $x_1\,(x_2)$ or $x_2\,(x_1)$ for generic C, which makes it inconvenient to use this formula for analysis of the solution behaviour. Below we describe the solutions of system (4.10) by applying methods from the qualitative theory of the dynamical systems (see [35], [53], [54]).

There are two isolated stationary points in the simplex Δ: $x_n = (0,0)$ and $x_s = \left(0, \frac{\Theta(1-\beta)}{\Omega-\Lambda}\right)$. Note that conditions (4.12) guarantee that $\frac{\Theta(1-\beta)}{\Omega-\Lambda} \in (0,1)$. There is also a set of non-isolated stationary points $\{(x_1, x_2) : x_1 + x_2 = 1\}$: each point $x_\alpha = (\alpha, 1-\alpha)$ with $\alpha \in [0,1]$ is a stationary point. Since all stationary points in the simplex Δ belong to the boundary of Δ, there are no limit cycles in the simplex Δ.

A standard linearisation about the stationary points produces the following results. The point x_n is a nodal attractive point with eigenvalues $\lambda_1^n = -\frac{\Theta}{1-\beta}$ and $\lambda_2^n = -\Theta$ which are negative due to (4.12). The eigenvalue λ_1^n corresponds to the eigenvector $(1,0)$ and the eigenvalue λ_2^n corresponds to the eigenvector $(0,1)$. The point x_s is a saddle point. Here the negative eigenvalue $\lambda_1^s = -\frac{\Psi\Theta\beta}{(\Omega-\Lambda)(1-\beta)}$ corresponds to the eigenvector $\left(1, \frac{\Lambda(1-\beta)}{\Omega-\Lambda}\right)$ and the positive eigenvalue $\lambda_2^s = \frac{\Psi\Theta}{\Omega-\Lambda}$ corresponds to the eigenvector $(0,1)$. The points $x_\alpha = (\alpha, 1-\alpha), \alpha \in [0,1]$ have the following eigenvalues.

1. A zero eigenvalue corresponding to the eigenvector $(-1,1)$.

2. An eigenvalue $\frac{\Theta\beta+\Omega}{1-\beta}\alpha - \frac{\Psi}{1-\beta}$ corresponding to the eigenvector $\left(1, -\frac{(1-\alpha)(\Psi-\alpha\Omega)}{\alpha(\alpha\Omega+\Lambda-\Omega+\Theta)}\right)$ if $\alpha \neq 0$ and $\alpha \neq -\frac{\Lambda-\Omega+\Theta}{\Omega}$ and to the eigenvector $(0,1)$ if $\alpha = 0$ or $\alpha = -\frac{\Lambda-\Omega+\Theta}{\Omega}$.

Defining $\alpha_0 = \frac{\Psi}{\Omega+\Theta\beta}$, which belongs to the interval $(0,1)$ by (4.12), we note that the second eigenvalue is positive for $\alpha > \alpha_0$, negative for $\alpha < \alpha_0$ and zero for $\alpha = \alpha_0$.

There are three other lines in the simplex Δ that are invariant under the dynamics of system (4.10). Two of these are boundary components of the simplex Δ: $\{(x_1, x_2) \in \Delta : x_1 = 0\}$ and $\{(x_1, x_2) \in \Delta : x_2 = 0\}$. The third line is

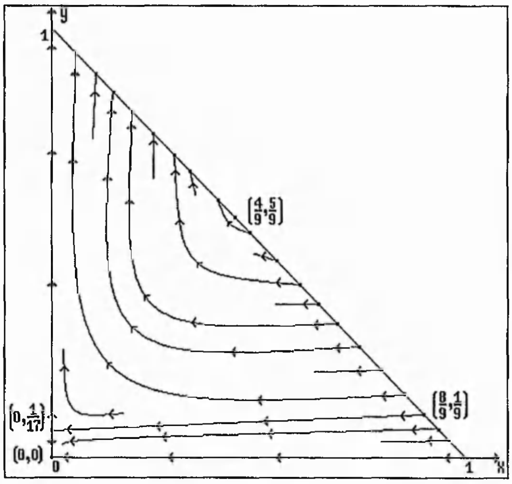
$$\left\{(x_1, x_2) \in \Delta : x_2 = \left(\frac{\Lambda(1-\beta)}{\Omega-\Lambda}\right) x_1 + \frac{\Theta(1-\beta)}{\Omega-\Lambda}\right\}$$

which intersects the boundary of the simplex at x_s and at $x_c = (c, 1-c)$ with $c = \frac{\Psi}{\Omega-\beta\Lambda}$. Note that conditions (4.12) guarantee $c \in (0,1)$. This line splits the simplex Δ into two regions: solutions in each region terminate at different sets.

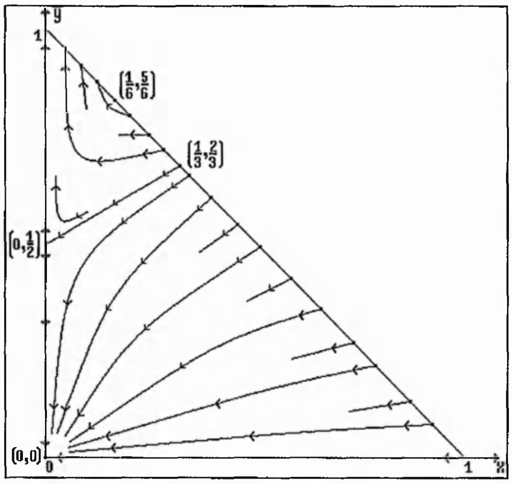
Using these results we can now draw a qualitative picture of the solutions for the dynamical system (4.10) given any set of parameters h_i $(i = 1,2,3,4)$ and β. In figure 4.1 we give four examples for different sets of parameter values. These sorts of pictures have been obtained in the past using computer simulation [15]. Here we have obtained them using only analytic techniques. Information about the parameters chosen is summarised in table 4.1.

Figure 4.1. Qualitative sketches of the dynamics of the Iterated Prisoners' Dilemma system.

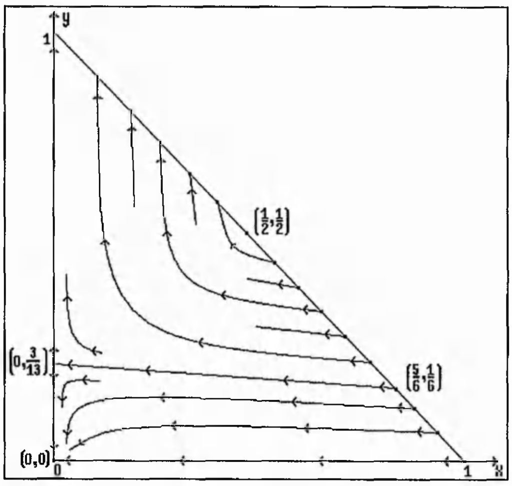
Examples 4.1 and 4.2 are related to the standard set of payoffs for the Iterated Prisoners' Dilemma, but two different values of the discount factor. The two other examples demonstrate how the dynamic changes depending on the values of the payoffs (see table 4.1 below for the precise parameter values).



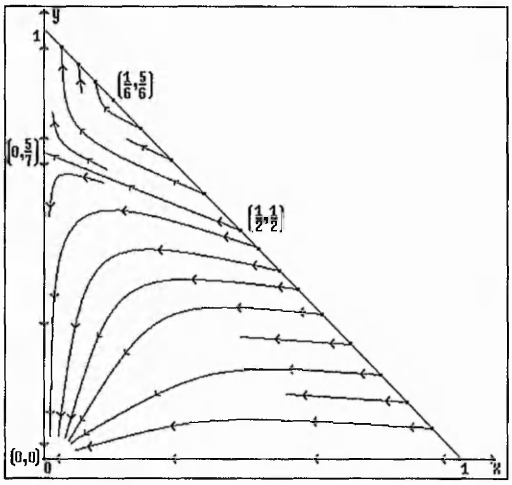
Example 4.1.



Example 4.2.



Example 4.3.



Example 4.4.

Table 4.1 Parameters of the Iterated Prisoners Dilemma dynamics.

example\parameter	h_1	h_2	h_3	h_4	β	Λ	Ω	Θ	Ψ
example 4.1:	3	0	5	1	$\frac{9}{10}$	1	$\frac{27}{10}$	1	$\frac{8}{5}$
example 4.2:	3	0	5	1	$\frac{3}{5}$	1	$\frac{9}{5}$	1	$\frac{2}{5}$
example 4.3:	3	0	4	$\frac{3}{2}$	$\frac{4}{5}$	$-\frac{1}{2}$	$\frac{4}{5}$	$\frac{3}{2}$	1
example 4.4:	3	0	4	$\frac{5}{2}$	$\frac{4}{5}$	$-\frac{3}{2}$	$-\frac{4}{5}$	$\frac{5}{2}$	$\frac{1}{5}$

Discussion. As we can see from figure 4.1 if the solution trajectories start at the region below the separatrix line they are attracted to the point $(0,0)$, which corresponds to a population of players who use the unconditional defection strategy. If the solution trajectories start at the region above the separatrix line they are attracted to a point $(\alpha, 1-\alpha)$ with $\alpha \in [0, \alpha_0)$ for some α_0 depending on h_i and β. The second set corresponds to populations which consist of various mixtures of unconditional cooperators and Tit for Tat players. This set demonstrates that cooperative behaviour may be an evolutionary outcome in the Iterated Prisoners' Dilemma. However, neither any point in the set nor the set as a whole is asymptotically or evolutionarily stable. In the next section we propose a new concept that will allow us to analyse such sets.

4.2 Setwise evolutionary attraction.

4.2.1 Definition.

To analyse the solution's behaviour in the neighbourhood of sets of non-isolated stationary points I prove the following useful theorem. This theorem generalises the direct Lyapounov method (see [35], [42] and [55]) for sets; the main difference of this theorem from standard results is that the set under the consideration is not assumed to be closed.

Theorem 4.2 *Let $\xi(t,x)$ be the solution trajectory which passes through the point x at time $t=0$. Let $\mathfrak{S}$ be a set of points in simplex Δ and suppose that for each point $x^0 \in \mathfrak{S}$ there exists a neighbourhood W_{x^0} in Δ and a continuous function $H_{x^0}(x)$ such that*

1. $H_{x^0}(x) \geq 0$ *for any* $x \in W_{x^0}$ *and* $H_{x^0}(x) = 0$ *if and only if* $x = x^0$*;*

2. *$H_{x^0}(\xi(t,x)) < H_{x^0}(x)$ for any $x \in W_{x^0}$ if $x \notin \mathfrak{S}, t > 0$, and $\xi(s,x) \in W_{x^0}$ for any $s \in [0,t]$.*

3. *$\dot{H}_{x^0}(x) = 0$ for some $x \in W_{x^0}$ if and only if $x \in \mathfrak{S}$.*

Then for every point $x^0 \in \mathfrak{S}$ and every neighbourhood U_{x^0} in Δ of the point x^0 there is a neighbourhood V_{x^0} in Δ such that for each $x \in V_{x^0}$ the following conditions hold: $\xi(t,x) \in U_{x^0} \cap \Delta$ for any $t \in [0,\infty)$ and there exists $\tilde{x}^0 \in \mathfrak{S} \cap U_{x^0}$ such that $\lim_{t\to\infty} \xi(t,x) = \tilde{x}^0$.

Proof. For each point $x^0 \in \mathfrak{S}$ consider the neighbourhoods W_{x^0} and U_{x^0} and find a neighbourhood B_{x^0} in Δ such that its boundary $\partial B_{x^0} \subset W_{x^0} \cap U_{x^0}$. Since ∂B_{x^0} is compact and the function H_{x^0} is continuous we can find $\min_{x\in\partial B_{x^0}} H_{x^0}(x) = h_{x^0} > 0$. For each point $x^0 \in \mathfrak{S}$ and for each h, $0 < h \leq h_{x^0}$, define a neighbourhood $V_h(x^0) = \{x \in B_{x^0} : H_{x^0}(x) < h\}$. Then

1. $V_h(x^0) \subset B_{x^0} \subset W_{x^0} \cap U_{x^0} \subset U_{x^0}$;

2. $V_{h_1}(x^0) \subset V_{h_2}(x^0)$, if $h_1 \leq h_2$;

3. neighbourhoods V_h, $0 < h \leq h_{x^0}$, are forward invariant. (The proof of this fact can be found in [35], in proof of theorem 6.2, p. 246.)

Let $V_{x^0} = V_{h_{x^0}}(x^0)$. Fix point $x^0 \in \mathfrak{S}$ and take $x \in V_{x^0} \backslash \mathfrak{S}$. Consider the $\omega - limit$, $\omega(x)$, of the solution $\xi(t,x)$:

$$\omega(x) = \left\{ x' \in \Delta : \exists t_k \underset{k\to+\infty}{\to} +\infty \text{ such that } \lim_{k\to+\infty} \xi(t_k,x) = x' \right\}.$$

Since V_{x^0} is compact and forward invariant, $\omega(x)$ is not empty. Take some $\tilde{x}^0 \in \omega(x)$. It has been shown in [55] (theorem 2.6.1, p. 19) that if $\tilde{x}^0 \in \omega(x)$ then $\dot{H}_{x^0}(\tilde{x}^0) = 0$ and therefore $\tilde{x}^0 \in \mathfrak{S}$ by condition 3.

Let us show that $\lim_{t\to\infty} \xi(t,x) = \tilde{x}^0$. This means that we need to show that for any $\varepsilon > 0$ there exists $T > 0$ such that for any $t > T$

$$\left|\xi(t,x) - \tilde{x}^0\right| < \varepsilon.$$

Denote $B_\varepsilon(\tilde{x}^0) = \left\{\widehat{x} \in \Delta : \left|\widehat{x} - \tilde{x}^0\right| < \varepsilon\right\}$ and choose $h \leq h_{\tilde{x}^0}$ such that $V_h(\tilde{x}^0) \subset B_\varepsilon(\tilde{x}^0)$. Since $\tilde{x}^0 \in \omega(x)$ there exists some sequence $t_k \underset{k\to+\infty}{\to} +\infty$ such that $\lim_{k\to+\infty} \xi(t_k,x) = \tilde{x}^0$. Then for any $\epsilon > 0$ there is $K \in N$ such that for any $k \leq K$

$$\left|\xi(t_k,x) - \tilde{x}^0\right| < \epsilon.$$

Therefore $\xi(t_k, x) \in B_\epsilon(\tilde{x}^0)$ for any $k \leq K$. Choose ϵ such that $B_\epsilon(\tilde{x}^0) \subset V_h(\tilde{x}^0) \subset B_\varepsilon(\tilde{x}^0)$. (This is possible because $V_h(\tilde{x}^0)$ is open.) Then $\xi(t_k, x) \in V_h(\tilde{x}^0)$ for any $t \geq t_K$ and, since $V_h(\tilde{x}^0)$ is forward invariant, $\xi(t, x) \in V_h(\tilde{x}^0)$ for any $t \geq t_K$. Hence $\xi(t, x) \in B_\varepsilon(\tilde{x}^0)$ for any $t \geq t_K$. The theorem is proven.

In chapter 1 we have discussed the notions of evolutionarily stable and asymptotically stable sets. The following proposition illustrates that conditions of theorem 4.2 are necessary in order for a set to be evolutionarily stable.

Proposition 4.3 *If set $\mathfrak{S}$ is evolutionarily stable then conditions of theorem 4.2 hold for this set.*

Proof. If a set $\mathfrak{S}$ is evolutionarily stable when each $x^0 \in \mathfrak{S}$ has a neighbourhood V_{x^0} such that $f_{x^0}(x) = (x^0 - x) A x^T > 0$ for all strategies $x \in V_{x^0} \backslash \mathfrak{S}$. Here A is a payoff matrix for the corresponding two-player game. Consider the Relative-Entropy Function

$$H_{x^0}(x) = \sum_{i=1}^{n} x_i^0 \log\left(\frac{x_i^0}{x_i}\right).$$

It is well known ([35], lemma 3.1, p. 98.) that there exists a neighbourhood in Δ such that condition 1 of theorem 4.2 is satisfied. Notice that since the derivative of the $H_{x^0}(x)$ is negative

$$\dot{H}_{x^0}(x) = -\left(x^0 - x\right) A x^T = -f_{x^0}(x) < 0,$$

the conditions 2 and 3 of theorem 4.2 follow. (For explanation of this fact see, for example, [35], p. 247).

The following proposition shows that conditions of theorem 4.2 are sufficient in order for a set to be asymptotically stable.

Proposition 4.4 *If conditions of theorem 4.2 hold for a closed set $\mathfrak{S}$ then this set is asymptotically stable.*

Proof. It is necessary to show that every neighbourhood B of $\mathfrak{S}$ contains a neighbourhood B^0 of $\mathfrak{S}$ such that for any $x \in B^0$ $\xi(t, x) \in B$ $\forall t > 0$ and there exists neighbourhood B^* of $\mathfrak{S}$ such that $\xi(t, x) \longrightarrow \mathfrak{S}$ for all $x \in B^*$. If conditions of theorem 4.2 hold then, since any neighbourhood of a set can be considered as a neighbourhood of any point in that set, we have that for every point $x^0 \in \mathfrak{S}$ there is a neighbourhood V_{x^0} in Δ such that for each

$x \in V_{x^0}$ the following conditions hold: $\xi(t,x) \in B \cap \Delta$ for any $t \in [0,\infty)$ and there exists $\tilde{x}^0 \in \mathfrak{S} \cap B$ such that $\lim_{t\to\infty} \xi(t,x) = \tilde{x}^0$. Let us take the union of neighbourhoods V_{x^0} there $x^0 \in \mathfrak{S}$. We obtain the neighbourhood $V = \bigcup_{x^0 \in \mathfrak{S}} V_{x^0}$. Now we can choose $B^0 = B^* = V$ and we obtain that the set $\mathfrak{S}$ is asymptotically stable.

Remark 4.4 *Even if the conditions of theorem 4.2 hold for an open set $\mathfrak{S}$ this set fails to be asymptotically stable (since by definition a set is required to be closed in order to be asymptotically stable). For example the cooperative behaviour set in the $\omega-$limit of the Iterated Prisoners' Dilemma is an open set and, therefore, fails to be asymptotically stable. The closure of this set also fails to be asymptotically stable. However, the pictures in figure 4.1 indicate that this failure may be "mild" in the sense that there is a neighbourhood of the set for which trajectories starting from almost all points in that neighbourhood converge to the set. In the Iterated Prisoners' Dilemma example it is only the points on the line $x_1 + x_2 = 1$ which do not lie on such trajectories, but they are neutrally stable and do not lie on trajectories leading away from the set either.*

This remark motivates us to give the following definition.

Definition 4.1 *Let $\xi(t,x)$ be the solution trajectory which passes through the point x at time $t = 0$. Then we call a closed set of stationary points, $\bar{\mathfrak{S}} \in \Delta$, evolutionarily attractive if every neighbourhood U of $\bar{\mathfrak{S}}$ contains a neighbourhood V such that for each $x \in V \cap \Delta$ one of the following conditions holds. Either*

- *$\xi(t,x) \in U \cap \Delta$ for any $t \in [0,\infty)$ and there exists a $\tilde{x}^0 \in \bar{\mathfrak{S}} \cap U$ such that $\lim_{t\to\infty} \xi(t,x) = \tilde{x}^0$,*
- *or $\xi(t,x) = x$ for any $t \in [0,\infty)$.*

Remark 4.5 *Definition 4.1 is satisfied for any asymptotically stable state or set.*

Remark 4.6 *Setwise Lyapounov stability is a necessary but not sufficient condition for definition 4.1 to hold for a particular set.*

Remark 4.7 *The set of points satisfying the second condition in definition 4.1 is either the whole of the (restricted) neighbourhood $U \cap \Delta$ or a subset W with $\dim(W) \leq \dim(U \cap \Delta) - 1$ [56].*

4.2.2 Example: the Iterated Prisoners' Dilemma game.

In this section we show that the set of cooperative behaviours in the Iterated Prisoners' Dilemma is an evolutionarily attractive set.

Let us consider again the dynamics for the Iterated Prisoners' Dilemma game. It is well-known that the only asymptotically stable set or point in the Iterated Prisoners' Dilemma is the state $x_n = (0,0)$ where every member of the population is using the strategy of unconditional defection, σ_D. The population states $x_\alpha = (\alpha, 1-\alpha)$ which are related to the cooperative Nash equilibria $[\sigma_\alpha, \sigma_\alpha]$ fail to be asymptotically stable, either individually or as a set. However, the preceding analysis of this game (see section 4.1.3) makes it intuitively obvious that definition 4.1 is satisfied for the set

$$\bar{\mathfrak{S}} = \{x_\alpha = (\alpha, 1-\alpha) : \alpha \in [0, \alpha_0]\},$$

where $\alpha_0 = \frac{\Psi}{\Omega + \Theta\beta}$.

Here we give a proof of this stability property. The open set of stationary points and the doubly degenerate end point are considered separately.

Lemma 4.5 *Let* $\mathfrak{S} = \{x_\alpha = (\alpha, 1-\alpha) : \alpha \in [0, \alpha_0)\}$. *For each point* $x_\alpha \in \mathfrak{S}$ *there exists a neighbourhood* W_{x_α} *in* Δ *such that for the function*

$$H_{x_\alpha}(x) = \alpha \log \frac{\alpha}{x_1} + (1-\alpha) \log \frac{(1-\alpha)}{x_2}$$

and for each population state $x \in W_{x_\alpha} \cap \Delta$ *the following conditions hold:*

1. $H_{x_\alpha}(x) \geq 0$ *and* $H_{x_\alpha}(x) = 0$ *if and only if* $x = x_\alpha$;

2. $H_{x_\alpha}(\xi(t,x)) < H_{x_\alpha}(x)$ *if* $x \notin \{x_\alpha : \alpha \in [0,1]\}, t > 0$, *and* $\xi(s,x) \in W_{x_\alpha}$ *for any* $s \in [0,t]$.

3. $\dot{H}_{x_\alpha}(x) = 0$ *for some* $x \in W_{x_\alpha}$ *if and only if* $x \in \mathfrak{S}$.

Proof. The proof of the first statement of the lemma is well known (see, for example, [35], p. 98). To verify the second and the third conditions let us calculate $\dot{H}_{x_\alpha}(x)$. It is easy to show that

$$\dot{H}_{x_\alpha}(x) = (1 - x_1 - x_2) \frac{\Lambda x_1 + (\Lambda - \Omega) x_2 + \beta\Theta\alpha + (1-\beta)\Theta}{1-\beta}.$$

Since

$$\left.\frac{\Lambda x_1 + (\Lambda - \Omega)\, x_2 + \beta\Theta\alpha + (1-\beta)\,\Theta}{1-\beta}\right|_{(\alpha,1-\alpha)} = \frac{(\Omega + \Theta\beta)\,\alpha - \Psi}{1-\beta} < 0$$

for any $x_\alpha \in \mathfrak{S}$, it is possible to choose a neighbourhood W_{x_α} in Δ such that $\dot{H}_{x_\alpha}(x) < 0$ on $W_{x_\alpha} \setminus \mathfrak{S}$ and $\dot{H}_{x_\alpha} = 0$ on $\mathfrak{S}$. This is then equivalent to condition 2 (see [35], p. 247).

Remark 4.8 *Since*

$$f_{x_{\alpha_0}}(x) = (x_{\alpha_0} - x)\, A\, x^T = -\dot{H}_{x_{\alpha_0}}(x)$$

and

$$\left.\frac{\Lambda x_1 + (\Lambda - \Omega)\, x_2 + \beta\Theta\alpha_0 + (1-\beta)\,\Theta}{1-\beta}\right|_{(\alpha_0,1-\alpha_0)} = \frac{(\Omega + \Theta\beta)\,\alpha_0 - \Psi}{1-\beta} = 0$$

it is not possible to find a neighbourhood $W_{x_{\alpha_0}}$ in Δ such that $\dot{H}_{x_{\alpha_0}}(x) < 0$ on $W_{x_{\alpha_0}} \setminus \bar{\mathfrak{S}}$. Therefore $\bar{\mathfrak{S}}$ is not evolutionarily stable.

Proposition 4.6 *For each point $x_\alpha \in \mathfrak{S}$ and any neighbourhood U_{x_α} of x_α in Δ there exists a neighbourhood V_{x_α} such that for each $x \in V_{x_\alpha} \cap \Delta$ the following condition hold: $\xi(t,x) \in U_{x_\alpha} \cap \Delta$ for any $t \in [0,\infty)$ and there exists $\tilde{x}_\alpha \in \mathfrak{S} \cap U_{x_\alpha}$ such that $\lim_{t\to\infty} \xi(t,x) = \tilde{x}_\alpha$.*

Proof. The proposition follows directly from lemma 4.5 and theorem 4.2.

Now we deal with the doubly degenerate end point $x_{\alpha_0} = (\alpha_0, 1-\alpha_0)$.

Proposition 4.7 *For point $x_{\alpha_0} = \left(\frac{\Psi}{\Omega+\Theta\beta}, 1 - \frac{\Psi}{\Omega+\Theta\beta}\right)$ and any neighbourhood $U_{x_{\alpha_0}}$ of x_{α_0} in Δ there exists a neighbourhood $V_{x_{\alpha_0}}$ such that for each $x \in V_{x_{\alpha_0}} \cap \Delta$ one of the following conditions holds. Either*

- *$\xi(t,x) \in U_{x_{\alpha_0}} \cap \Delta$ for any $t \in [0,\infty)$ and there exists $\tilde{x}_{\alpha_0}(x) \in \mathfrak{S} \cap U_{x_{\alpha_0}}$ such that $\lim_{t\to\infty} \xi(t,x) = \tilde{x}_{\alpha_0}(x)$, or*
- *$\xi(t,x) = x$ for any $t \in [0,\infty)$.*

Proof. Note that, since

$$\left(\Lambda x_1 + (\Lambda - \Omega)\, x_2 + \Theta\right)|_{(\alpha_0,1-\alpha_0)} = \frac{\beta\Theta(\Lambda+\Theta)}{\Omega+\beta\Theta} > 0$$

and

$$\left(\Lambda x_1 + (\Lambda - \Omega)\, x_2 + \Theta\,(1-\beta)\right)|_{(\alpha_0,1-\alpha_0)} = \frac{-\beta\Theta\Psi}{\Omega+\beta\Theta} < 0$$

there exists neighbourhood $Q_{x_{\alpha_0}}$ in Δ of the point x_{α_0} such that $\dot{x}_1 < 0$ and $\dot{x}_2 > 0$ on $Q_{x_{\alpha_0}} \backslash \partial\Delta$. Take the intersection $Q_{x_{\alpha_0}} \cap U_{x_{\alpha_0}} \cap \Delta$ and find an $\varepsilon > 0$ such that the neighbourhood $B_\varepsilon = \{x \in \Delta : |x - x_{\alpha_0}| < \varepsilon\}$ belongs to $Q_{x_{\alpha_0}} \cap U_{x_{\alpha_0}} \cap \Delta$. Define the neighbourhood

$$\widetilde{V}_{x_{\alpha_0}} = \left\{x = (x_1, x_2) : \left\{x_2 > 1 - \frac{\Psi}{\Omega + \Theta\beta} - \frac{\varepsilon}{2}\right\} \cap \left\{x_1 > \frac{\Psi}{\Omega + \Theta\beta} - \frac{\varepsilon}{2}\right\} \cap \{x_1 + x_2 \leq 1\}\right\}.$$

Consider the point $x_{\alpha_0 - \frac{\varepsilon}{2}}$. This point has a negative eigenvalue, hence there exists a solution trajectory ξ approaching this point as $t \longrightarrow +\infty$. Consider the dynamics when $t \longrightarrow -\infty$. Then x_1 is growing along ξ and x_2 is decreasing. Therefore, using the Poincaré-Bendixson theorem we have two possibilities for the trajectory ξ : it either terminates at some point x_α, with $\alpha \in \left(\alpha_0, \alpha_0 + \frac{\varepsilon}{2}\right]$, or it crosses the interval

$$\left\{x = (x_1, x_2) : \left\{x_2 = 1 - \tfrac{\Psi}{\Omega+\Theta\beta} - \tfrac{\varepsilon}{2}\right\} \cap \left\{\tfrac{\Psi}{\Omega+\Theta\beta} - \tfrac{\varepsilon}{2} \leq x_1 \leq \tfrac{\Psi}{\Omega+\Theta\beta} + \tfrac{\varepsilon}{2}\right\}\right\}.$$

In both of these cases ξ splits $\widetilde{V}_{x_{\alpha_0}}$ into two regions. The region that contains point x_{α_0} is a forward invariant neighbourhood. We denote it as $V_{x_{\alpha_0}}$. Then any trajectory $\xi(t, x)$ such that $x \in V_{x_{\alpha_0}}$ will remain in $V_{x_{\alpha_0}}$ for any moment of time t. Taking into account that there are no limit cycles in the interior of Δ and applying the Poincaré-Bendixson theorem, we can conclude that the trajectory $\xi(t, x)$ terminates at some stationary point in $V_{x_{\alpha_0}}$. Since the only attractive stationary points in $V_{x_{\alpha_0}}$ are from the set $\mathfrak{S}$ we obtain proposition 4.7.

Proposition 4.8 *The set* $\bar{\mathfrak{S}} = \{x_\alpha = (\alpha, 1 - \alpha) : \alpha \in [0, \alpha_0]\}$, *where* $\alpha_0 = \frac{\Psi}{\Omega+\Theta\beta}$ *is evolutionarily attractive.*

Proof. It is necessary to show that every neighbourhood U of $\bar{\mathfrak{S}}$ contains a neighbourhood V such that for each $x \in V \cap \Delta$ the conditions of definition 4.1 hold. Since any neighbourhood of a set can be considered as a neighbourhood of any point in that set, we have that it follows from propositions 4.6 and 4.7 that for every point $x_\alpha \in \bar{\mathfrak{S}}$ there is a neighbourhood V_{x_α} in Δ such that conditions of definition 4.1 hold. If we take the union of neighbourhoods V_{x_α} there $x_\alpha \in \bar{\mathfrak{S}}$ we obtain the necessary neighbourhood as following $V = \bigcup_{\alpha \in [0,\alpha_0]} V_{x_\alpha}$. The proposition is proven.

The above analysis shows that populations composed of cooperative individuals may be the end point of the evolutionary dynamics. Although these populations are neither evolutionarily stable in the sense of [13] nor asymptotically stable points from a dynamical systems perspective, the failure of stability is minor. Small deviations from cooperative populations typically lead to evolutionary trajectories which take the system back to a (possibly different) cooperative population. Only a negligible set of deviations lead to neutrally stable populations, and no deviations lead to trajectories which diverge from cooperative behaviour.

4.2.3 Example: Entry Deterrence Game.

We will now consider a game in which there exists a set of non-isolated stationary points for which the conditions of theorem 4.2 are satisfied but that set is not evolutionarily attractive in the sense of definition 4.1. Consider a two player game with payoffs given by the following matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Denote

$$a_1 = a_{11} - a_{21}, \qquad a_2 = a_{22} - a_{12},$$
$$b_1 = b_{11} - b_{12}, \qquad b_2 = b_{22} - b_{21}.$$

The standard (multi-population) Replicator Dynamics (see, for example, [35]) for this game is the following system

$$\begin{cases} \dot{x}_1 = \frac{dx_1}{dt} = ((a_1 + a_2)\, x_2 - a_2)\, x_1\, (1 - x_1) \\ \dot{x}_2 = \frac{dx_2}{dt} = ((b_1 + b_2)\, x_1 - b_2)\, x_2\, (1 - x_2) \end{cases}$$

Solution trajectories can be found by integrating

$$\frac{dx_1}{dx_2} = \frac{((a_1+a_2)x_2-a_2)x_1(1-x_1)}{((b_1+b_2)x_1-b_2)x_2(1-x_2)}.$$

This can be done analytically to obtain

$$x_1^{b_2}\, (1 - x_1)^{b_1} = C x_2^{a_2}\, (1 - x_2)^{a_1}\,,$$

where C is a constant that depends on the initial conditions.

Let us choose

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 6 & 6 \end{bmatrix}, \qquad \begin{matrix} a_1 = 2, & a_2 = 1, \\ b_1 = 3, & b_2 = 0. \end{matrix}$$

This game can be represented by the bi-matrix given in the following table

		Player 2	
		Y	F
Player 1	E	3,3	0,0
	A	1,6	1,6

and can be interpreted as an entry deterrence model (see [35] and [57]). Here player 1 can be interpreted as a potential competitor in the second player's market. Player 2 earns its highest payoff of 6 if player 1 stays out. If player 1 decides to enter the market then player 2 has two choices: either to yield (share the market) or to fight.

This game has an isolated Nash equilibrium $[E, Y]$ and a set of Nash equilibria $[A, \sigma_\alpha]$ where σ_α is any mixed strategy which plays Y with probability less than or equal to one third. The standard two-population Replicator Dynamics for this game is

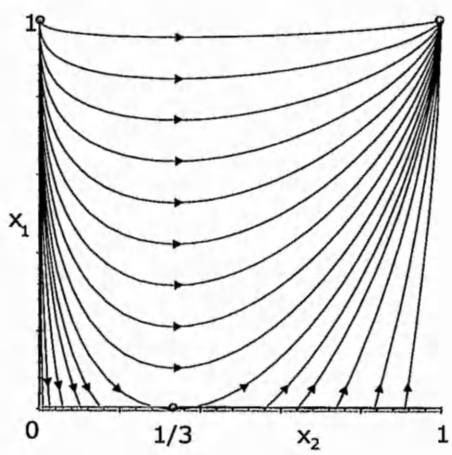
$$\begin{cases} \dot{x}_1 = x_1(1 - x_1)(3x_2 - 1) \\ \dot{x}^2 = 3x_1x_2(1 - x_2) \end{cases}$$

where x_1 is the proportion of E-players and x_2 is the proportion of Y-players. The solutions of this game are

$$x_1 = 1 - Cx_2^{\frac{1}{3}} (1 - x_2)^{\frac{2}{3}}.$$

A sketch of the solutions for a few values of the constant C is given in figure 4.2 below.

Figure 4.2. Qualitative sketch of the dynamics for the entry deterrence game.



From this figure it is immediately apparent that definition 4.1 is satisfied only for the isolated fixed point. It can be shown that for the fixed points in the set $\mathfrak{K} = \{x^\alpha = (0, \alpha) : 0 \leq \alpha < \frac{1}{3}\}$ the conditions of theorem 4.2 are satisfied. In this sense the set is similar to the set $\mathfrak{S}$ in the Iterated Prisoner's Dilemma game. However, $\bar{\mathfrak{K}}$ is not evolutionarily attractive: since $\dot{x}_2 > 0$ for $x_1 > 0$, some small deviations from the end-point of the set lead to trajectories which diverge from the set.

4.3 Resolving singularities: the $\sigma - process$.

In this section we illustrate a technique, known either as "blowing up" or "the sigma process", which can be used to determine the stability properties of non-hyperbolic fixed points in the Replicator Dynamics. Such points often appear in the Replicator Dynamics when the "pairwise contest" of repeated games is represented by a game in a normal form. The payoff bi-matrix for such games is commonly non-generic (see, for example, the payoffs for the Iterated Prisoners' Dilemma given in (4.7)). We illustrate the use of "blowing up" technique by considering a Replicator Dynamics system which is qualitatively similar to the dynamics for model of social interactions obtained by using multi-state games (considered in section 7.1).

4.3.1 The standard Replicator Dynamics.

We consider a symmetric two-person game with the following payoff bi-matrix.

$P_1 \backslash P_2$	s_1	s_2	s_3	s_4
s_1	11, 11	5, 10	4, 8	6, 6
s_2	10, 5	9, 9	3, 7	6, 6
s_3	8, 4	7, 3	2, 2	6, 6
s_4	6, 6	6, 6	6, 6	6, 6

(4.15)

These payoffs could be viewed as having come from a stochastic or repeated game. However, this interpretation is only incidental in this section; the important point is that the resulting normal form game is non-generic.

There are four Nash Equilibria for this game:

(i) both players use pure strategy s_1;

(ii) both players use pure strategy s_2;

(iii) both players use pure strategy s_4;

(iv) a mixed strategy equilibrium in which each player chooses s_1 with probability $\frac{4}{5}$ and s_2 with probability $\frac{1}{5}$.

Let x_i be the proportion of individuals who adopt strategies s_i for $i = 1,2,3,4$. The standard Replicator Dynamics (Taylor and Jonker [28]) gives the dynamics of the i^{th} component of the population state as

$$\dot{x}_i = x_i \left(\left(AX^T\right)_i - XAX^T \right) \ , \quad i = 1,2,3,4. \tag{4.16}$$

where

$$A = \begin{bmatrix} 11 & 5 & 4 & 6 \\ 10 & 9 & 3 & 6 \\ 8 & 7 & 2 & 6 \\ 6 & 6 & 6 & 6 \end{bmatrix} \quad \text{and} \quad X = (x_1, x_2, x_3, x_4)\,.$$

By incorporating the constraint $x_4 = 1 - x_1 - x_2 - x_3$ we can reduce the system of equations to one describing the evolution of a point $x = (x_1, x_2, x_3)$ in the domain

$$\Delta = \left\{ (x_1, x_2, x_3) : \bigcap_{i=1,2,3} (x_i \geq 0) \bigcap ((x_1 + x_2 + x_3) \leq 1) \right\}. \tag{4.17}$$

Equations (4.16) become

$$\dot{x}_i = G_i \left(x_1, x_2, x_3\right) \tag{4.18}$$

with

$$\begin{aligned} G_1 \left(x_1, x_2, x_3\right) &= x_1 \left(5x_1 - x_2 - 2x_3 - 5x_1^2 - 3x_2^2 + 4x_3^2 - 3x_1x_2 + 2x_2x_3\right); \\ G_2 \left(x_1, x_2, x_3\right) &= x_2 \left(4x_1 + 3x_2 - 3x_3 - 5x_1^2 - 3x_2^2 + 4x_3^2 - 3x_1x_2 + 2x_2x_3\right); \\ G_3 \left(x_1, x_2, x_3\right) &= x_3 \left(2x_1 + x_2 - 4x_3 - 5x_1^2 - 3x_2^2 + 4x_3^2 - 3x_1x_2 + 2x_2x_3\right). \end{aligned}$$

There are five stationary points for the system (4.18). The coordinates of these points are given in table 4.2 below together with the results of a standard linearisation analysis in the neighbourhood of each stationary point.

Table 4.2. The eigenvectors and associated eigenvalues for each of the stationary point of RD system (4.18) as determined by local linearisation.

point	eigenvectors and eigenvalues
$(0,0,0)$	$[1,0,0] \leftrightarrow 0, \quad [0,1,0] \leftrightarrow 0, \quad [0,0,1] \leftrightarrow 0.$
$(1,0,0)$	$[1,0,0] \leftrightarrow -5, \quad [-1,1,0] \leftrightarrow -1, \quad [-1,0,1] \leftrightarrow -3.$
$(0,1,0)$	$[0,1,0] \leftrightarrow -3, \quad [-1,1,0] \leftrightarrow -4, \quad [0,-1,1] \leftrightarrow -2.$
$(0,0,1)$	$[0,0,1] \leftrightarrow 4, \quad [-1,0,1] \leftrightarrow 2, \quad [0,-1,1] \leftrightarrow 1.$
$(\frac{4}{5},\frac{1}{5},0)$	$[4,1,0] \leftrightarrow -\frac{19}{5}, \quad [6,1,-7] \leftrightarrow -2, \quad [-1,1,0] \leftrightarrow \frac{4}{5}.$

From the analysis of the eigenvalues of each point we can conclude that $(1,0,0)$ and $(0,1,0)$ are attractive nodal points, $(0,0,1)$ is a repulsive nodal point and $(\frac{4}{5},\frac{1}{5},0)$ is a saddle point. The stability properties of the point $(0,0,0)$ are indeterminate.

4.3.2 Coordinate transformations.

To describe the dynamics at the non-hyperbolic fixed point $(0,0,0)$ we use three singular coordinate transformations. This procedure is called blowing up or the $\sigma - process$ (see, for example, Arrowsmith and Place [54], p.102). We introduce three new coordinate systems U, V and W defined as follows.

$$\begin{aligned} U &: \left\{u_1 = x_1, u_2 = \frac{x_2}{x_1}, u_3 = \frac{x_3}{x_1}\right\}; \\ V &: \left\{v_1 = \frac{x_1}{x_2}, v_2 = x_2, v_3 = \frac{x_3}{x_2}\right\}; \\ W &: \left\{w_1 = \frac{x_1}{x_3}, w_2 = \frac{x_2}{x_3}, w_3 = x_3\right\}. \end{aligned} \tag{4.19}$$

In these coordinate systems the point $(0,0,0)$ in the (x_1,x_2,x_3) coordinates corresponds to the planes $u_1 = 0, v_2 = 0$ and $w_3 = 0$ respectively.

Remark 4.9 *The geometric interpretation of the coordinate changes (4.19) is that the point* $(0,0,0)$ *is transformed into the projective plane* RP^2 *consisting of all the directions of lines passing through this point (see e.g. Harris [58], p. 81). But, since we only considering dynamics restricted to the domain* Δ *given in (4.17), we can think that the point* $(0,0,0)$ *is transformed into the positive octant of a sphere.*

To find the behaviour of solutions near the point $(0,0,0)$ we analyse the dynamical system using each of the new coordinate systems in turn.

In the coordinate system U, the dynamics (4.18) has the form

$$\begin{cases} \dot{u}_1 = -u_1^2\left(-5+u_2+2u_3+5u_1+3u_1u_2^2-4u_1u_3^2+3u_1u_2-2u_1u_2u_3\right) \\ \dot{u}_2 = u_1u_2\left(-1+4u_2-u_3\right) \\ \dot{u}_3 = u_1u_3\left(-3+2u_2-2u_3\right) \end{cases}.$$

In order to remove the degeneracy, we make the time substitution $d\tau = |u_1|\,dt$, and obtain the following topologically equivalent system

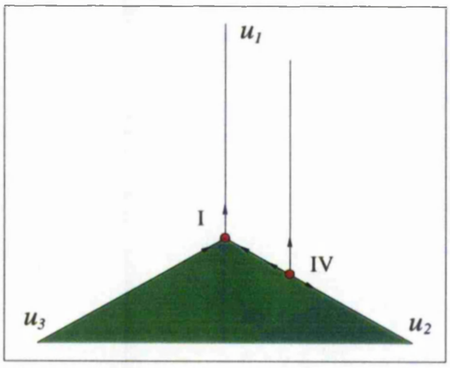
$$\begin{cases} \dot{u}_1 = -u_1\left(-5+u_2+2u_3+5u_1+3u_1u_2^2-4u_1u_3^2+3u_1u_2-2u_1u_2u_3\right) \\ \dot{u}_2 = u_2\left(-1+4u_2-u_3\right) \\ \dot{u}_3 = u_3\left(-3+2u_2-2u_3\right) \end{cases}.$$

The stationary points for this system that are on the plane $u_1 = 0$ and also in the domain Δ (4.17) are given below.

point	eigenvectors and eigenvalues
$\mathrm{I} = (0,0,0)$	$[1,0,0] \leftrightarrow 5,\quad [0,1,0] \leftrightarrow -1,\quad [0,0,1] \leftrightarrow -3.$
$\mathrm{IV} = \left(0,\frac{1}{4},0\right)$	$[1,0,0] \leftrightarrow \frac{19}{4},\quad [0,1,0] \leftrightarrow 1,\quad [0,1,14] \leftrightarrow -\frac{5}{2}.$

Thus we obtain the picture for the dynamics in the coordinates U given in figure 4.3 below.

Figure 4.3. The dynamics near the point $(0,0,0)$ in the coordinates U.
The third eigenvector for point IV has been ignored because it lies
in the plane $u_1 = 0$ (which corresponds, in its entirety,
to the point $(0,0,0)$ in the original coordinates.)



In the coordinates V we have the following dynamical system.

$$\begin{cases} \dot{v}_1 = v_1\left(v_1 - 4 + v_3\right) \\ \dot{v}_2 = v_2\left(4v_1 + 3 - 3v_3 - 5v_1^2 v_2 - 3v_2 + 4v_2 v_3^2 - 3v_1 v_2 + 2v_2 v_3\right) \\ \dot{v}_3 = -v_3\left(2v_1 + 2 + v_3\right) \end{cases}$$

The stationary points for this system that are on the plane $v_2 = 0$ and also in the domain Δ (4.17) are as follows.

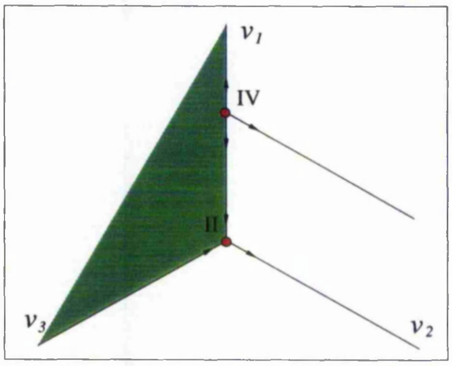
point	eigenvectors and eigenvalues
$\mathrm{II} = (0,0,0)$	$[1,0,0] \leftrightarrow -4, \quad [0,1,0] \leftrightarrow 3, \quad [0,0,1] \leftrightarrow -2.$
$\mathrm{IV} = (4,0,0)$	$[1,0,0] \leftrightarrow 4, \quad [0,1,0] \leftrightarrow 19, \quad [1,0,-\frac{7}{2}] \leftrightarrow -10.$

Thus we obtain the picture for the dynamics in the coordinates V given in figure 4.4 below.

Figure 4.4. The dynamics near the point $(0,0,0)$ in the coordinates V.

The third eigenvector for point IV has been ignored

because it lies in the plane $v_2 = 0$.



In the coordinates W the dynamics is given by

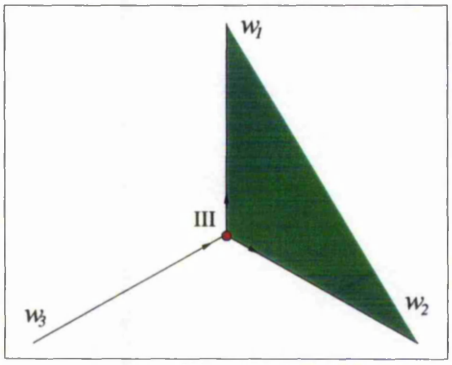
$$\begin{cases} \dot{w}_1 = w_1\left(3w_1 - 2w_2 + 2\right) \\ \dot{w}_2 = w_2\left(2w_1 + 2w_2 + 1\right) \\ \dot{w}_3 = -w_3\left(-2w_1 - w_2 + 4 + 5w_1^2 w_3 + 3w_2^2 w_3 - 4w_3 + 3w_1 w_3 w_2 - 2w_2 w_3\right) \end{cases}$$

There is only one stationary point on the plane $w_3 = 0$ that belongs to the domain Δ (4.17).

point	eigenvectors and eigenvalues
$\mathrm{III} = (0,0,0)$	$[1,0,0] \leftrightarrow 2, \quad [0,1,0] \leftrightarrow 1, \quad [0,0,1] \leftrightarrow -4.$

The dynamics in the coordinates W shown in figure 4.5.

Figure 4.5. The dynamics near the point $(0,0,0)$ **in the coordinates** W.

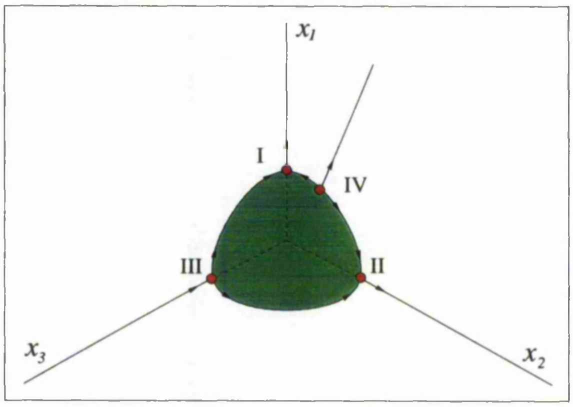


4.3.3 Overall picture of the dynamics.

Now, let us return to the original coordinates. Every stationary point in the coordinates U, V or W corresponds to a direction in the original coordinates. Points I, II and III, correspond to the basis directions $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$, respectively. Point IV corresponds to the direction $[4,1,0]$. Transferring the results of the analysis back to the original coordinate system, we obtain dynamics in the neighbourhood of the point $(0,0,0)$ as shown in figure 4.6 below.

Figure 4.6. The dynamics near the non-hyperbolic point $(0,0,0)$.

The point $(0,0,0)$ is "blown up".



One result obtained from the coordinate transformation is that there is a solution leaving the point $(0,0,0)$ along the vector $[4,1,0]$. There is also a stationary point $\left(\frac{4}{5}, \frac{1}{5}, 0\right)$ for the system (4.16). This suggests that the line

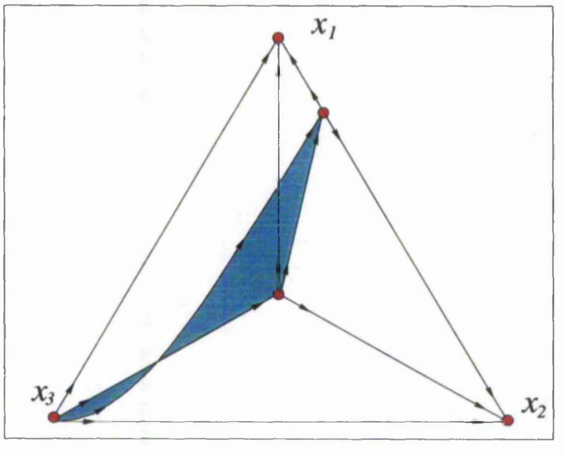
$$\begin{cases} x_1 = 4x_2 \\ x_3 = 0 \end{cases} \tag{4.20}$$

passing through the origin and the point $\left(\frac{4}{5}, \frac{1}{5}, 0\right)$ might be an invariant line under the dynamics (4.16). This can be confirmed by explicit calculation.

It follows from the center manifold theorem for flows (see section 4.1.2 or [50]) that in the neighbourhood of the point $\left(\frac{4}{5}, \frac{1}{5}, 0\right)$ there exists a two-dimensional stable invariant manifold M tangent to the eigenspace generated by the vectors $[4, 1, 0]$ and $[6, 1, -7]$. If we consider this dynamical system when $t \to -\infty$, we find that all trajectories starting from an interior point of the domain (4.17) terminate at the point $(0, 0, 1)$. Taking this into account, we can conclude that the invariant manifold M passes through the point $(0, 0, 1)$. On the other hand, due to the comment made above, invariant line (4.20) also belongs to the manifold M. Therefore we obtain a qualitative picture of the invariant manifold M as shown in figure 4.7 below.

Figure 4.7. The complete picture of the dynamics for the system (4.18).

The invariant manifold M is shaded.



This manifold separates the domain Δ into two regions with different behaviour of the solutions. Any solution trajectory with the initial condition lying between M and $(0, 1, 0)$ ends at the point $(0, 1, 0)$. Solution trajectories starting on the other side of the manifold end at the point $(1, 0, 0)$.

4.4 Summary.

In this chapter we have illustrated how ideas from the theory of qualitative analysis of differential equations can be used to obtain qualitative pictures of evolutionary dynamics. Using the example of the Iterated Prisoners' Dilemma we have shown that there exists a set of strategies, corresponding to populations which consist of various mixtures of unconditional cooperators and Tit for Tat players, which demonstrates that cooperative behaviour may be an evolutionary outcome in the Iterated Prisoners' Dilemma. However, this set fails to be asymptotically or evolutionarily stable. This motivated us to introduce a concept of evolutionarily attractive sets. We will show in chapters 6 and 7 that such sets are common if the population analysed includes a proportion of individuals who use "punishing" strategies (such as Tit for Tat) and also a proportion of individuals who use "non-punishing" equivalents of these "punishing" strategies. (For example, unconditional cooperation would be the "non-punishing" equivalent of Tit for Tat strategy since the observed behaviour for unconditional cooperation and Tit for Tat is the same.)

We have also described the singular coordinate transformation ("blowing up") technique. We have shown how singular coordinate transformations can be used to determine the stability properties of non-hyperbolic fixed points which may occur in the Replicator Dynamics of non-generic games. Under this transformation a non-hyperbolic point (usually assumed to be at the origin of the coordinate system) is substituted by an invariant manifold. In the particular example considered we used directional blow-ups as these seem to be the simplest transformations for resolving the dynamics. While these stability properties may also be determined by other methods (e.g. Lyapounov functions), the coordinate transformation technique has the added advantage of elucidating the form of the stability or instability. This may then provide information about invariant manifolds, as it did in the particular example we have considered.

We use these ideas and techniques in chapters 6 and 7 to analyse Replicator Dynamics of the multi-state game model.

Chapter 5

Multi-state games: Nash Equilibria.

Now we propose one more way of generalising the Prisoners' Dilemma model of interaction and in the next three chapters consider a multi-state interaction between two individuals. The multi-state models allow for the possibility that individuals may interact in more than one context. We will analyse a repeated interaction between two individuals in which the game played at any particular time is randomly selected from a specified set of two games G_1 and G_2. For example, individuals may interact either to hunt for food or to defend a jointly held territory. In addition to examining the behaviour of the individuals in these two context-games, we will also allow them to decide whether or not they wish to continue their long-term association. This will be modelled by a game G_0, which we will refer to as "association game". In this chapter we concentrate on a Nash Equilibrium analysis.

Firstly, we consider a multi-state interaction as a memoryless stochastic game and show that such an approach does not help to overcome the restriction of the Prisoners' Dilemma. Modelling games G_1 and G_2 by Prisoners' Dilemma games, we demonstrate that the presence of other games in the interaction and the possibility of discontinuing an association do not change the solution: cooperative behavior is not rational in such a model.

Then we consider the one-stage memory model for the multi-state game. Considering a memory model gives us the possibility of using "punishing strategies". The multi-state model provides us with two new ways of modelling behaviour. Firstly, the players are allowed to discontinue association and therefore the "punishment" for such models can consist not in defecting but in breaking the long term association if a partner does not cooperate at some state. This is the type of model which we will analyse in this thesis. Secondly, we can introduce a new type of behaviour. Since we have two activity games G_1 and G_2 we can introduce strategies based on the idea of the division of labour or allocating tasks. That is one player cooperates in game G_1 and defects in game G_2 while the other player defects in G_1 and cooperates in G_2. We will show that under certain conditions on the parameters of the model such strategy is a Nash Equilibrium. Note that, although we have already obtained cooperative Nash Equilibria using the Iterated Prisoners' Dilemma model, the allocating task strategies are different. The observed type of behavior in any one game for such strategies is: one player cooperates while the other defects. Therefore such strategies may shed light on the explanation of the apparently altruistic behaviour among unrelated individuals. This is particularly important since the understanding of evolutionary mechanisms producing altruistic behaviour in animals has been regarded as "the central theoretical problem in sociobiology" [1].

We also consider some other interesting strategies that we believe are relevant to the understanding of the cooperative behaviour between individuals. We show that in this model (as well as in the Iterated Prisoners' Dilemma game) punishing cooperative strategies are rational. Moreover, depending on the structure of the association game G_0 (for example, if we model the association game by a Hawk-Dove game), it is possible that strategies similar to *All Defect* of the Prisoners' Dilemma game are no longer Nash Equilibria for the one-stage memory model. In comparison, for the models based on the Iterated Prisoners' Dilemma *All Defect* is a Nash Equilibrium for any set of parameters.

Below we use the term "context-game" to refer to a single interaction two-player game in the normal form.

5.1 Description of the model: the multi-state game.

We consider an interaction which can be described as the following multi-state game. There are four possible context-games G_0, G_1, G_2 and G_3.

- The interaction starts with context-game G_0, that is the decision of individuals about whether or not they wish to continue their long-term association. The first player and the second player choose between the possible actions A="associate" or B="break up". The payoffs in context-game G_0 can be considered as the background rewards (or costs) obtained from association. Payoffs in context-game G_0 can also include rewards (or costs) obtained from other activities, such as games which are not explicitly considered or non-interactive behaviour which the individuals may undertake during the association.

- Context-games G_1 and G_2 are related to some specific activities in which the individuals can participate together. For example, if we model animal behaviour, these games might correspond to territory defence and hunting. We will consider games where the players choose from a set of two possible actions. We denote these actions by C and D. In modelling some types of behaviour (for example territory defence), it is appropriate to interpret these actions as cooperation and defection; so we sometimes will refer to the action C as "cooperation" and to the action D as "defection".
- Context-game G_3 can be considered as a background state representing the situation when there is no interaction or association between the players. There is only one possible action to choose L="be alone".

Remark 5.1 *For simplicity of analysis in this model the players are not allowed to reform the association. This restriction is not crucial since payoffs in context-game G_3 could include payoffs obtained from association formed with other individuals after the current one breaks up. For consistency these payoffs should be derived from the equilibrium behaviour in context-games G_0, G_1 and G_2 but for simplicity we will treat them as fixed. We will also assume that there is a constant discount factor β between all states of the model. This assumption imposes certain restrictions on the model. For example the effects of different contexts of interaction on survival probability (which is a standard interpretation of the discount factor) can be different. Nevertheless, for the simplicity of the analysis, we will assume that there is the same discount factor after each state.*

The actions chosen define both immediate payoffs to the individuals and future transition probabilities. The immediate payoffs collected by the players are given in table 5.1. The first entry in each payoff pair contains the payoff to the player P_1, who selects the row action, the second is for the player P_2, who selects the column action. Transition probabilities, which are determined by the choice of actions are presented in table 5.1 as a set of four numbers. This set of numbers appears in square brackets in each cell of the matrices. Here the first, second, third or fourth number is, respectively, the probability that context-game G_0, G_1, G_2 or G_3 is played at the next round. The probabilities are defined by the following rules.

G_0 : If context-game G_0 is played

A: and action A ="associate" is chosen by both players, at the next round context-game G_1 is played with probability p and context-game G_2 is played with probability $1-p$;

B: if action B="break up" is chosen by at least one player, context-game G_3 is played at the next round with probability 1.

G_1 or G_2 : Whatever actions are chosen while context-games G_1 or G_2 are played, context-game G_0 is played at the next round with probability 1.

G_3 : If context-game G_3 is played, at the next round context-game G_3 is played again with probability 1.

Table 5.1. The multi-state game.

G_0:

P1\P2	A	B
A	$(c_1,c_1)/[0,p,1-p,0]$	$(c_2,c_3)/[0,0,0,1]$
B	$(c_3,c_2)/[0,0,0,1]$	$(c_4,c_4)/[0,0,0,1]$

G_3:

P1\P2	L
L	$(z,z)/[0,0,0,1]$

G_1:

P1\P2	C	D
C	$(h_1,h_1)/[1,0,0,0]$	$(h_2,h_3)/[1,0,0,0]$
D	$(h_3,h_2)/[1,0,0,0]$	$(h_4,h_4)/[1,0,0,0]$

G_2:

P1\P2	C	D
C	$(t_1,t_1)/[1,0,0,0]$	$(t_2,t_3)/[1,0,0,0]$
D	$(t_3,t_2)/[1,0,0,0]$	$(t_4,t_4)/[1,0,0,0]$

5.2 A model without memory.

In this section I will prove an intuitively clear fact that if one of the games G_1 or G_2 are modeled by a Prisoners' Dilemma and there is no memory assumed in the model then the defection in a Prisoners' Dilemma state still is the only possible outcome of the interaction. The existence of other games and the possibility of discontinuing the association do not affect this solution.

We consider the game defined by table 5.1 as a standard stochastic game. For such a model the strategies which the players use in the game are stationary, that is they do not depend on time point at which the game is played, but depend only on the context-games played. We use the standard approach of the discounted Markov decision processes discussed in chapter 3 and find Nash equilibrium solutions. We describe the Markov decision process related to this game as follows.

- The states of the Markov process are the context-games and we have the four state process: $\mathbf{S} = \{0,1,2,3\} = \{G_0, G_1, G_2, G_3\}$.
- The sets of actions $\mathbf{A}^i(s)$, which can be chosen by the first ($i=1$) and the second ($i=2$) players at some round, are

$$\mathbf{A}^i(0) = \{A, B\}, \qquad \mathbf{A}^i(s) = \{C, D\}, s = 1,2, \qquad \mathbf{A}^i(3) = \{L\}, \qquad i = 1,2.$$

- Immediate rewards and transition probabilities are given by table 5.1.
- The i^{th} player's strategy can be described as $\mathbf{f}_i = (\mathbf{f}_{i,0}, \mathbf{f}_{i,1}, \mathbf{f}_{i,2}, \mathbf{f}_{i,3})$, $i = 1,2$. Here

$$\begin{aligned}
&\mathbf{f}_i(0) = (f_{i,0}, 1 - f_{i,0}), && \text{where } f_{i,0} \text{ is the probability of choosing action } A,\\
&\mathbf{f}_i(s) = (f_{i,s}, 1 - f_{i,s}), s = 1,2, && \text{where } f_{i,s} \text{ is the probability of choosing action } C,\\
&\mathbf{f}_i(3) = (1).
\end{aligned}$$

For the components of the immediate expected reward vector we have

$$\begin{aligned}
r^1(0, \mathbf{f}_1, \mathbf{f}_2) &= c_1 f_{1,0} f_{2,0} + c_2 f_{1,0}(1 - f_{2,0}) + c_3 (1 - f_{1,0}) f_{2,0} + c_4 (1 - f_{1,0})(1 - f_{2,0}),\\
r^1(1, \mathbf{f}_1, \mathbf{f}_2) &= h_1 f_{1,1} f_{2,1} + h_2 f_{1,1}(1 - f_{2,1}) + h_3 (1 - f_{1,1}) f_{2,1} + h_4 (1 - f_{1,1})(1 - f_{2,1}),\\
r^1(2, \mathbf{f}_1, \mathbf{f}_2) &= t_1 f_{1,2} f_{2,2} + t_2 f_{1,2}(1 - f_{2,2}) + t_3 (1 - f_{1,2}) f_{2,2} + t_4 (1 - f_{1,2})(1 - f_{2,2}),\\
r^1(3, \mathbf{f}_1, \mathbf{f}_2) &= z
\end{aligned}$$

and

$$\begin{aligned}
r^2(0, \mathbf{f}_1, \mathbf{f}_2) &= c_1 f_{1,0} f_{2,0} + c_3 f_{1,0}(1 - f_{2,0}) + c_2 (1 - f_{1,0}) f_{2,0} + c_4 (1 - f_{1,0})(1 - f_{2,0}),\\
r^2(1, \mathbf{f}_1, \mathbf{f}_2) &= h_1 f_{1,1} f_{2,1} + h_3 f_{1,1}(1 - f_{2,1}) + h_2 (1 - f_{1,1}) f_{2,1} + h_4 (1 - f_{1,1})(1 - f_{2,1}),\\
r^2(2, \mathbf{f}_1, \mathbf{f}_2) &= t_1 f_{1,2} f_{2,2} + t_3 f_{1,2}(1 - f_{2,2}) + t_2 (1 - f_{1,2}) f_{2,2} + t_4 (1 - f_{1,2})(1 - f_{2,2}),\\
r^2(3, \mathbf{f}_1, \mathbf{f}_2) &= z.
\end{aligned}$$

Notice that $r^i(s, \mathbf{f}_1, \mathbf{f}_2)$, $i = 1, 2$, $s = 0, 1, 2$, depend only on s^{th} components of strategies $f_{1,s}$ and $f_{2,s}$. That is $r^i(s, \mathbf{f}_1, \mathbf{f}_2) = r^i(s, f_{1,s}, f_{2,s})$.

We assume that the probability of participating in the next round does not depend on the context-game played, the actions chosen in this context-game or the moment at which the context-game is played. This means that there is a constant discount factor β which is the same between all context-games.

The strategies $\mathbf{f}_1$, $\mathbf{f}_2$ define a probability transition matrix as following

$$P(\mathbf{f}_1, \mathbf{f}_2) = \begin{pmatrix} 0 & pf_{1,0}f_{2,0} & (1-p)f_{1,0}f_{2,0} & (1 - f_{1,0}f_{2,0}) \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence the vector $\mathbf{v}^i_\beta(\mathbf{f}_1, \mathbf{f}_2)$ can be calculated as

$$\mathbf{v}^i_\beta(\mathbf{f}_1, \mathbf{f}_2) = [I_4 - \beta P(\mathbf{f}_1, \mathbf{f}_2)]^{-1} \mathbf{r}^i(\mathbf{f}_1, \mathbf{f}_2).$$

For the model considered we find that

$$[I_4 - \beta P(\mathbf{f}_1, \mathbf{f}_2)]^{-1} = \begin{pmatrix} \frac{1}{(1-\beta^2 f_{1,0}f_{2,0})} & \frac{p\beta f_{1,0}f_{2,0}}{(1-\beta^2 f_{1,0}f_{2,0})} & \frac{(1-p)\beta f_{1,0}f_{2,0}}{(1-\beta^2 f_{1,0}f_{2,0})} & \frac{\beta(1-f_{1,0}f_{2,0})}{(1-\beta)(1-\beta^2 f_{1,0}f_{2,0})} \\ \frac{\beta}{(1-\beta^2 f_{1,0}f_{2,0})} & 1 + \frac{p\beta^2 f_{1,0}f_{2,0}}{(1-\beta^2 f_{1,0}f_{2,0})} & \frac{(1-p)\beta^2 f_{1,0}f_{2,0}}{(1-\beta^2 f_{1,0}f_{2,0})} & \frac{\beta^2(1-f_{1,0}f_{2,0})}{(1-\beta)(1-\beta^2 f_{1,0}f_{2,0})} \\ \frac{\beta}{(1-\beta^2 f_{1,0}f_{2,0})} & \frac{p\beta^2 f_{1,0}f_{2,0}}{(1-\beta^2 f_{1,0}f_{2,0})} & 1 + \frac{(1-p)\beta^2 f_{1,0}f_{2,0}}{(1-\beta^2 f_{1,0}f_{2,0})} & \frac{\beta^2(1-f_{1,0}f_{2,0})}{(1-\beta)(1-\beta^2 f_{1,0}f_{2,0})} \\ 0 & 0 & 0 & \frac{1}{(1-\beta)} \end{pmatrix}.$$

Then, since $r^i(s, \mathbf{f}_1, \mathbf{f}_2) = r^i(s, f_{1,s}, f_{2,s})$, we obtain the discounted value vectors as follows

$$\begin{aligned} \mathbf{v}^i_\beta(0, \mathbf{f}_1, \mathbf{f}_2) &= \frac{\left(\beta(1-\beta)\left(pr^i(1,f_{1,1},f_{2,1}) + (1-p)r^i(2,f_{1,2},f_{2,2})\right) - \beta z\right) f_{1,0}f_{2,0} + (1-\beta)r^i(0,f_{1,0},f_{2,0}) + \beta z}{(1-\beta)(1-\beta^2 f_{1,0}f_{2,0})}, \\ \mathbf{v}^i_\beta(1, \mathbf{f}_1, \mathbf{f}_2) &= \beta \mathbf{v}^i_\beta(0, \mathbf{f}_1, \mathbf{f}_2) + r^i(1, f_{1,1}, f_{2,1}), \\ \mathbf{v}^i_\beta(2, \mathbf{f}_1, \mathbf{f}_2) &= \beta \mathbf{v}^i_\beta(0, \mathbf{f}_1, \mathbf{f}_2) + r^i(2, f_{1,2}, f_{2,2}), \\ \mathbf{v}^i_\beta(3, \mathbf{f}_1, \mathbf{f}_2) &= \frac{z}{1-\beta}. \end{aligned} \tag{5.1}$$

Now, let us find Nash equilibrium solutions for this stochastic game. We shall analyse conditions (3.2) from definition 3.5 of a Nash equilibrium for stochastic games.

Notice that the expression for $\mathbf{v}^i_\beta(0, \mathbf{f}_1, \mathbf{f}_2)$ we can represent as a sum

$$\mathbf{v}^i_\beta(0, \mathbf{f}_1, \mathbf{f}_2) = \Psi \frac{\beta(1-\beta) f_{1,0}f_{2,0}}{(1-\beta)(1-\beta^2 f_{1,0}f_{2,0})} + \frac{-\beta z f_{1,0}f_{2,0} + (1-\beta)r^i(0,f_{1,0},f_{2,0}) + \beta z}{(1-\beta)(1-\beta^2 f_{1,0}f_{2,0})},$$

where $\Psi = \left(pr^i(1, f_{1,1}, f_{2,1}) + (1-p)\, r^i(2, f_{1,2}, f_{2,2})\right).$

To maximise $\mathbf{v}_\beta^i(0, \mathbf{f}_1, \mathbf{f}_2)$, first consider fixed $f_{1,0}$ and $f_{2,0}$. For any fixed $f_{1,0}$ and $f_{2,0}$ the value vector attains its maximum when Ψ is maximised, since $\frac{\beta(1-\beta)f_{1,0}f_{2,0}}{(1-\beta)(1-\beta^2 f_{1,0}f_{2,0})} > 0$. Maximising Ψ determines the strategy components $f^*_{1,1}, f^*_{2,1}, f^*_{1,2}$ and $f^*_{2,2}$. Once these are known, the maximum value vector can be determined by considering only differing values of $f_{1,0}$ and $f_{2,0}$.

The problem of maximising Ψ is equivalent to finding Nash Equilibria for states G_1 and G_2 considered as single games: we must find the probabilities $f^*_{1,1}, f^*_{2,1}, f^*_{1,2}$ and $f^*_{2,2}$ such that

$$\begin{cases} r^1(s, f^*_{1,s}, f^*_{2,s}) \geq r^1(s, f_{1,s}, f^*_{2,s}) \\ r^2(s, f^*_{1,s}, f^*_{2,s}) \geq r^2(s, f^*_{1,s}, f_{2,s}) \end{cases} \qquad s = 1, 2. \tag{5.2}$$

We can see from conditions (5.2) that, disregarding the strategies used in the context-game G_0, players defect in context-games G_1 and G_2 if these games represent the Prisoners' Dilemma type interaction (since mutual defection is the only Nash Equilibrium in the Prisoners' Dilemma). If only stationary strategies are used in the multi-state game it is not possible to conclude that cooperative behaviour is rational. Therefore more complex models need to be considered in an attempt to explain cooperative behaviour. In the next section we consider a model with non-stationary strategies.

5.3 One-stage memory model for the multi-state game.

5.3.1 Model of interaction. Role dependent modelling of states G_1 and G_2.

Now we would like to draw attention to the following point. Since for the model considered in section 5.1 there are two context-games G_1 and G_2 related to some specific activities in which individuals can participate together, we wish to introduce a possibility for players to divide their responsibilities. For example the first player may "prefer" to "cooperate" in context-game G_1 and "defect" in context-game G_2. In return the second player "cooperates" in context-game G_2 and "defects" in context-game G_1. Such a strategy can be interpreted as the first player liking to participate in the activity modelled by context-game G_1 and not liking G_2. On the other hand, the second player does not take a part in G_1 but carries out all work in G_2. But, since we assume the two players to be equal, there should be some independent mechanism that determines the preferences of the players. We will formalise it as follows.

In modelling the interaction, we suppose that

- each player can be assigned one of the two roles (denoted by $\mathcal{A}$ and $\mathcal{B}$);
- there exists a rule that assigns a role to a player in such a way that if one of the players is assigned role $\mathcal{A}$, then the other is assigned role $\mathcal{B}$;
- the players are equal with respect to the assigning rule; the probabilities of being assigned role $\mathcal{A}$ or role $\mathcal{B}$ are independent of the strategy the player uses and are, therefore, equal $\frac{1}{2}$;
- the players are certain about which role they have been assigned;
- when context-games G_1 or G_2 are played the players may choose different actions in these games if they use the same strategy but are assigned different roles.

As an example of role assignment we can consider the right of ownership on a resource. Such an interaction is discussed in [14] where the so-called "Bourgeois" strategy for the Hawk-Dove game was considered. This strategy prescribes playing Hawk if the player is the owner of a resource and Dove if an intruder. Another example can be an interaction between a male and a female, or between a small and a large individual. In relation to the multi-state game considered here we can suppose, for instance, that the rule prescribing a role depends on ownership of a territory.

If we wish to calculate the total payoff obtained by such a role dependent strategy we should calculate the payoff to a player in role $\mathcal{A}$, multiply it by the probability of being assigned role $\mathcal{A}$ and add it to the payoff to a player in role $\mathcal{B}$ multiplied by the probability of being assigned role $\mathcal{B}$. This can involve a lot of calculation if we are analysing a number of different strategies. To avoid such calculations we construct a special Markov process: we duplicate states G_1 and G_2 to account for information about a role assigned to the first player (then second player is assigned the other role). From calculational point of view there is no difference whether a role prescribed to a player at the beginning of the whole interaction or before state G_1 or G_2 is played since the role prescription is probabilistic. We only have to adjust the appropriate transition probabilities. The probability to be transferred to state $G_1(\mathcal{A})$, $G_2(\mathcal{A})$, $G_1(\mathcal{B})$ or $G_2(\mathcal{B})$ after the context-game G_0 was played now become $\frac{p}{2}$, $\frac{1-p}{2}$, $\frac{p}{2}$ or $\frac{1-p}{2}$, correspondingly. The Markov process in this instance gives us the advantage of using the techniques described in chapter 4: we can use a software program (we have used 'Maple' to perform calculations in this thesis) to calculate the total payoffs for strategies, we also will be able to obtain the Nash Equilibrium conditions. Table 5.2 summarises this model.

Table 5.2. Role dependent modelling of states G_1 and G_2.

PLAYER I (PLAYER II)			
$\frac{1}{2} \swarrow$		$\searrow \frac{1}{2}$	
ROLE $\mathcal{A}$ (ROLE $\mathcal{B}$)		ROLE $\mathcal{B}$ (ROLE $\mathcal{A}$)	
$p \swarrow$	$\searrow 1-p$	$p \swarrow$	$\searrow 1-p$
STATE $G_1(\mathcal{A})$	STATE $G_2(\mathcal{A})$	STATE $G_1(\mathcal{B})$	STATE $G_2(\mathcal{B})$

Here if the first player is assigned role $\mathcal{A}$ then the second player is assigned role $\mathcal{B}$ and vice versa.

Remark 5.2 *If we assume that the players are not equal with respect to the assigning rule and the probabilities of being assigned role $\mathcal{A}$ to the first player is q (and, therefore, to be assigned role $\mathcal{B}$ is $1-q$). Then the probability to be transferred to state $G_1(\mathcal{A})$, $G_2(\mathcal{A})$, $G_1(\mathcal{B})$ or $G_2(\mathcal{B})$ after the context-game G_0 become pq, $(1-p)q$, $p(1-q)$ or $(1-p)(1-q)$, correspondingly. In this work we only consider the case of $q=\frac{1}{2}$. Since assignment rule does not depend on any attributes of the players, the $q=\frac{1}{2}$ seems to be a natural choice.*

To complete the description of the Markov process we should explain how "punishment" will be modeled. Using multi-state models allow us a wide range of opportunities for modeling "punishing" strategies. Since for multi-state models there are more than one context-game, the "punishment" can be placed on another context-game (not necessarily the one in which the player wishes to ensure cooperation). For example, in the Iterated Prisoners' Dilemma game the only possibility for modelling the "punishment" was to swap from cooperation to defection in the corresponding Prisoners' Dilemma. The players did not have a chance to discontinue an association whatever they do. In the model that we consider below, the "punishment" is placed on the game G_0 and consists in breaking the long-term association if a partner did not cooperate at some state in the past.

To describe the model we assume the following.

- Whenever states $G_1(\mathcal{A})$, $G_2(\mathcal{A})$, $G_1(\mathcal{B})$ or $G_2(\mathcal{B})$ are visited the players use the same actions at these states.

- The players may use different actions in context-game G_0 depending on which actions have been chosen at the previous round in one of the states $G_1(\mathcal{A})$, $G_2(\mathcal{A})$, $G_1(\mathcal{B})$ or $G_2(\mathcal{B})$.
- There is a constant discounting factor β between all states of the process.

In the next section we describe the Markov process related to this model and obtain a formula for the total payoff values.

5.3.2 The Markov Process and the Total Payoff Values: The two-person game.

To obtain a Markov process with stationary strategies we introduce seventeen different information states for the context-game G_0 depending on past history (restricted to one-stage memory as described in the previous section). The set of all possible information states for the context-game G_0 in this case is $I = \{i_0, i_1, ..., i_{16}\}$, where

$i_1=\{G_1(\mathcal{A}),C,C\}$, $i_2=\{G_1(\mathcal{A}),C,D\}$, $i_3=\{G_1(\mathcal{A}),D,C\}$, $i_4=\{G_1(\mathcal{A}),D,D\}$,

$i_5=\{G_2(\mathcal{A}),C,C\}$, $i_6=\{G_2(\mathcal{A}),C,D\}$, $i_7=\{G_2(\mathcal{A}),D,C\}$, $i_8=\{G_2(\mathcal{A}),D,D\}$,

$i_9=\{G_1(\mathcal{B}),C,C\}$, $i_{10}=\{G_1(\mathcal{B}),C,D\}$, $i_{11}=\{G_1(\mathcal{B}),D,C\}$, $i_{12}=\{G_1(\mathcal{B}),D,D\}$,

$i_{13}=\{G_2(\mathcal{B}),C,C\}$, $i_{14}=\{G_2(\mathcal{B}),C,D\}$, $i_{15}=\{G_2(\mathcal{B}),D,C\}$, $i_{16}=\{G_2(\mathcal{B}),D,D\}$.

Here $\{G_n(\mathcal{A}), a_1, a_2\}$ or $\{G_n(\mathcal{B}), a_1, a_2\}$ is information that context-game $G_n(\mathcal{A})$ or $G_n(\mathcal{B})$, respectively, $n = 1, 2$, was played at the previous stage, action a_1 was chosen by the first player and action a_2 was chosen by the second player. For example, $i_6 = \{G_2(\mathcal{A}), C, D\}$ means that at the previous state game $G_2(\mathcal{A})$ was played, first player was cooperating and second player defected. The final state i_0 is the information that neither G_1 or G_2 has yet been played during the game. Therefore, for context-game G_0 seventeen states $G_0(i_k)$, $k = 0, \ldots, 16$, are introduced. It is necessary to introduce five more states in the process to include context-games $G_1(\mathcal{A})$, $G_2(\mathcal{A})$, $G_1(\mathcal{B})$, $G_2(\mathcal{B})$ and G_3. Therefore, the set of different states of the process is

$$\begin{aligned} \mathrm{S} &= \{0, 1, 2, \ldots, 21\} \\ &= \{G_0(i_0), G_0(i_1), \ldots, G_0(i_{16}), G_1(\mathcal{A}), G_2(\mathcal{A}), G_1(\mathcal{B}), G_2(\mathcal{B}), G_3\}. \end{aligned}$$

The sets of actions $\mathbf{A}^i(s)$, which can be chosen by the i^{th} player at state s, are

$$\begin{aligned}
\mathbf{A}^i(s) &= \{1,2\} = \{A,B\}, \quad s=0,\ldots,16, \quad i=1,2,\\
\mathbf{A}^i(s) &= \{1,2\} = \{C,D\}, \quad s=17,\ldots,20, \quad i=1,2,\\
\mathbf{A}^i(21) &= \{1\} = \{L\}, \quad i=1,2.
\end{aligned}$$

The i^{th} player's strategy $\mathbf{f}_i = (\mathbf{f}_{i,0}, \mathbf{f}_{i,1}, \ldots, \mathbf{f}_{i,20}, \mathbf{f}_{i,21})$. Here $\mathbf{f}_{i,s} = (f_{i,s}, 1-f_{i,s})$, $s = 0,1,\ldots,20$, where $f_{i,s}$, $i=1,2$, is the probability that action A is chosen in the state s by the i^{th} player if $s=0,\ldots,16$, and the probability that action C is chosen in the state s by the i^{th} player if $s=17,\ldots,20$. Note that $\mathbf{f}_{i,21} = (1)$.

The immediate expected rewards then are

$$\mathbf{r}^i(\mathbf{f}_1,\mathbf{f}_2) = \left(r^i(0,\mathbf{f}_1,\mathbf{f}_2), r^i(1,\mathbf{f}_1,\mathbf{f}_2), \ldots, r^i(21,\mathbf{f}_1,\mathbf{f}_2)\right)^T,$$

where for $s=0,\ldots,16$

$$\begin{aligned}
r^1(s,\mathbf{f}_1,\mathbf{f}_2) &= c_1 f_{1,s} f_{2,s} + c_2 f_{1,s}(1-f_{2,s}) + c_3(1-f_{1,s}) f_{2,s} + c_4(1-f_{1,s})(1-f_{2,s}),\\
r^2(s,\mathbf{f}_1,\mathbf{f}_2) &= c_1 f_{1,s} f_{2,s} + c_3 f_{1,s}(1-f_{2,s}) + c_2(1-f_{1,s}) f_{2,s} + c_4(1-f_{1,s})(1-f_{2,s}),
\end{aligned}$$

for $s=17,19$

$$\begin{aligned}
r^1(s,\mathbf{f}_1,\mathbf{f}_2) &= h_1 f_{1,s} f_{2,s} + h_2 f_{1,s}(1-f_{2,s}) + h_3(1-f_{1,s}) f_{2,s} + h_4(1-f_{1,s})(1-f_{2,s}),\\
r^2(s,\mathbf{f}_1,\mathbf{f}_2) &= h_1 f_{1,s} f_{2,s} + h_3 f_{1,s}(1-f_{2,s}) + h_2(1-f_{1,s}) f_{2,s} + h_4(1-f_{1,s})(1-f_{2,s}),
\end{aligned}$$

for $s=18,20$

$$\begin{aligned}
r^1(s,\mathbf{f}_1,\mathbf{f}_2) &= t_1 f_{1,s} f_{2,s} + t_2 f_{1,s}(1-f_{2,s}) + t_3(1-f_{1,s}) f_{2,s} + t_4(1-f_{1,s})(1-f_{2,s}),\\
r^2(s,\mathbf{f}_1,\mathbf{f}_2) &= t_1 f_{1,s} f_{2,s} + t_3 f_{1,s}(1-f_{2,s}) + t_2(1-f_{1,s}) f_{2,s} + t_4(1-f_{1,s})(1-f_{2,s}),
\end{aligned}$$

and $r^1(21,\mathbf{f}_1,\mathbf{f}_2) = z$, $r^2(21,\mathbf{f}_1,\mathbf{f}_2) = z$. (See table 5.1 for definition of c_i, h_i and t_i.)

Now the total payoffs $\mathbf{v}^i_\beta(s,\mathbf{f}_1,\mathbf{f}_2)$ for the i^{th} player, if the game starts from an initial state s and strategies $\mathbf{f}_1$ and $\mathbf{f}_2$ are used by the first and second players, respectively, can be found using the approach discussed in chapter 3:

$$\mathbf{v}^i_\beta(s,\mathbf{f}_1,\mathbf{f}_2) = \left[I_{22} - \beta P(\mathbf{f}_1,\mathbf{f}_2)\right]^{-1} \mathbf{r}^i(\mathbf{f}_1,\mathbf{f}_2). \tag{5.3}$$

The probability transition matrix $P(\mathbf{f}_1,\mathbf{f}_2) = \left(p(s'|s,\mathbf{f}_1,\mathbf{f}_2)\right)_{s,s'=0}^{21}$, is given by

$$\begin{aligned}
p(s'|s,\mathbf{f}_1,\mathbf{f}_2) &= p(s'|s,A,A) f_{1,s} f_{2,s} + p(s'|s,A,B) f_{1,s}(1-f_{2,s})\\
&\quad + p(s'|s,B,A)(1-f_{1,s}) f_{2,s} + p(s'|s,B,B)(1-f_{1,s})(1-f_{2,s}),
\end{aligned}$$

for $s = 0, \ldots, 16,$

$$\begin{aligned} p\left(s'|s,\mathrm{f}_1,\mathrm{f}_2\right) &= p\left(s'|s,C,C\right) f_{1,s} f_{2,s} + p\left(s'|s,C,D\right) f_{1,s}\left(1-f_{2,s}\right) \\ &\quad + p\left(s'|s,D,C\right)\left(1-f_{1,s}\right) f_{2,s} + p\left(s'|s,D,D\right)\left(1-f_{1,s}\right)\left(1-f_{2,s}\right), \end{aligned}$$

for $s = 17, \ldots, 20$ and $p\left(s'|21,\mathrm{f}_1,\mathrm{f}_2\right) = p\left(s'|21,L,L\right)$. The immediate transition probabilities in this case are defined as follows.

$$\begin{aligned} &p\left(s'|s,A,A\right) = \tfrac{p}{2}, \quad s = 0,\ldots,16; \quad s' = 17,19. \\ &p\left(s'|s,A,A\right) = \tfrac{1-p}{2}, \quad s = 0,\ldots,16; \quad s' = 18,20. \\ &p\left(21|s,A,B\right) = p\left(21|s,B,A\right) = p\left(21|s,B,B\right) = 1, \quad s = 0,\ldots,16. \\ &p\left(1|17,C,C\right) = p\left(2|17,C,D\right) = p\left(3|17,D,C\right) = p\left(4|17,D,D\right) = 1; \\ &p\left(5|18,C,C\right) = p\left(6|18,C,D\right) = p\left(7|18,D,C\right) = p\left(8|18,D,D\right) = 1; \\ &p\left(9|19,C,C\right) = p\left(10|19,C,D\right) = p\left(11|19,D,C\right) = p\left(12|19,D,D\right) = 1; \\ &p\left(13|20,C,C\right) = p\left(14|20,C,D\right) = p\left(15|20,D,C\right) = p\left(16|20,D,D\right) = 1; \\ &p\left(21|21,L,L\right) = 1; \end{aligned}$$

and all other immediate transition probabilities $p\left(s'|s,a^1,a^2\right) = 0.$

Therefore

$$P(\mathrm{f}_1,\mathrm{f}_2) = \begin{pmatrix} 0 & 0 & \Lambda_1 & \Lambda_2 \\ 0 & \Lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\Lambda_1 = \begin{pmatrix} \frac{p f_{1,0} f_{2,0}}{2} & \frac{(1-p) f_{1,0} f_{2,0}}{2} & \frac{p f_{1,0} f_{2,0}}{2} & \frac{(1-p) f_{1,0} f_{2,0}}{2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{p f_{1,16} f_{2,16}}{2} & \frac{(1-p) f_{1,16} f_{2,16}}{2} & \frac{p f_{1,16} f_{2,16}}{2} & \frac{(1-p) f_{1,16} f_{2,16}}{2} \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 1 - f_{1,0} f_{2,0} \\ \vdots \\ 1 - f_{1,16} f_{2,16} \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} \Theta_{17} & 0 & 0 & 0 \\ 0 & \Theta_{18} & 0 & 0 \\ 0 & 0 & \Theta_{19} & 0 \\ 0 & 0 & 0 & \Theta_{20} \end{pmatrix},$$

where

$$\Theta_s = \left(\begin{array}{cccc} f_{1,s}f_{2,s} & f_{1,s}\left(1-f_{2,s}\right) & \left(1-f_{1,s}\right)f_{2,s} & \left(1-f_{1,s}\right)\left(1-f_{2,s}\right) \end{array} \right), \quad s = 17, \ldots 20.$$

In the next section we consider some strategies which can be used by players under this memory model. We use the formulae obtained to calculate the total payoff values of these strategies in order to determine the conditions under which these strategies are Nash equilibria when played against themselves. We also use these formulae in the next chapter where we analyse the Replicator Dynamics for the multi-state game.

5.3.3 Strategies.

When a Markov process constructed for a particular game-model has many different states it can be difficult to find all Nash equilibria. For example there are 2^{21} different pure strategies for the process constructed in the previous section and we can see that the difficulty of the problem increases exponentially even if we wish to consider only pure strategies. It seems that the difficulties can be avoided if we can guess that some particular pair of strategies is a Nash equilibrium and then prove it. Below we introduce a few strategies that will be analysed in the next section. These strategies cover different types of behaviour from being a "sucker", which is similar to *All C* strategy of the Iterated Prisoners' Dilemma game, to "unsociable", which ignores any attempt to establish an association. A completely new type of behaviour is also introduced using which the players divide their responsibilities in context-games G_1 and G_2. The strategies considered are the following.

S : "Sucker" (denoted by S). The player, who adopts this strategy chooses the following actions:

(1) A="associate" in context-game G_0 regardless;

(2) C="cooperate" in context-games G_1 or G_2 regardless.

CP : "Cooperator with punishment" (denoted by CP). Adopting this strategy, the player follows the instructions given below.

(1) Choose A="associate" in context-game G_0 and then

(a) if context-game G_1 or G_2 is played, play C="cooperate",

   (b) then follow the instructions given in paragraph (2);

(2) depending on the action chosen by the partner in context-game G_1 or G_2 at the previous round, do the following:

   (a) if the partner has chosen C="cooperate", follow the instructions given in paragraph (1),

   (b) if the partner has chosen D="defect", choose B="break up" in context-game G_0.

AT : "Allocating tasks" (denoted by AT). This strategy is as follows.

(1) Choose A="associate" in context-game G_0 and then follow the instructions given in paragraph (2);

(2) depending on the role, do the following:

   (a) if in role $\mathcal{A}$

      (i) choose C="cooperate" if context-game G_1 is played, then follow the instructions given in paragraph (1);

      (ii) choose D="defect" if context-game G_2 is played, then follow the instructions given in paragraph (1);

   (b) if in role $\mathcal{B}$

      (i) choose C="cooperate" if context-game G_2 is played, then follow the instructions given in paragraph (1);

      (ii) choose D="defect" if context-game G_1 is played, then follow the instructions given in paragraph (1);

ATP : "Allocating tasks with punishment" (denoted by ATP). This strategy is as follows.

(1) Choose A="associate" in context-game G_0 and then follow the instructions given in paragraph (2);

(2) depending on the role, do the following:

   (a) if in role $\mathcal{A}$

      (i) choose C="cooperate" if context-game G_1 is played, then follow the instructions given in paragraph (1);

(ii) choose D="defect" if context-game G_2 is played, then follow the instructions given in paragraph (3);

(b) if in role B

(i) choose C="cooperate" if context-game G_2 is played, then follow the instructions given in paragraph (1);

(ii) choose D="defect" if context-game G_1 is played, then follow the instructions given in paragraph (3);

(3) depending on the action chosen by the partner in context-game G_1 or G_2 at the previous round do the following:

(a) if the partner has chosen C="cooperate", follow the instructions given in paragraph (1).

(b) if the partner has chosen D="defect", choose B="break up" in the context-game G_0.

P : "Pathological" (denoted by P). The player, who adopts this strategy chooses action

(1) A="associate" in context-game G_0 regardless,

(2) D="defect" in context-games G_1 or G_2 regardless.

LFS : "Looking for a sucker" (denoted by LFS). Adopting this strategy, a player follows the instructions given below.

(1) Choose A="associate" in the context-game G_0 and then

(a) if context-game G_1 or G_2 is played, choose D="defect",

(b) then follow the instructions given in paragraph (2);

(2) depending on the action chosen by the partner in context-game G_1 or G_2 at the previous round do the following:

(a) if the partner has chosen C="cooperate", follow the instructions given in paragraph (1),

(b) if the partner has chosen D="defect", then choose B="break up" in the context-game G_0.

U : "Unsociable" (denoted by U). This strategy is to play B="break up" in context-game G_0 regardless.

In the next section we find Nash Equilibrium conditions that guarantee that a strategy is a Nash Equilibrium in a class of all strategies allowed by the model. To do so let us obtain the state description of the strategies given above. Since there is no choice of action in state 21 (G_3), this state is not included in the state description of any strategy. If the first player has adopted some strategy $\mathbf{f} = str$, then f_s can be found in the body of table 5.3. For convenience the probabilities that correspond to the states $G_1(\mathcal{A})$, $G_2(\mathcal{A})$, $G_1(\mathcal{B})$ and $G_2(\mathcal{B})$ are given in bold.

Table 5.3. State description of the strategies when used by the first player.

$str\backslash s$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
S	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	**1**	**1**	**1**	**1**
CP	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	**1**	**1**	**1**	**1**
AT	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	**1**	**0**	**0**	**1**
ATP	1	1	1	1	1	1	0	1	0	1	0	1	0	1	1	1	1	**1**	**0**	**0**	**1**
P	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	**0**	**0**	**0**	**0**
LFS	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	**0**	**0**	**0**	**0**
U	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**0**	**0**	**0**	**0**

The strategies CP, AT, ATP or LFS are not symmetric under permutation of the players. The state representation of these strategies when used by the second player can be found in table 5.4.

Table 5.4. State description of the strategies when used by the second player.

$str\backslash s$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
CP	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	**1**	**1**	**1**	**1**
AT	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	**0**	**1**	**1**	**0**
ATP	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	0	0	**0**	**1**	**1**	**0**
LFS	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	**0**	**0**	**0**	**0**

In the next section we analyse the strategies introduced. Using the dynamic programming equations we determine the conditions under which these strategies are Nash Equilibria in a class of all memory-one strategies. Note that, although we have only introduced above seven different strategies, the dynamic programming approach allows us to determine whether or not a strategy is a Nash Equilibrium compared to all memory-one (that is pure and mixed) strategies.

5.4 Nash Equilibria for one-stage memory model.

5.4.1 Dynamic programming equations.

In this section we obtain the formulae for the dynamic programming equations for the one-stage memory model. The dynamic programming equations in a general case are given by formula (3.5) reproduced below for convenience.

$$\begin{aligned}&\mathbf{V}^1_\beta(s,\mathrm{f})\\ &= \max_{a^1\in\mathbf{A}^1(s)}\left\{\sum_{a^2\in\mathbf{A}^2(s)} r^1(s,a^1,a^2)f(s,a^2)+\beta\sum_{s'\in\bar{\mathbf{S}}}\sum_{a^2\in\mathbf{A}^2(s)} p\left(s'|s,a^1,a^2\right)f(s,a^2)\mathbf{V}^1_\beta(s',\mathrm{f})\right\},\end{aligned}$$

For the one-stage memory model of the multi-state game considered, these equations have the following form. If $s=0,\dots,16$, then $\mathbf{V}^1_\beta(s,\mathrm{f})=\max\{u_{f_s}(A),u_{f_s}(B)\}$, where

$$\begin{aligned}u_{f_s}(A) &= f_s c_1+(1-f_s)\,c_2\\ &\quad+\beta f_s\left[\frac{p}{2}\mathbf{V}^1_\beta(17,\mathrm{f})+\frac{1-p}{2}\mathbf{V}^1_\beta(18,\mathrm{f})+\frac{p}{2}\mathbf{V}^1_\beta(19,\mathrm{f})+\frac{1-p}{2}\mathbf{V}^1_\beta(20,\mathrm{f})\right]\\ &\quad+\beta\,(1-f_s)\,\mathbf{V}^1_\beta(21,\mathrm{f}),\\ u_{f_s}(B) &= f_s c_3+(1-f_s)\,c_4+\beta\mathbf{V}^1_\beta(21,\mathrm{f}),\end{aligned}$$

If $s = 17, \ldots, 20$, then $\mathbf{V}_\beta^1(s, \mathbf{f}) = \max\{u_{f_s}(C), u_{f_s}(D)\}$, where

$$u_{f_{17}}(C) = f_{17}h_1 + (1 - f_{17})\, h_2 + \beta \left(f_{17}\mathbf{V}_\beta^1(1, \mathbf{f}) + (1 - f_{17})\, \mathbf{V}_\beta^1(2, \mathbf{f}) \right),$$

$$u_{f_{17}}(D) = f_{17}h_3 + (1 - f_{17})\, h_4 + \beta \left(f_{17}\mathbf{V}_\beta^1(3, \mathbf{f}) + (1 - f_{17})\, \mathbf{V}_\beta^1(4, \mathbf{f}) \right);$$

$$u_{f_{18}}(C) = f_{18}t_1 + (1 - f_{18})\, t_2 + \beta \left(f_{18}\mathbf{V}_\beta^1(5, \mathbf{f}) + (1 - f_{18})\, \mathbf{V}_\beta^1(6, \mathbf{f}) \right),$$

$$u_{f_{18}}(D) = f_{18}t_3 + (1 - f_{18})\, t_4 + \beta \left(f_{18}\mathbf{V}_\beta^1(7, \mathbf{f}) + (1 - f_{18})\, \mathbf{V}_\beta^1(8, \mathbf{f}) \right);$$

$$u_{f_{19}}(C) = f_{19}h_1 + (1 - f_{19})\, h_2 + \beta \left(f_{19}\mathbf{V}_\beta^1(9, \mathbf{f}) + (1 - f_{19})\, \mathbf{V}_\beta^1(10, \mathbf{f}) \right),$$

$$u_{f_{19}}(D) = f_{19}h_3 + (1 - f_{19})\, h_4 + \beta \left(f_{19}\mathbf{V}_\beta^1(11, \mathbf{f}) + (1 - f_{19})\, \mathbf{V}_\beta^1(12, \mathbf{f}) \right);$$

$$u_{f_{20}}(C) = f_{20}t_1 + (1 - f_{20})\, t_2 + \beta \left(f_{20}\mathbf{V}_\beta^1(13, \mathbf{f}) + (1 - f_{20})\, \mathbf{V}_\beta^1(14, \mathbf{f}) \right),$$

$$u_{f_{20}}(D) = f_{20}t_3 + (1 - f_{20})\, t_4 + \beta \left(f_{20}\mathbf{V}_\beta^1(15, \mathbf{f}) + (1 - f_{20})\, \mathbf{V}_\beta^1(16, \mathbf{f}) \right).$$

We also have that $\mathbf{V}_\beta^1(21, \mathbf{f}) = \frac{z}{1-\beta}$.

5.4.2 "Allocating tasks" and "Allocating tasks with punishment" strategies.

In this section we solve the dynamic programming equations and obtain the Nash Equilibrium conditions for "Allocating tasks" and "Allocating tasks with punishment" strategies. These strategies represent a new type of behaviour, which allows players "divide their labour" in the game. If we are able to obtain such Nash Equilibrium conditions in the case that games G_1 and G_2 are modelled by Prisoners' Dilemmas, then the observed type of behaviour would be that one player cooperates while the other defects. (It is possible for "Allocating tasks with punishment" strategy as we demonstrate in section 5.5.) We, therefore, will be able to use the strategy "Allocating tasks with punishment" in an explanation of cooperative behaviour. Below we find the Nash Equilibrium conditions.

Consider strategy ATP "allocating tasks with punishment". For this strategy when it is played by the second player $f_s = 0$, if $s = 3, 4, 15, 16, 17, 20$ and $f_s = 1$ at all other states. Therefore if $s = 0, 1, 2, 5, \ldots, 14$ then

$$\mathbf{V}_\beta^1(s, \mathbf{f}) = \max \left\{ \begin{array}{l} c_1 + \beta \left[\frac{p}{2} \left(\mathbf{V}_\beta^1(17, \mathbf{f}) + \mathbf{V}_\beta^1(19, \mathbf{f}) \right) + \frac{1-p}{2} \left(\mathbf{V}_\beta^1(18, \mathbf{f}) + \mathbf{V}_\beta^1(20, \mathbf{f}) \right) \right] \\ c_3 + \beta \mathbf{V}_\beta^1(21, \mathbf{f}) \end{array} \right\},$$

and if $s = 3, 4, 15, 16$ then $\mathbf{V}^1_\beta(s, \mathbf{f}) = \max\left\{c_2 + \beta\mathbf{V}^1_\beta(21, \mathbf{f}), c_4 + \beta\mathbf{V}^1_\beta(21, \mathbf{f})\right\}$. For $s = 17, \ldots, 20$ we have

$$\mathbf{V}^1_\beta(17, \mathbf{f}) = \max\left\{h_2 + \beta\mathbf{V}^1_\beta(2, \mathbf{f}), h_4 + \beta\mathbf{V}^1_\beta(4, \mathbf{f})\right\},$$
$$\mathbf{V}^1_\beta(18, \mathbf{f}) = \max\left\{t_1 + \beta\mathbf{V}^1_\beta(5, \mathbf{f}), t_3 + \beta\mathbf{V}^1_\beta(7, \mathbf{f})\right\},$$
$$\mathbf{V}^1_\beta(19, \mathbf{f}) = \max\left\{h_1 + \beta\mathbf{V}^1_\beta(9, \mathbf{f}), h_3 + \beta\mathbf{V}^1_\beta(11, \mathbf{f})\right\},$$
$$\mathbf{V}^1_\beta(20, \mathbf{f}) = \max\left\{t_2 + \beta\mathbf{V}^1_\beta(14, \mathbf{f}), t_4 + \beta\mathbf{V}^1_\beta(16, \mathbf{f})\right\}.$$

Using the following notation

$$\mathbf{V}^1_\beta(s, \mathbf{f}) = \mathbf{A}, \text{ if } s = 0, 1, 2, 5, \ldots, 14, \quad \mathbf{V}^1_\beta(s, \mathbf{f}) = \mathbf{B}, \text{ if } s = 3, 4, 15, 16,$$
$$\mathbf{V}^1_\beta(17, \mathbf{f}) = \mathbf{C}, \quad \mathbf{V}^1_\beta(18, \mathbf{f}) = \mathbf{D}, \quad \mathbf{V}^1_\beta(19, \mathbf{f}) = \mathbf{E}, \quad \mathbf{V}^1_\beta(20, \mathbf{f}) = \mathbf{F},$$

we find that the system of dynamic programming equations is as follows.

$$\begin{cases} \mathbf{A} = \max\left\{c_1 + \beta\left[\frac{p(\mathbf{C}+\mathbf{E})+(1-p)(\mathbf{D}+\mathbf{F})}{2}\right], c_3 + \beta\frac{z}{1-\beta}\right\} \\ \mathbf{B} = \max\left\{c_2 + \beta\frac{z}{1-\beta}, c_4 + \beta\frac{z}{1-\beta}\right\} \\ \mathbf{C} = \max\left\{h_2 + \beta\mathbf{A}, h_4 + \beta\mathbf{B}\right\} \\ \mathbf{D} = \max\left\{t_1 + \beta\mathbf{A}, t_3 + \beta\mathbf{A}\right\} \\ \mathbf{E} = \max\left\{h_1 + \beta\mathbf{A}, h_3 + \beta\mathbf{A}\right\} \\ \mathbf{F} = \max\left\{t_2 + \beta\mathbf{A}, t_4 + \beta\mathbf{B}\right\} \end{cases}.$$

Now, in order to be a Nash Equilibrium the strategy ATP played by first player must prescribe choosing the optimal action at every empty memory state. There are five empty memory states for this model, which are $s = 0 = G_0(i_0)$, $s = 17 = G_1(\mathcal{A})$, $s = 18 = G_2(\mathcal{A})$, $s = 19 = G_1(\mathcal{B})$ and $s = 20 = G_2(\mathcal{B})$. The probability $f_s = 1$, if $s = 0, 17, 20$, therefore choosing cooperation at these states must give higher or equal payoff than choosing defection. The probability $f_s = 0$, if $s = 18, 19$, therefore choosing defection at these states must give higher or equal payoff than choosing cooperation. Hence, we have the following necessary and sufficient conditions under which the strategy "allocating tasks with punishment" is a Nash Equilibrium.

$$\begin{cases} c_1 + \beta\left[\frac{p(\mathbf{C}+\mathbf{E})+(1-p)(\mathbf{D}+\mathbf{F})}{2}\right] \geq c_3 + \beta\frac{z}{1-\beta} \\ h_2 + \beta\mathbf{A} \geq h_4 + \beta\mathbf{B} \\ t_1 + \beta\mathbf{A} \leq t_3 + \beta\mathbf{A} \\ h_1 + \beta\mathbf{A} \leq h_3 + \beta\mathbf{A} \\ t_2 + \beta\mathbf{A} \geq t_4 + \beta\mathbf{B} \end{cases}. \tag{5.4}$$

If these conditions hold we have that

$$\begin{cases} \mathbf{A} = c_1 + \beta\left(\frac{p(\mathbf{C}+\mathbf{E})+(1-p)(\mathbf{D}+\mathbf{F})}{2}\right) \\ \mathbf{B} = \max\{c_2, c_4\} + \beta\frac{z}{1-\beta} \\ \mathbf{C} = h_2 + \beta\mathbf{A} \\ \mathbf{D} = t_3 + \beta\mathbf{A} \\ \mathbf{E} = h_3 + \beta\mathbf{A} \\ \mathbf{F} = t_2 + \beta\mathbf{A} \end{cases},$$

which can be solved to obtain

$$\mathbf{A} = \tfrac{1}{2}\frac{2c_1+\beta(p(h_2+h_3)+(1-p)(t_3+t_2))}{1-\beta^2}, \quad \mathbf{D} = t_3 + \tfrac{\beta}{2}\frac{2c_1+\beta(p(h_2+h_3)+(1-p)(t_3+t_2))}{1-\beta^2},$$
$$\mathbf{B} = \max\{c_2, c_4\} + \beta\frac{z}{1-\beta}, \quad \mathbf{E} = h_3 + \tfrac{\beta}{2}\frac{2c_1+\beta(p(h_2+h_3)+(1-p)(t_3+t_2))}{1-\beta^2},$$
$$\mathbf{C} = h_2 + \tfrac{\beta}{2}\frac{2c_1+\beta(p(h_2+h_3)+(1-p)(t_3+t_2))}{1-\beta^2}, \quad \mathbf{F} = t_2 + \tfrac{\beta}{2}\frac{2c_1+\beta(p(h_2+h_3)+(1-p)(t_3+t_2))}{1-\beta^2}.$$

Finally, substituting these expressions to the system (5.4) we obtain the necessary and sufficient conditions under which the strategy "allocating tasks with punishment" is a Nash Equilibrium, when played against itself, in a class of one-stage memory strategies.

$$t_1 \le t_3, \quad h_1 \le h_3,$$
$$h_2 + \tfrac{\beta}{2}\frac{2c_1+\beta(p(h_2+h_3)+(1-p)(t_3+t_2))}{1-\beta^2} \ge h_4 + \beta\left(\max\{c_2, c_4\} + \beta\frac{z}{1-\beta}\right),$$
$$t_2 + \tfrac{\beta}{2}\frac{2c_1+\beta(p(h_2+h_3)+(1-p)(t_3+t_2))}{1-\beta^2} \ge t_4 + \beta\left(\max\{c_2, c_4\} + \beta\frac{z}{1-\beta}\right),$$
$$2c_1 + \beta p(h_2+h_3) + \beta(1-p)(t_3+t_2) \ge 2\left(1-\beta^2\right)c_3 + 2(1+\beta)\beta z.$$

In the same way we can find the necessary and sufficient conditions under which the strategy "allocating tasks" is a Nash Equilibrium. These conditions are as follows.

$$2c_1 + \beta p(h_2+h_3) + \beta(1-p)(t_3+t_2) \ge 2\left(1-\beta^2\right)c_3 + 2(1+\beta)\beta z,$$
$$h_2 \ge h_4, \quad t_1 \le t_3, \quad h_1 \le h_3, \quad t_2 \ge t_4.$$

5.4.3 "Cooperator with punishment" and "Sucker" strategies.

In this section we consider "Cooperator with punishment" and "Sucker" strategies which are similar to Tit for Tat and All C strategies, respectively, in the Iterated Prisoners' Dilemma.

For the strategy CP "cooperator with punishment" when it is played by the second player $f_s = 0$, if $s = 3, 4, 7, 8, 11, 12, 15, 16$ and $f_s = 1$ at all other states. Therefore if

$s = 0, 1, 2, 5, 6, 9, 10, 13, 14$

$$\mathbf{V}_{\beta}^{1}(s,\mathbf{f}) = \max\left\{\begin{array}{l} c_1 + \beta\left[\frac{p}{2}\left(\mathbf{V}_{\beta}^{1}(17,\mathbf{f}) + \mathbf{V}_{\beta}^{1}(19,\mathbf{f})\right) + \frac{1-p}{2}\left(\mathbf{V}_{\beta}^{1}(18,\mathbf{f}) + \mathbf{V}_{\beta}^{1}(20,\mathbf{f})\right)\right] \\ c_3 + \beta\mathbf{V}_{\beta}^{1}(21,\mathbf{f}) \end{array}\right\},$$

and if $s = 3, 4, 7, 8, 11, 12, 15, 16$

$$\mathbf{V}_{\beta}^{1}(s,\mathbf{f}) = \max\left\{c_2 + \beta\mathbf{V}_{\beta}^{1}(21,\mathbf{f}), c_4 + \beta\mathbf{V}_{\beta}^{1}(21,\mathbf{f})\right\}.$$

For $s = 17, 18, 19, 20$ we have

$$\begin{aligned} \mathbf{V}_{\beta}^{1}(17,\mathbf{f}) &= \max\left\{h_1 + \beta\mathbf{V}_{\beta}^{1}(1,\mathbf{f}), h_3 + \beta\mathbf{V}_{\beta}^{1}(3,\mathbf{f})\right\}, \\ \mathbf{V}_{\beta}^{1}(18,\mathbf{f}) &= \max\left\{t_1 + \beta\mathbf{V}_{\beta}^{1}(5,\mathbf{f}), t_3 + \beta\mathbf{V}_{\beta}^{1}(7,\mathbf{f})\right\}, \\ \mathbf{V}_{\beta}^{1}(19,\mathbf{f}) &= \max\left\{h_1 + \beta\mathbf{V}_{\beta}^{1}(9,\mathbf{f}), h_3 + \beta\mathbf{V}_{\beta}^{1}(11,\mathbf{f})\right\}, \\ \mathbf{V}_{\beta}^{1}(20,\mathbf{f}) &= \max\left\{t_1 + \beta\mathbf{V}_{\beta}^{1}(13,\mathbf{f}), t_3 + \beta\mathbf{V}_{\beta}^{1}(15,\mathbf{f})\right\}. \end{aligned}$$

Let us introduce the following notation

$$\begin{aligned} &\mathbf{V}_{\beta}^{1}(s,\mathbf{f}) = \mathbf{A}, \text{ if } s = 0, 1, 2, 5, 6, 9, 10, 13, 14, \\ &\mathbf{V}_{\beta}^{1}(s,\mathbf{f}) = \mathbf{B}, \text{ if } s = 3, 4, 7, 8, 11, 12, 15, 16, \\ &\mathbf{V}_{\beta}^{1}(17,\mathbf{f}) = \mathbf{V}_{\beta}^{1}(19,\mathbf{f}) = \mathbf{C}, \quad \mathbf{V}_{\beta}^{1}(18,\mathbf{f}) = \mathbf{V}_{\beta}^{1}(20,\mathbf{f}) = \mathbf{D}. \end{aligned}$$

Then we can rewrite the equations as follows.

$$\begin{aligned} &\mathbf{A} = \max\left\{c_1 + \beta p\mathbf{C} + \beta(1-p)\mathbf{D}, c_3 + \beta\tfrac{z}{1-\beta}\right\}, \\ &\mathbf{B} = \max\left\{c_2 + \beta\tfrac{z}{1-\beta}, c_4 + \beta\tfrac{z}{1-\beta}\right\}, \\ &\mathbf{C} = \max\left\{h_1 + \beta\mathbf{A}, h_3 + \beta\mathbf{B}\right\}, \quad \mathbf{D} = \max\left\{t_1 + \beta\mathbf{A}, t_3 + \beta\mathbf{B}\right\}. \end{aligned} \tag{5.5}$$

Now, in order to be a Nash Equilibrium the strategy CP played by the first player must prescribe choosing the optimal action at every empty memory state. There are five empty memory states for this model, which are $s = 0 = G_0(i_0)$, $s = 17 = G_1(\mathcal{A})$, $s = 18 = G_2(\mathcal{A})$, $s = 19 = G_1(\mathcal{B})$ and $s = 20 = G_2(\mathcal{B})$. The probability $f_s = 1$, if $s = 0, 17, 18, 19, 20$, and therefore choosing cooperation at these states must give higher or equal payoff than choosing defection. Since if $s = 0$ then $\mathbf{V}_{\beta}^{1}(s,\mathbf{f}) = \mathbf{A}$, if $s = 17, 19$ then $\mathbf{V}_{\beta}^{1}(s,\mathbf{f}) = \mathbf{C}$, and if $s = 18, 20$ then $\mathbf{V}_{\beta}^{1}(s,\mathbf{f}) = \mathbf{D}$, and, noticing that $u_{f_s}(C)$ in this case is given by the first expression in the brackets (5.5), we have the following necessary and sufficient conditions under which the strategy "cooperator with punishment" is a Nash Equilibrium.

$$\begin{cases} h_1 + \beta\mathbf{A} \geq h_3 + \beta\mathbf{B} \\ t_1 + \beta\mathbf{A} \geq t_3 + \beta\mathbf{B}, \\ c_1 + \beta p\mathbf{C} + \beta(1-p)\mathbf{D} \geq c_3 + \beta\frac{z}{1-\beta} \end{cases}. \tag{5.6}$$

If these conditions are satisfied then we have that

$$\begin{cases} \mathbf{A} = c_1 + \beta p\mathbf{C} + \beta(1-p)\mathbf{D} \\ \mathbf{B} = \max\{c_2, c_4\} + \beta\frac{z}{1-\beta} \\ \mathbf{C} = h_1 + \beta\mathbf{A} \\ \mathbf{D} = t_1 + \beta\mathbf{A} \end{cases}.$$

Solving this system we obtain

$$\begin{array}{ll} \mathbf{A} = \frac{c_1+\beta(ph_1+(1-p)t_1)}{1-\beta^2}, & \mathbf{C} = h_1 + \beta\frac{c_1+\beta(ph_1+(1-p)t_1)}{1-\beta^2}, \\ \mathbf{B} = \max\{c_2, c_4\} + \beta\frac{z}{1-\beta}, & \mathbf{D} = t_1 + \beta\frac{c_1+\beta(ph_1+(1-p)t_1)}{1-\beta^2}. \end{array} \tag{5.7}$$

Substituting expressions (5.7) to the system (5.6) we obtain the following necessary and sufficient conditions under which the strategy "cooperator with punishment" is a Nash Equilibrium, when played against itself, in a class of one-stage memory strategies for the multi-state game.

$$\begin{array}{l} h_1 + \beta\frac{c_1+\beta(ph_1+(1-p)t_1)}{1-\beta^2} \geq h_3 + \beta\left(\max\{c_2, c_4\} + \beta\frac{z}{1-\beta}\right), \\ t_1 + \beta\frac{c_1+\beta(ph_1+(1-p)t_1)}{1-\beta^2} \geq t_3 + \beta\left(\max\{c_2, c_4\} + \beta\frac{z}{1-\beta}\right), \\ c_1 + \beta(ph_1 + (1-p)t_1) \geq (\beta z + c_3(1-\beta))(1+\beta). \end{array}$$

In the same way we can obtain the following sufficient and necessary conditions under which the strategy "sucker" is a Nash Equilibrium. This conditions are as follows.

$$c_1 + \beta(ph_1 + (1-p)t_1) \geq (1+\beta)((1-\beta)c_3 + \beta z), \quad h_1 \geq h_3, \quad t_1 \geq t_3.$$

5.4.4 "Looking for a sucker", "Pathological" and "Unsociable" strategies.

In the same way as it has been done for strategies "cooperator with punishment" and "allocating tasks with punishment" we can solve the dynamic programming equations and obtain Nash Equilibrium conditions for every strategy introduced in section 5.3.3. Table 5.5 contains the summary of the results obtained.

Table 5.5. Conditions on Nash equilibria for non-cooperative strategies.

Strategy	The following conditions must be satisfied simultaneously in order for a strategy to be a Nash Equilibrium when played against itself.
P	$c_1 + \beta(ph_4 + (1-p)t_4) \geq (1+\beta)((1-\beta)c_3 + \beta z), \quad h_2 \leq h_4, \quad t_2 \leq t_4.$
LFS	$h_2 + \beta\frac{c_1+\beta(ph_4+(1-p)t_4)}{1-\beta^2} \leq h_4 + \beta\left(\max\{c_2, c_4\} + \beta\frac{z}{1-\beta}\right),$ $t_2 + \beta\frac{c_1+\beta(ph_4+(1-p)t_4)}{1-\beta^2} \leq t_4 + \beta\left(\max\{c_2, c_4\} + \beta\frac{z}{1-\beta}\right),$ $c_1 + \beta(ph_4 + (1-p)t_4) \geq (1+\beta)((1-\beta)c_3 + \beta z).$
U	$h_2 \leq h_4, \quad t_2 \leq t_4, \quad c_2 \leq c_4$

5.5 Examples.

In the this section we consider examples which demonstrate how the Nash Equilibrium conditions obtained in the previous section depend on the parameters of the game.

Example 5.1. Consider the game given by table 5.6.

Table 5.6. The multi-state game: example 5.1.

G_0:

$P_1 \backslash P_2$	A	B
A	$(3,3)/[0,p,1-p,0]$	$(0,6)/[0,0,0,1]$
B	$(6,0)/[0,0,0,1]$	$(-\frac{1}{2},-\frac{1}{2})/[0,0,0,1]$

G_3:

$P_1 \backslash P_2$	L
L	$(z,z)/[0,0,0,1]$

G_1:

$P_1 \backslash P_2$	C	D
C	$(h_1,h_1)/[1,0,0,0]$	$(h_2,h_3)/[1,0,0,0]$
D	$(h_3,h_2)/[1,0,0,0]$	$(h_4,h_4)/[1,0,0,0]$

G_2:

$P_1 \backslash P_2$	C	D
C	$(t_1,t_1)/[1,0,0,0]$	$(t_2,t_3)/[1,0,0,0]$
D	$(t_3,t_2)/[1,0,0,0]$	$(t_4,t_4)/[1,0,0,0]$

Where payoffs in games G_1 and G_2 are such that

$$ph_1 + t_1(1-p) = 3, \quad ph_3 + t_3(1-p) = 5, \quad ph_2 + t_2(1-p) = 0, \quad ph_4 + t_4(1-p) = 1.$$

For example, the payoffs in games G_1 and G_2 can be equal, that is

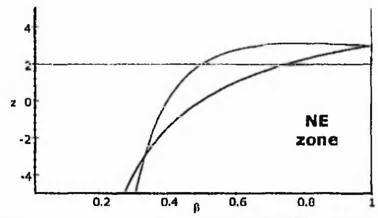
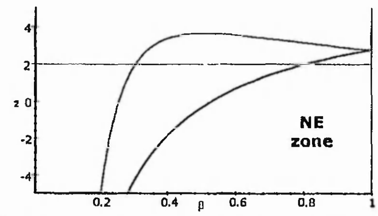
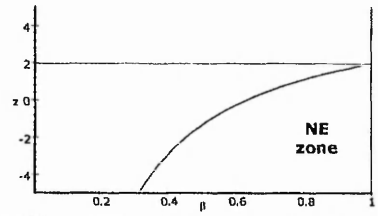
$$h_1 = 3, \quad h_2 = 0, \quad h_3 = 5, \quad h_4 = 1; \quad t_1 = 3, \quad t_2 = 0, \quad t_3 = 5, \quad t_4 = 1.$$

The association game G_0 is modeled by a Hawk-Dove game which is commonly used when the sharing aspect of interaction is considered. Such a game can be interpreted in the following way.

> **"Sharing Animals" interpretation.** There are two animals who have to decide whether or not to share a common resource, for example, a hunting territory. This is modeled by game G_0. If both animals decide on sharing (choose action A in game G_0) then they have to protect jointly held territory from two different kinds of intruders (for example, protecting territory from invasion by the same kind of animals and by the animals of another species). This is represented by two Prisoners' Dilemmas: G_1 and G_2. If one of the players chooses B and the other chooses A in the association game it is interpreted as the first player being prepared to fight for the possession of the resource and the second player choosing to run away. If both players choose B, then they fight and each one has equal probability $\frac{1}{2}$ of gaining the resource. The cost of fighting is supposed to be higher than the value of the resource. Once the association is broken the player's future payoffs are then equal to z. It is supposed that neither of the players has sufficient power to protect the resource from intruders on its own.

In this case strategies S, AT, LFS and U are not Nash Equilibria for any value of the parameter z and discount factor $\beta \in [0, 1)$. For strategies CP, ATP and P there is a range of values for z and β for which these strategies are Nash Equilibria. The conditions on the parameters for these strategies are given in the table 5.7 below together with plots that show the range of the acceptable parameters. In chapter 7 we will also analyse the corresponding Replicator Dynamics for this model. In particular we will concentrate on the case when parameter $z = 2$. This value is shown in green on the plots below.

Table 5.7. Nash Equlibrim conditions: example 5.1.

Strategy	Conditions	Plot
CP	$z \le \frac{5\beta-2}{\beta^2}$ – *blue* $z \le 3\frac{2\beta-1}{\beta}$ – *black*	
ATP	$z \le \frac{1}{2}\frac{7\beta^2+6\beta-2}{\beta^2(\beta+1)}$ – *blue* $z \le \frac{1}{2}\frac{-6+12\beta^2+5\beta}{\beta(\beta+1)}$ – *black*	
P	$z \le \frac{6\beta^2-3+\beta}{\beta(\beta+1)}$ – *blue*	

Analysing plots in Table 5.7 we can obtain the following results.

1. It is possible to choose the parameters z and β in such a way that the "allocating tasks with punishment" strategy is a Nash Equilibrium. Using this strategy one player cooperates in game G_1 while the other player defects and the first player defects in game G_2 while the other cooperates. This is a type of behaviour which cannot be observed if the interaction is modeled by a single Prisoners' Dilemma and it may be viewed as relevant to the explanation of altruistic behaviour.

2. Notice that the Nash Equilibrium conditions do not depend on value of the probability p if the payoffs in games G_1 and G_2 are equal. We therefore can consider that p is equal to zero or one. For example, if game G_1 represents protecting territory from invasion by the animals of the same kind and G_2 by the animals of the other species and the animals of the other species become extinct when the probability of playing G_2 becomes zero, which corresponds to the value of p becoming equal to one. Surprisingly

the strategy "allocating tasks with punishment" continues to be a Nash Equilibrium in this situation. This is due to the role dependant definition of the strategy: the strategy carries in itself both the program of actions for cooperating and for defecting player and at the beginning of the multi-state game a rule determines which role each player is assigned, but the players have equal probabilities to be assigned either of the roles. The observed behavior in this case is non reciprocal cooperation: one player cooperates while the other defects. Such types of behaviour cannot arise in the Iterated Prisoners' Dilemma game.

3. It is possible to choose parameters in such a way that non cooperative strategies such as LFS, P and U are no longer Nash Equilibria. These strategies are generalisations of the $All\ Defect$ strategy for the Iterated Prisoners' Dilemma game which is a Nash Equilibrium for any combination of the parameters (see chapter 4). It seems that the possibility to discontinue the association is the main factor that changes the situation. These results do not mean that there is no non cooperative strategy that can be a Nash Equilibrium (since we are unable to analyse even all one-stage memory strategies) but they open an interesting direction that may be useful when an attempt to explain the evolution of an altruistic behaviour is made.

4. It is possible to choose the parameters z and β in such a way that the "cooperating with punishment" strategy is a Nash Equilibrium. This strategy is similar to Tit for Tat strategy of the Prisoners' Dilemma models and therefore is relevant to the explanation of cooperative behaviour.

5. Notice that the range of parameters for which the necessary conditions are satisfied for all the above cases at the same time is quite narrow: β should be quite high and z should not be too small or too high.

Example 5.2. Consider the game given by table 5.8. Here the payoffs in the context-game G_0 may represent an extra cost of the association.

Table 5.8. The multi-state game: example 5.2.

G_0:

P1\P2	A	B
A	$(-\frac{1}{2},-\frac{1}{2})/[0,p,1-p,0]$	$(0,0)/[0,0,0,1]$
B	$(0,0)/[0,0,0,1]$	$(0,0)/[0,0,0,1]$

G_3:

P1\P2	L
L	$(z,z)/[0,0,0,1]$

G_1:

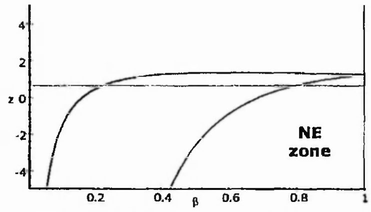
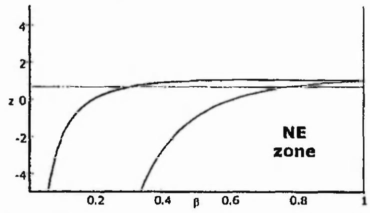
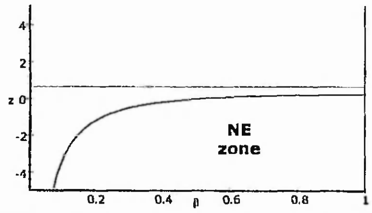
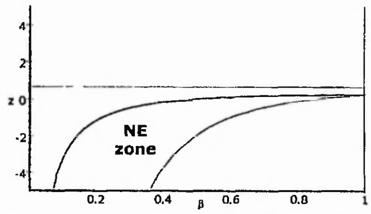
P1\P2	C	D
C	$(3,3)/[1,0,0,0]$	$(0,5)/[1,0,0,0]$
D	$(5,0)/[1,0,0,0]$	$(1,1)/[1,0,0,0]$

G_2:

P1\P2	C	D
C	$(3,3)/[1,0,0,0]$	$(0,5)/[1,0,0,0]$
D	$(5,0)/[1,0,0,0]$	$(1,1)/[1,0,0,0]$

In this example the payoffs in games G_1 and G_2 are equal, so the Nash equilibrium conditions do not depend on value of the probability p. For this example we have that S and AT are not Nash Equilibria for any value of the parameter z and discount factor $\beta \in [0,1)$. Strategy U earns the same total value against every strategy and every strategy earns the same total value against U, therefore U is a Nash Equilibrium for any value of parameter z and $\beta \in [0,1)$. For strategies CP, ATP, LFS and P there are ranges of values for z and β for which these strategies are Nash Equilibria (see table 5.9). In chapter 7 we will analyse the Replicator Dynamics for the populations that consist of proportions of CP, ATP, P and U players. In particular we will consider the case when parameter $z = \frac{2}{3}$. This value is shown in green on the plots below.

Table 5.9. Nash Equlibrim conditions: example 5.2.

Strategy	Conditions	Plot
CP	$z \leq \frac{1}{2}\frac{10\beta^2-\beta-4}{\beta^2(\beta+1)}$ – *blue* $z \leq \frac{1}{2}\frac{-1+6\beta}{\beta(\beta+1)}$ – *black*	
ATP	$z \leq \frac{1}{2}\frac{7\beta^2-\beta-2}{\beta^2(\beta+1)}$ – *blue* $z \leq \frac{1}{2}\frac{-1+5\beta}{\beta(\beta+1)}$ – *black*	
P	$z \leq \frac{1}{2}\frac{-1+2\beta}{\beta(\beta+1)}$ – *blue*	
LFS	$z \geq \frac{1}{2}\frac{4\beta^2-\beta-2}{\beta^2(\beta+1)}$ – *blue* $z \leq \frac{1}{2}\frac{-1+2\beta}{\beta(\beta+1)}$ – *black*	

The conclusions obtained for this example are similar to those drawn for example 5.1, with the exception that strategy U is a Nash Equilibrium for any value of parameter z and $\beta \in [0, 1)$. Since it earns the same total value against every strategy and every strategy earns the same total value against U, it is not an evolutionarily stable. The evolutionary properties of this strategy can be determined by considering the Replicator Dynamics and using the "blowing up" technique described in chapter 3. The results of such analysis for this example can be found in chapter 7.1.

Example 5.3. Consider the game given by table 5.10. This example is similar to example 5.2 and the payoffs in the context-game G_0 again represent an extra cost of the association.

Table 5.10. The multi-state game: example 5.3.

G_0:

$P_1 \backslash P_2$	A	B
A	$(-1,-1)/[0,p,1-p,0]$	$(0,0)/[0,0,0,1]$
B	$(0,0)/[0,0,0,1]$	$(0,0)/[0,0,0,1]$

G_3:

$P_1 \backslash P_2$	L
L	$(1,1)/[0,0,0,1]$

G_1:

$P_1 \backslash P_2$	C	D
C	$(3,3)/[1,0,0,0]$	$(0,5)/[1,0,0,0]$
D	$(5,0)/[1,0,0,0]$	$(1,1)/[1,0,0,0]$

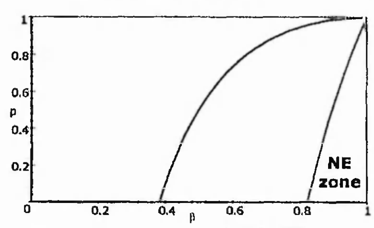
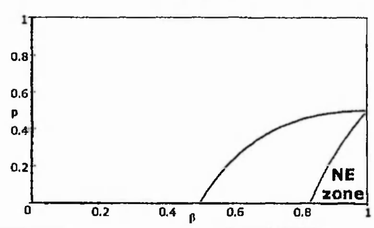
G_2:

$P_1 \backslash P_2$	C	D
C	$(4,4)/[1,0,0,0]$	$(1,6)/[1,0,0,0]$
D	$(6,1)/[1,0,0,0]$	$(2,2)/[1,0,0,0]$

In this example the payoffs in games G_1 and G_2 are different, and therefore the Nash equilibrium conditions depend on value of the probability p. In this example we fix the value of the payoff z to be equal 1, and investigate how the Nash equilibrium conditions depend on value of the probability p and discount factor β.

In this case strategies S, AT, LFS and P are not Nash Equilibria for any values of the probability $p \in [0,1]$ and discount factor $\beta \in [0,1)$. Strategy U again earns the same total value against every strategy and every strategy earns the same total value against U, therefore U is a Nash Equilibrium for any value of $p \in [0,1]$ and $\beta \in [0,1)$. For strategies CP and ATP there are ranges of values for p and β for which these strategies are Nash Equilibria (see table 5.11).

Table 5.11. Nash Equlibrim conditions: example 5.3.

Strategy	Conditions	Plot
CP	$\begin{cases} p \le \frac{-\beta^3+5\beta^2-\beta-2}{\beta^2} & - \textit{blue} \\ p \le \frac{3\beta-\beta^2-1}{\beta} & - \textit{black} \end{cases}$	
ATP	$\begin{cases} p \le \frac{7\beta^2-2\beta^3-2\beta-2}{2\beta^2} & - \textit{blue} \\ p \le \frac{5\beta-2\beta^2-2}{2\beta} & - \textit{black} \end{cases}$	

To obtain strategies CP and ATP as Nash Equilibria the value of the discount factor β must be quite close to one, which means that the expected time of association is long. For every fixed β there is a barrier values p^{β}_{CP} and p^{β}_{ATP} of probability p such that if p is greater than p^{β}_{CP} or p^{β}_{ATP} then strategies CP or ATP, respectively, are not Nash Equilibria. The barrier values appear because the payoffs in game G_1 are lower than in game G_2, so if game G_1 is played with high probability it becomes inefficient to continue an association. The barrier values also depend on the value of the parameter z and decrease as z increases.

Example 5.4. Consider the game given by table 5.12. This example is similar to the example 5.1. Here context-game G_0 is modelled again by a Hawk-Dove Game. Again we fix the value of the payoff z to be equal 1, and investigate how the Nash equilibrium conditions depend on value of the probability p and discount factor β.

Table 5.12. The multi-state game: example 5.4.

G_0:

P1\P2	A	B
A	$(\frac{1}{2},\frac{1}{2})/[0,p,1-p,0]$	$(0,1)/[0,0,0,1]$
B	$(1,0)/[0,0,0,1]$	$(-\frac{1}{2},-\frac{1}{2})/[0,0,0,1]$

G_3:

P1\P2	L
L	$(1,1)/[0,0,0,1]$

G_1:

P1\P2	C	D
C	$(3,3)/[1,0,0,0]$	$(0,5)/[1,0,0,0]$
D	$(5,0)/[1,0,0,0]$	$(1,1)/[1,0,0,0]$

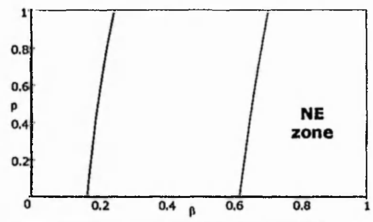
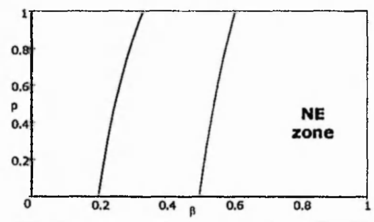
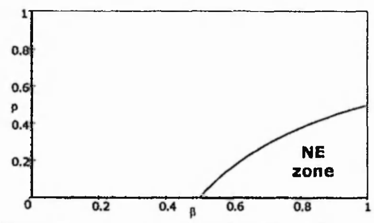
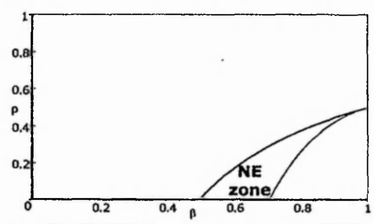
G_2:

P1\P2	C	D
C	$(4,4)/[1,0,0,0]$	$(1,6)/[1,0,0,0]$
D	$(6,1)/[1,0,0,0]$	$(2,2)/[1,0,0,0]$

In this case strategies S, AT, and U are not Nash Equilibria for any values of the probability $p \in [0,1]$ and discount factor $\beta \in [0,1)$. For strategies CP, ATP, LFS and P we can determine a range of values for p and β for which these strategies are Nash Equilibria. The conditions on the parameters for these strategies are summarised in the table 5.13 below.

The conclusions in this case are similar to those for example 5.1. The ranges of parameters p and β for which strategies CP and ATP are Nash Equilibria depend on the value of the payoff z. The ranges are wide since the value of z is quite low (equal to 1). As the value of z increases these ranges will become narrower.

Table 5.13. Nash Equlibrim conditions: example 5.4.

Strategy	Conditions	Plot
CP	$\begin{cases} p \leq \frac{10\beta^2+\beta-4-2\beta^3}{2\beta^2} - blue \\ p \leq \frac{6\beta-1}{2\beta} - black \end{cases}$	
ATP	$\begin{cases} p \leq \frac{7\beta^2+\beta-2-2\beta^3}{2\beta^2} - blue \\ p \leq \frac{5\beta-1}{2\beta} - black \end{cases}$	
P	$p \leq \frac{2\beta-1}{2\beta} - black$	
LFS	$\begin{cases} p \geq \frac{4\beta^2+\beta-2-2\beta^3}{2\beta^2} - blue \\ p \leq \frac{2\beta-1}{2\beta} - black \end{cases}$	

5.6 Summary.

In this chapter we have introduced a multi-state game model that allows for the possibility that individuals may interact in more than one context. We have constructed the Markov process for the one-stage memory model and obtained the expression for the total payoffs values which we will use in the next chapter to define the Replicator Dynamics for this model.

We introduced a new "allocating tasks" type of behaviour. It was shown that the strategy "allocating tasks with punishment" can be a symmetric Nash Equilibrium for a certain range of parameters. Since this strategy may be viewed as relevant to explanation of both reciprocal and non-reciprocal (see example 5.1 for more details) cooperative behaviour this result is particularly important.

It has also been shown that there exists a range of parameters of the model for which non-cooperative strategies such as "looking for a sucker", "pathological" or "unsociable" are not Nash Equilibria. In all models based on the Iterated Prisoners' Dilemma the uncooperative type of behaviour is a strict Nash Equilibrium and, therefore, cooperative behavior cannot evolve from a population where all players use the *All Defect* strategy. In this instance, the result obtained for multi-state model means that there is the possibility of evolving to cooperative or altruistic population from anarchy. This possibility will be examined in the next two chapters where we analyse the Replicator Dynamics and evolutionary stability properties of the strategies introduced in this chapter.

Chapter 6

Multi-state games: Replicator Dynamics.

In this chapter we start the analysis of the Replicator Dynamics for the multi-state game model introduced in the previous chapter. We obtain the conditions for the asymptotic stability and instability of the non cooperative population states. We will also obtain sufficient conditions for which there are subsets of the intervals corresponding to the cooperative or allocating tasks populations such that the conditions of theorem 4.2 holds (this theorem deals with conditions of the setwise evolutionary attraction). This will be done for a generic set of parameters. We will use these results in the next chapter where we consider specific examples.

6.1 Introduction.

6.1.1 The two-person game.

The Replicator Dynamics which we will analyse in this chapter describes changes of a population state in a population whose members are playing a symmetric two-person game. The individuals in the population are allowed to adopt the behaviours which correspond to the seven strategies considered in the previous chapter. That is: S, CP, ATP, AT, P, LFS and U. To calculate the total payoff value obtained by these strategies we use the Markov Decision Process constructed for the multi-state game in chapter 5. Using formula (5.3), we calculate the total reward $\pi(f_1,f_2) = v^1_\beta(1,f_1,f_2)$ for the first player. Information about these rewards is summarised in matrix A. "Maple" software was used to obtain the exact formulae in this case.

$$A = \begin{bmatrix} \pi(\text{S,S}) & \pi(\text{S,CP}) & \pi(\text{S,ATP}) & \pi(\text{S,AT}) & \pi(\text{S,P}) & \pi(\text{S,LFS}) & \pi(\text{S,U}) \\ \pi(\text{CP,S}) & \pi(\text{CP,CP}) & \pi(\text{CP,ATP}) & \pi(\text{CP,AT}) & \pi(\text{CP,P}) & \pi(\text{CP,LFS}) & \pi(\text{CP,U}) \\ \pi(\text{ATP,S}) & \pi(\text{ATP,CP}) & \pi(\text{ATP,ATP}) & \pi(\text{ATP,AT}) & \pi(\text{ATP,P}) & \pi(\text{ATP,LFS}) & \pi(\text{ATP,U}) \\ \pi(\text{AT,S}) & \pi(\text{AT,CP}) & \pi(\text{AT,ATP}) & \pi(\text{AT,AT}) & \pi(\text{AT,P}) & \pi(\text{AT,LFS}) & \pi(\text{AT,U}) \\ \pi(\text{P,S}) & \pi(\text{P,CP}) & \pi(\text{P,ATP}) & \pi(\text{P,AT}) & \pi(\text{P,P}) & \pi(\text{P,LFS}) & \pi(\text{P,U}) \\ \pi(\text{LFS,S}) & \pi(\text{LFS,CP}) & \pi(\text{LFS,ATP}) & \pi(\text{LFS,AT}) & \pi(\text{LFS,P}) & \pi(\text{LFS,LFS}) & \pi(\text{LFS,U}) \\ \pi(\text{U,S}) & \pi(\text{U,CP}) & \pi(\text{U,ATP}) & \pi(\text{U,AT}) & \pi(\text{U,P}) & \pi(\text{U,LFS}) & \pi(\text{U,U}) \end{bmatrix}$$

$$= \begin{bmatrix} 2\varkappa & 2\varkappa & \psi+\varkappa & \psi+\varkappa & 2\psi & 2\psi & C_2 \\ 2\varkappa & 2\varkappa & \frac{2(1-\beta^2)(\psi+\varkappa)+\beta^2 C_3}{(2-\beta^2)} & \frac{2(1-\beta^2)(\psi+\varkappa)+\beta^2 C_3}{(2-\beta^2)} & 2(1-\beta^2)\psi+\beta^2 C_3 & 2(1-\beta^2)\psi+\beta^2 C_3 & C_2 \\ \varkappa+\omega & \frac{2(1-\beta^2)(\varkappa+\omega)+\beta^2 C_2}{(2-\beta^2)} & \psi+\omega & \psi+\omega & \frac{2(1-\beta^2)(\psi+\chi)+\beta^2 C_3}{(2-\beta^2)} & \frac{2(1-\beta^2)(\psi+\chi)+\beta^2 C_4}{(2-\beta^2)} & C_2 \\ \varkappa+\omega & \frac{2(1-\beta^2)(\varkappa+\omega)+\beta^2 C_2}{(2-\beta^2)} & \psi+\omega & \psi+\omega & \psi+\chi & \frac{2(1-\beta^2)(\psi+\chi)+\beta^2 C_2}{(2-\beta^2)} & C_2 \\ 2\omega & 2(1-\beta^2)\omega+\beta^2 C_2 & \frac{2(1-\beta^2)(\chi+\omega)+\beta^2 C_2}{(2-\beta^2)} & \omega+\chi & 2\chi & 2(1-\beta^2)\chi+\beta^2 C_2 & C_2 \\ 2\omega & 2(1-\beta^2)\omega+\beta^2 C_2 & \frac{2(1-\beta^2)(\chi+\omega)+\beta^2 C_4}{(2-\beta^2)} & \frac{2(1-\beta^2)(\chi+\omega)+\beta^2 C_3}{(2-\beta^2)} & 2(1-\beta^2)\chi+\beta^2 C_3 & 2(1-\beta^2)\chi+\beta^2 C_4 & C_2 \\ C_3 & C_3 & C_3 & C_3 & C_3 & C_3 & C_4 \end{bmatrix} \tag{6.1}$$

Here

$$\varkappa = \frac{1}{2}\frac{c_1+\beta(ph_1+t_1(1-p))}{1-\beta^2}; \quad \omega = \frac{1}{2}\frac{c_1+\beta(ph_3+t_3(1-p))}{1-\beta^2};$$
$$\psi = \frac{1}{2}\frac{c_1+\beta(ph_2+t_2(1-p))}{1-\beta^2}; \quad \chi = \frac{1}{2}\frac{c_1+\beta(ph_4+t_4(1-p))}{1-\beta^2};$$
$$C_2 = c_2 + \frac{\beta z}{(1-\beta)}; \qquad C_3 = c_3 + \frac{\beta z}{(1-\beta)}; \qquad C_4 = c_4 + \frac{\beta z}{(1-\beta)}.$$

Notice that some payoffs given in matrix A are equal. For example,

$$\pi(S,S) = \pi(S,CP) = \pi(CP,S) = \pi(CP,CP),$$
$$\pi(AT,AT) = \pi(AT,ATP) = \pi(ATP,AT) = \pi(ATP,ATP).$$

Equal payoffs commonly appear in situations where the only difference between some strategies present in the population is that one uses punishment and another does not. This may explain the fact that the cooperative strategy with punishment (such as *TFT*, *CP* or *ATP*) can not be evolutionarily stable in the way described by J. Maynard Smith and G. R. Price [13]. Therefore other methods are used in this situation to investigate the outcome of the evolution process. The Replicator Dynamics provides us with the necessary technique.

Remark 6.1 *To simplify the analysis in further consideration it is assumed that the payoffs which are given by different expressions in each column of the matrix A are not equal. This assumptions holds for generic choice of the parameters of the multi-state game.*

6.1.2 Cooperative, relatively cooperative and non-cooperative strategies.

An exact definition of cooperative behaviour in a game-theoretic sense is given below. Definition 6.1 formalises the concept used in [52] for describing cooperative and non-cooperative strategies in a pairwise contest.

Definition 6.1 *Strategy* f_1 *(a type of behaviour) is called*

- *cooperative with respect to strategy* f_2 *if the following conditions hold*

$$\pi(\mathrm{f}_1, \mathrm{f}_1) > \pi(\mathrm{f}_2, \mathrm{f}_2) \tag{6.2}$$

  and

$$2\pi(\mathrm{f}_1, \mathrm{f}_1) > \pi(\mathrm{f}_1, \mathrm{f}_2) + \pi(\mathrm{f}_2, \mathrm{f}_1) \tag{6.3}$$

  or

$$\pi(\mathrm{f}_1, \mathrm{f}_1) = \pi(\mathrm{f}_1, \mathrm{f}_2) = \pi(\mathrm{f}_2, \mathrm{f}_1) = \pi(\mathrm{f}_2, \mathrm{f}_2),$$

  where $\pi(\mathrm{f}_i, \mathrm{f}_j)$ *is the total reward for strategy* f_i *playing against* f_j*;*

- *non-cooperative with respect to strategy* f_2 *if at least one of the above conditions is not satisfied.*

Definition 6.2 generalises Definition 6.1 so it can be applied to a population of players.

Definition 6.2 *Consider a population whose members are playing a symmetric two-person game. Strategy* f_1 *(a type of behaviour) is called*

- *cooperative with respect to the population if it is cooperative with respect to every strategy* $\mathbf{f}_2$ *that may be present in the population;*
- *partially cooperative with respect to the population if there exists a strategy* $\mathbf{f}_2 \neq \mathbf{f}_1$ *in the population such that* $\mathbf{f}_1$ *is a cooperative strategy with respect to the strategy* $\mathbf{f}_2$;
- *non-cooperative with respect to the population if it is non-cooperative with respect to any strategy* $\mathbf{f}_2 \neq \mathbf{f}_1$ *that is present in the population;*

Proposition 6.1 *Let us consider the population whose members are playing a symmetric two-person game with the payoffs represented by matrix A (see formula (6.1)). Then*

- *S and CP are cooperative strategies with respect to any strategy in this population if*

$$4\varkappa > \max\{2(\psi+\omega), 4\chi, 2C_4, C_2+C_3\}. \tag{6.4}$$

- *Assuming that 6.4 holds, AT and ATP are cooperative strategies with respect to themselves and P, LFS and U if*

$$2(\psi+\omega) > \max\{4\chi, 2C_4, C_2+C_3\}. \tag{6.5}$$

- *Assuming that 6.4 and 6.5 hold,*

  P is cooperative strategy with respect to LFS and U if

$$4\chi > \max\{2C_4, C_2+C_3\}.$$

  LFS is cooperative strategy with respect to P if

$$2C_4 > \max\{4\chi, C_2+C_3\}$$

  and with respect to U if

$$\begin{cases} 2\chi > C_4 \\ 4\left(1-\beta^2\right)\chi + 2\beta^2 C_4 > C_2 + C_3 \end{cases}.$$

  U is cooperative strategy with respect to P and LFS if

$$2C_4 > \max\{4\chi, C_2+C_3\}.$$

Therefore

- *S and CP are cooperative strategies with respect to the population;*
- *AT and ATP are partially cooperative strategies with respect to the population;*
- *P, LFS and U are non-cooperative strategies with respect to the population*

if and only if the following conditions hold

$$4\varkappa > 2(\psi + \omega) > C_2 + C_3 > \max\{2C_4, 4\chi\}. \qquad (6.6)$$

Proof. Proposition 6.1 can be obtained by verifying conditions (6.2) and (6.3) for each particular strategy.

6.1.3 Dynamical system.

The Replicator Dynamics for the multi-state game describe changes of a population state $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in a population whose members are playing a symmetric two-person game with the payoffs given by the matrix A (6.1). Here we denote by x_1, x_2, x_3, x_4, x_5, x_6 and x_7 the proportion of the individuals in the population who adopt behaviour S, CP, ATP, AT, P, LFS and U, respectively.

In this case the number of equations of the Replicator Dynamics (4.2) can be reduced to six, giving

$$\dot{x}_i = \frac{dx_i}{dt} = G_i(x) = x_i\left(\left\{\sum_{j=1}^{6}(a_{ij} - a_{i7})\, x_j + a_{i7}\right\} - \mu(x)\right), \qquad i = 1, \dots 6. \qquad (6.7)$$

Here a_{ij} is an element of the matrix A (6.1), $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ and

$$\mu(x) = (x_1, \dots, x_6, 1 - x_1 - \dots - x_6)\, A\, (x_1, \dots, x_6, 1 - x_1 - \dots - x_6)^T.$$

6.2 Stability properties of vertices.

6.2.1 Conditions of asymptotic stability for strategies P, LFS and U.

In this section we formulate necessary and sufficient conditions under which the non-cooperative types of behaviour P, LFS and U are asymptotically stable.

Proposition 6.2 *The necessary and sufficient conditions of asymptotic stability for the points* $(0,0,0,0,0,0)$, $(0,0,0,0,1,0)$ *and* $(0,0,0,0,0,1)$ *are the following.*

Point x	(0,0,0,0,0,0)	(0,0,0,0,1,0)	(0,0,0,0,0,1)
Strategy	U	P	LFS
Condition	$C_4-C_2>0$	$\begin{cases} \chi-\psi>0 \\ (C_3-2\psi)(1-\beta^2)+2\chi-C_3>0 \\ 2\chi-C_3>0 \end{cases}$	$\begin{cases} (C_4-2\chi)\beta^2+2(\chi-\psi)>0 \\ (2\chi-2\psi)(1-\beta^2)+(C_4-C_3)\beta^2>0 \\ \beta^2((2-\beta^2)(C_4-2\chi)+(2\psi-C_2))+2(\chi-\psi)>0 \\ C_4-C_2>0 \\ \beta^2(C_4-2\chi)+2\chi-C_3>0 \end{cases}$

(6.8)

Proof. The proof of Proposition 6.2 consists of verifying the fact that all eigenvalues of the Jacobian $J_G|_x$ (see section 4.1.1) are negative for corresponding point x.

Remark 6.2 *If conditions given in Proposition 6.2 are not satisfied then, taking in to account the restriction on the parameters described in Remark 6.1, we will have that there is at least one positive eigenvalue for each point. Then these points are not asymptotically and, therefore, not evolutionarily stable.*

6.2.2 Interval $I_1 = \{(1-\alpha_1, \alpha_1, 0, 0, 0, 0) : \alpha_1 \in [0,1]\}$: cooperative behaviours.

For points $(1,0,0,0,0,0)$ (corresponding to the population in which all members use strategy S), $(0,1,0,0,0,0)$ (corresponding to the population in which all members use strategy CP), $(0,0,1,0,0,0)$ (corresponding to the population in which all members use strategy ATP) and $(0,0,0,1,0,0)$ (corresponding to the population in which all members use strategy AT) there are eigenvalues of the Jacobian $J_G|_x$ that are equal to zero. These vertex points are the end points of the sets of non-isolated stationary points $I_1 = \{(1-\alpha_1, \alpha_1, 0, 0, 0, 0) : \alpha_1 \in [0,1]\}$ (corresponding to the populations which consist of various mixtures of players who use strategy S or CP) and $I_2 = \{(0, 0, 1-\alpha_2, \alpha_2, 0, 0) : \alpha_2 \in [0,1]\}$ (corresponding to the populations which consist of various mixtures of players who use strategy AT or ATP). In this section we formulate sufficient conditions for existence of a subset $\mathfrak{S}_1$ of the interval I_1 such that conditions of theorem 4.2 holds. Analysis of the set I_2 is given in the next sections.

Theorem 6.3 *If*

$$\begin{cases} 2\varkappa-C_3>0 \\ \left[\begin{array}{l} (\varkappa-\omega)>0 \\ 2(\varkappa-\omega)+\beta^2(2\omega-C_2)>0 \end{array} \right. \end{cases} \tag{6.9}$$

then there exists a non-empty subset $\mathfrak{S}_1$ *of the set* $I_1 = \{(1-\alpha_1, \alpha_1, 0, 0, 0, 0) : \alpha_1 \in [0,1]\}$ *such that conditions of Theorem 4.2 holds for this subset.*

1. *If* $\begin{cases} 2\varkappa - C_3 > 0 \\ (\varkappa - \omega) > 0 \\ 2(\varkappa-\omega)+\beta^2(2\omega - C_2) > 0 \end{cases}$ *then* $\mathfrak{S}_1 = \{(1-\alpha_1,\alpha_1,0,0,0,0):\alpha_1\in[0,1]\}$.

2. *If* $\begin{cases} 2\varkappa - C_3 > 0 \\ (\varkappa - \omega) < 0 \\ 2(\varkappa-\omega)+\beta^2(2\omega - C_2) > 0 \end{cases}$ *then* $\mathfrak{S}_1 = \left\{(1-\alpha_1,\alpha_1,0,0,0,0):\alpha_1\in\left(\frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)},1\right]\right\}$.

3. *If* $\begin{cases} 2\varkappa - C_3 > 0 \\ (\varkappa - \omega) > 0 \\ 2(\varkappa-\omega)+\beta^2(2\omega - C_2) < 0 \end{cases}$ *then* $\mathfrak{S}_1 = \left\{(1-\alpha_1,\alpha_1,0,0,0,0):\alpha_1\in\left[0,\frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)}\right)\right\}$.

Remark 6.3 *Here we use a square bracket [in description of conditions to denote logical "or" operation and a curly bracket { to denote logical "and" operation. For example,*

$\left[\begin{array}{l} a \\ b \end{array}\right.$ *means that either a or b or both a and b are true, and*

$\left\{\begin{array}{l} a \\ b \end{array}\right.$ *means that both a and b are true at the same time.*

Proof. For each point $x^0 = (1-\alpha_1, \alpha_1, 0, 0, 0, 0), \alpha_1 \in [0,1]$, let us consider the function

$$H_{x^0}(x) = (1-\alpha_1)\log\frac{(1-\alpha_1)}{x_1} + \alpha_1 \log\frac{\alpha_1}{x_2}.$$

The proof of the following facts can be found in [35], p. 98.

(a) If $\alpha_1 \in (0,1)$ then $H_{x^0}(x) \geq 0$ for any $x \in \Delta$ such that $x_1 \neq 0$ and $x_2 \neq 0$.
If $\alpha_1 = 0$ then $H_{x^0}(x) \geq 0$ for any $x \in \Delta$ such that $x_1 \neq 0$.
If $\alpha_1 = 1$ then $H_{x^0}(x) \geq 0$ for any $x \in \Delta$ such that $x_2 \neq 0$.

(b) $H_{x^0}(x) = 0$ if and only if $x = x^0$.

(c)

$$\dot{H}_{x^0}(x) = \begin{bmatrix} x_1 - x_1^0 \\ \vdots \\ x_6 - x_6^0 \\ x_1^0 - x_1 + \ldots + x_6^0 - x_6 \end{bmatrix}^T A \begin{bmatrix} x_1 \\ \vdots \\ x_6 \\ 1 - x_1 - \ldots - x_6 \end{bmatrix}$$

for $x \in \Delta$ described in (a).

Since conditions (2) and (3) of the theorem 4.2 will follow (see [35] p.247) if $\dot{H}_{x^0}(x)$ is negative in a neighbourhood of every point of the set $\mathfrak{S}_1$, let us calculate function $\dot{H}_{x^0}(x)$ at any point $x^0 = \{1-\alpha_1, \alpha_1, 0, 0, 0, 0\}$, $\alpha_1 \in [0,1]$. We can represent simplex Δ as a collection of planes

$$Pl^1_{\mu,\eta,\nu,\vartheta} : \begin{cases} x_3 = \mu\,(1-x_1-x_2) \\ x_4 = \eta\,(1-x_1-x_2) \\ x_5 = \nu\,(1-x_1-x_2) \\ x_6 = \vartheta\,(1-x_1-x_2) \end{cases}, \quad \begin{array}{c} \mu+\eta+\nu+\vartheta \le 1, \\ 0 \le \mu,\ 0 \le \eta, \\ 0 \le \nu,\ 0 \le \vartheta. \end{array} \tag{6.10}$$

Then on the plane $Pl^1_{\mu,\eta,\nu,\vartheta}$, μ, η, ν and ϑ are fixed constants, and the function $\dot{H}_{x^0}(x)$ can be factorised as follows.

$$\dot{H}_{x^0}(x)\Big|_{Pl^1_{\mu,\eta,\nu,\vartheta}} = -\,(1-x_1-x_2)\,L^1_{\mu+\eta,\nu+\vartheta}\,(x_1,x_2) = -\,(1-x_1-x_2)\,L^1_{k,l}(x_1,x^2).$$

Here $k = \mu+\eta$, $l = \nu+\vartheta$ and $L^1_{\mu+\eta,\nu+\vartheta}\,(x_1,x_2) = L^1_{k,l}\,(x_1,x_2)$ is a function which is linear in x_1 and linear in x_2. Since μ, η, ν and ϑ satisfy conditions (6.10) then k and l belong to

$$Q^1 = \{(k,l) : 0 \le k, \quad 0 \le l, \quad k+l \le 1\}.$$

Now, let us find points $x^0 = \{1-\alpha_1, \alpha_1, 0, 0, 0, 0\}$, $\alpha_1 \in [0,1]$, such that

$$L^1_{k,l}\,(x_1,x_2)\big|_{\{x_1=1-\alpha_1, x_2=\alpha_1\}} > 0$$

for any $(k,l) \in Q^1$. Performing direct calculations, we find that

$$\begin{aligned} L^1_{k,l}\,(x_1,x_2)\Big|_{\{x_1=1-\alpha_1,x_2=\alpha_1\}} &= L^1_{k,l}(1-\alpha_1,\alpha_1) \\ &= -\frac{\beta^2(((2-\beta^2)(C_2-2\omega))l+(C_2-\omega-\varkappa)k)\alpha_1+(2-\beta^2)((2\omega-C_3)l+(-C_3+\varkappa+\omega)k+C_3-2\varkappa)}{2-\beta^2} \end{aligned} \tag{6.11}$$

Since $L^1_{k,l}(1-\alpha_1,\alpha_1)$ is a linear function in α_1, it is positive(negative) on the interval $[0,1]$ if its values at the end points $\alpha_1 = 0$ and $\alpha_1 = 1$ are both positive(negative). We also notice that for any fixed α_1 the value of $L^1_{k,l}(1-\alpha_1,\alpha_1)$ is a linear function in k and l. Therefore we obtain that this value is positive(negative) for any $(k,l) \in Q^1$ if it is positive(negative) at the end points $(0,0)$, $(1,0)$ and $(0,1)$. In the table below these values are given.

$L^1_{k,l}(1-\alpha_1,\alpha_1)$	$\alpha_1 = 0$	$\alpha_1 = 1$
$k=0, l=0$	$2\varkappa - C_3$	$2\varkappa - C_3$
$k=1, l=0$	$\varkappa - \omega$	$\frac{2(\varkappa-\omega)+\beta^2(2\omega-C_2)}{(2-\beta^2)}$
$k=0, l=1$	$2\,(\varkappa-\omega)$	$2\,(\varkappa-\omega)+\beta^2\,(2\omega-C_2)$

(6.12)

If

$$\begin{cases} 2\varkappa - C_3 > 0 \\ (\varkappa - \omega) > 0 \\ 2(\varkappa - \omega) + \beta^2 (2\omega - C_2) > 0 \end{cases}$$

then, analysing (6.12), we have that $L^1_{k,l}(1-\alpha_1, \alpha_1) > 0$ for any α_1 and for any $(k,l) \in Q^1$. Therefore in this case $\mathfrak{S}_1 = I_1 = \{(1-\alpha_1, \alpha_1, 0, 0, 0, 0) : \alpha_1 \in [0,1]\}$.

If

$$\begin{cases} 2\varkappa - C_3 > 0 \\ (\varkappa - \omega) < 0 \\ 2(\varkappa - \omega) + \beta^2 (2\omega - C_2) > 0 \end{cases},$$

we obtain that

$$0 < 2(\varkappa - \omega) + \beta^2 (2\omega - C_2) = 2\left(1-\beta^2\right)(\varkappa - \omega) + \beta^2 (2\varkappa - C_2),$$
$$0 < 2\left(1-\beta^2\right)(\omega - \varkappa) < \beta^2 (2\varkappa - C_2),$$
$$0 < (2\varkappa - C_2) < (\varkappa + \omega - C_2) < (2\omega - C_2).$$

Therefore for any $\alpha_1 \in [0,1]$ such that

$$\alpha_1 > \max_{(k,l)\in Q^1} \frac{\left(2-\beta^2\right)}{\beta^2} \frac{(2\omega - C_3)\, l + (\varkappa + \omega - C_3)\, k + C_3 - 2\varkappa}{\left(2-\beta^2\right)(2\omega - C_2)\, l + (\omega + \varkappa - C_2)\, k}$$

$L^1_{k,l}(1-\alpha_1, \alpha_1) > 0$.

Let us find this maximum. The function

$$M^1(k,l) = \frac{\left(2-\beta^2\right)}{\beta^2} \frac{(2\omega - C_3)\, l + (\varkappa + \omega - C_3)\, k + C_3 - 2\varkappa}{\left(2-\beta^2\right)(2\omega - C_2)\, l + (\omega + \varkappa - C_2)\, k}$$

on Q^1 can reach its maximum values only on the boundary of the domain Q^1. The maxima of $M^1(k,l)$ on the three components of the boundary are found below.

$$l = 0: \quad \max_{(k,0)\in Q^1} M^1(k,0) = \max_k \frac{\left(2-\beta^2\right)\left((\varkappa+\omega-C_3)k + C_3 - 2\varkappa\right)}{\beta^2 (\omega+\varkappa-C_2) k}$$
$$= \max_k \frac{\left(2-\beta^2\right)}{\beta^2}\left(\frac{(\varkappa+\omega-C_3)}{(\omega+\varkappa-C_2)} + \frac{C_3-2\varkappa}{\omega+\varkappa-C_2}\frac{1}{k}\right)$$
$$= \frac{\left(2-\beta^2\right)}{\beta^2}\left(\frac{\omega-\varkappa}{\omega+\varkappa-C_2}\right),$$

$$k = 0: \quad \max_{(0,l)\in Q^1,} M^1(0,l) = \max_l \frac{1}{\beta^2}\frac{(2\omega-C_3)l + C_3 - 2\varkappa}{(2\omega-C_2)l}$$
$$= \max_l \frac{1}{\beta^2}\left(\frac{(2\omega-C_3)}{(2\omega-C_2)} + \frac{C_3-2\varkappa}{(2\omega-C_2)l}\right) = \frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)},$$

$$\begin{aligned} k=1-l: \max_{(1-l,l)\in Q^1,} M^1(1-l,l) &= \max_l \frac{(2-\beta^2)}{\beta^2}\frac{(2\omega-C_3)l+(\varkappa+\omega-C_3)(1-l)+C_3-2\varkappa}{(2-\beta^2)(2\omega-C_2)l+(\omega+\varkappa-C_2)(1-l)} \\ &= \max_l \frac{(2-\beta^2)}{\beta^2}\frac{(\omega-\varkappa)(l+1)}{((2-\beta^2)(2\omega-C_2)-(\omega+\varkappa-C_2))l+(\omega+\varkappa-C_2)} \\ &= \max\left\{\frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)}, \frac{(2-\beta^2)}{\beta^2}\frac{(\omega-\varkappa)}{(\omega+\varkappa-C_2)}\right\}. \end{aligned}$$

Since

$$\begin{aligned} &\frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)} > \frac{(2-\beta^2)}{\beta^2}\frac{(\omega-\varkappa)}{(\omega+\varkappa-C_2)}, \\ &\frac{2}{(2\omega-C_2)} > \frac{(2-\beta^2)}{(\omega+\varkappa-C_2)}, \\ &2(\omega+\varkappa-C_2) > (2-\beta^2)(2\omega-C_2), \\ &2(\varkappa-\omega)+\beta^2(2\omega-C_2) > 0, \end{aligned}$$

we find that

$$\begin{aligned} \max_{(k,l)\in Q^1} M^1(k,l) &= \max\left\{\begin{array}{l} \max_k \frac{(2-\beta^2)((\varkappa+\omega-C_3)k+C_3-2\varkappa)}{\beta^2(\omega+\varkappa-C_2)k}, \\ \max_l \frac{1}{\beta^2}\frac{(2\omega-C_3)l+C_3-2\varkappa}{(2\omega-C_2)l}, \\ \max_l \frac{(2-\beta^2)}{\beta^2}\frac{(2\omega-C_3)l+(\varkappa+\omega-C_3)(1-l)+C_3-2\varkappa}{(2-\beta^2)(2\omega-C_2)l+(\omega+\varkappa-C_2)(1-l)} \end{array}\right\} \\ &= \max\left\{\frac{(2-\beta^2)}{\beta^2}\frac{(\omega-\varkappa)}{(\omega+\varkappa-C_2)}, \frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)}\right\} = \frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)}. \end{aligned}$$

Therefore, since

$$\frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)} < 1,$$

for any $\alpha_1 \in \left(\frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)}, 1\right]$ we have that $L^1_{k,l}(1-\alpha_1,\alpha_1) > 0$. Hence, in this case

$$\mathfrak{S}_1 = \left\{(1-\alpha_1,\alpha_1,0,0,0,0) : \alpha_1 \in \left(\frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)}, 1\right]\right\}.$$

If

$$\left\{\begin{array}{l} 2\varkappa - C_3 > 0 \\ (\varkappa-\omega) > 0 \\ 2(\varkappa-\omega)+\beta^2(2\omega-C_2) > 0 \end{array}\right. ,$$

we obtain that

$$\begin{aligned} 0 > 2(\varkappa-\omega)+\beta^2(2\omega-C_2) &= 2(1-\beta^2)(\varkappa-\omega)+\beta^2(2\varkappa-C_2), \\ 0 > 2(1-\beta^2)(\omega-\varkappa) &> \beta^2(2\varkappa-C_2), \\ 0 > (2\varkappa-C_2) > (\varkappa+\omega-C_2) &> (2\omega-C_2). \end{aligned}$$

Therefore for any $\alpha_1 \in [0,1]$ such that

$$\alpha_1 < \min_{(k,l)\in Q^1} \frac{\left(2-\beta^2\right)}{\beta^2} \frac{(2\omega - C_3)\,l + (\varkappa+\omega-C_3)\,k + C_3 - 2\varkappa}{\left(2-\beta^2\right)(2\omega-C_2)\,l + (\omega+\varkappa-C_2)\,k}$$

$L^1_{k,l}(1-\alpha_1,\alpha_1) > 0$.

Let us find this minimum. In this case function $M^1(k,l)$ on Q^1 can again reach its minimum values only on the boundary of the domain Q^1. To find the minima of $M^1(k,l)$ on the three components of the boundary we perform analogous considerations and, since

$$\begin{aligned}
&\tfrac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)} < \tfrac{\left(2-\beta^2\right)}{\beta^2}\tfrac{(\omega-\varkappa)}{(\omega+\varkappa-C_2)},\\
&\tfrac{2}{(2\omega-C_2)} < \tfrac{\left(2-\beta^2\right)}{(\omega+\varkappa-C_2)},\\
&2\,(\omega+\varkappa-C_2) < \left(2-\beta^2\right)(2\omega-C_2)\,,\\
&2\,(\varkappa-\omega) + \beta^2\,(2\omega-C_2) < 0,
\end{aligned}$$

in this case, we obtain that

$$\begin{aligned}
\min_{(k,l)\in Q^1} M^1(k,l) &= \min\left\{\begin{array}{l}
\min\limits_{k} \frac{\left(2-\beta^2\right)((\varkappa+\omega-C_3)k+C_3-2\varkappa)}{\beta^2(\omega+\varkappa-C_2)k},\\
\min\limits_{l} \frac{1}{\beta^2}\frac{(2\omega-C_3)l+C_3-2\varkappa}{(2\omega-C_2)l},\\
\min\limits_{l} \frac{\left(2-\beta^2\right)}{\beta^2}\frac{(2\omega-C_3)l+(\varkappa+\omega-C_3)(1-l)+C_3-2\varkappa}{\left(2-\beta^2\right)(2\omega-C_2)l+(\omega+\varkappa-C_2)(1-l)}
\end{array}\right\}\\
&= \min\left\{\frac{\left(2-\beta^2\right)}{\beta^2}\frac{(\omega-\varkappa)}{(\omega+\varkappa-C_2)}, \frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)}\right\} = \frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)}.
\end{aligned}$$

Therefore, since

$$0 < \frac{2\,(\omega-\varkappa)}{\beta^2\,(2\omega-C_2)},$$

for any $\alpha_1 \in \left[0, \frac{2(\omega-\varkappa)}{\beta^2(2\omega-C_2)}\right)$ we have that $L^1_{k,l}(1-\alpha_1,\alpha_1) > 0$ and

$$\mathfrak{S}_1 = \left\{(1-\alpha_1,\alpha_1,0,0,0,0) : \alpha_1 \in \left[0, \frac{2\,(\omega-\varkappa)}{\beta^2\,(2\omega-C_2)}\right)\right\}.$$

Theorem 6.3 is proved.

Remark 6.4 *It follows from the proof of theorem 6.3 that if*

$$\left\{\begin{array}{l} 2\varkappa-C_3>0\\ (\varkappa-\omega)>0\\ 2(\varkappa-\omega)+\beta^2(2\omega-C_2)>0 \end{array}\right.$$

then the whole interval $I_1 = \{(1-\alpha_1,\alpha_1,0,0,0,0) : \alpha_1 \in [0,1]\}$ (corresponding to the populations which consist of various mixtures of players who use strategy S or CP) is evolutionarily stable.

Remark 6.5 *If*

$$\begin{cases} 2\varkappa - C_3 > 0 \\ (\varkappa - \omega) < 0 \\ 2(\varkappa - \omega) + \beta^2(2\omega - C_2) > 0 \end{cases} \quad or \quad \begin{cases} 2\varkappa - C_3 > 0 \\ (\varkappa - \omega) > 0 \\ 2(\varkappa - \omega) + \beta^2(2\omega - C_2) > 0 \end{cases}$$

then further analysis at the point corresponding to $\alpha_1 = \frac{2(\omega-\varkappa)}{\beta^2(2\omega - C_2)}$ *is necessary in order to prove that* $\mathfrak{S}_1$ *is evolutionarily attractive (see definition 4.1 from chapter 4). This analysis is, in general, quite involved.*

6.2.3 Interval $I_2 = \{(0, 0, \alpha_2, 1 - \alpha_2, 0, 0) : \alpha_2 \in [0, 1]\}$: allocating tasks behaviours.

In this section we formulate sufficient conditions for existence of a subset $\mathfrak{S}_2$ of the interval $I_2 = \{(0, 0, \alpha_2, 1 - \alpha_2, 0, 0) : \alpha_2 \in [0, 1]\}$ such that conditions of theorem 4.2 hold. Recall that the interval I_2 corresponds to the populations which consist of various mixtures of players who use strategy AT or ATP. For this interval the following theorem is true.

Theorem 6.4 *If*

$$\begin{cases} \omega + \psi - C_3 > 0 \\ \omega - \varkappa > 0 \\ \left[\begin{array}{l} \psi - \chi > 0 \\ \begin{cases} \psi - \chi < 0 \\ \left(2 - \beta^2\right)(\psi - \chi) + \beta^2(\omega + \chi - C_2) > 0 \\ \left(2 - \beta^2\right)(\psi - \chi) + \beta^2(\omega + \chi - C_4) > 0 \end{cases} \\ \begin{cases} \psi - \chi < 0 \\ 2(\psi - \chi)\left(1 - \beta^2\right) + (\psi + \omega - C_3)\beta^2 > 0 \\ \left(2 - \beta^2\right)(\psi - \chi) + \beta^2(\omega + \chi - C_2) > 0 \\ \left(2 - \beta^2\right)(\psi - \chi) + \beta^2(\omega + \chi - C_4) < 0 \\ \frac{(2-\beta^2)(\psi-\chi)}{\beta^2(C_2 - \omega - \chi)} < \frac{2(1-\beta^2)(\psi-\chi) + (\psi+\omega-C_3)\beta^2}{\beta^2(C_4 - C_3)} \end{cases} \end{array} \right. \end{cases} \tag{6.13}$$

then there exists a non-empty subset $\mathfrak{S}_2$ *of the set* $I_2 = \{(0, 0, \alpha_2, 1 - \alpha_2, 0, 0) : \alpha_2 \in [0, 1]\}$ *such that conditions of theorem 4.2 holds for this subset.*

1. *If* $\begin{cases} \omega + \psi - C_3 > 0 \\ \omega - \varkappa > 0 \\ \psi - \chi > 0 \\ \left(2 - \beta^2\right)(\psi - \chi) + \beta^2(\omega + \chi - C_2) > 0 \\ \left(2 - \beta^2\right)(\psi - \chi) + \beta^2(\omega + \chi - C_4) > 0 \end{cases}$ *then* $\mathfrak{S}_2 = \{(0,0,\alpha_2,1-\alpha_2,0,0) : \alpha_2 \in [0,1]\}$.

2. *If* $\begin{cases} \omega + \psi - C_3 > 0 \\ \omega - \varkappa > 0 \\ \psi - \chi > 0 \\ \left(2 - \beta^2\right)(\psi - \chi) + \beta^2(\omega + \chi - C_2) < 0 \\ \left(2 - \beta^2\right)(\psi - \chi) + \beta^2(\omega + \chi - C_4) > 0 \end{cases}$ *then* $\mathfrak{S}_2 = \left\{ (0,0,\alpha_2,1-\alpha_2,0,0) : \alpha_2 \in \left[0, \frac{(2-\beta^2)(\psi-\chi)}{\beta^2(C_2-\omega-\chi)}\right) \right\}$.

3. *If*
$$\left\{\begin{array}{l}\omega+\psi-C_3>0\\ \omega-\varkappa>0\\ \psi-\chi>0\\ \left(2-\beta^2\right)(\psi-\chi)+\beta^2(\omega+\chi-C_2)>0\\ \left(2-\beta^2\right)(\psi-\chi)+\beta^2(\omega+\chi-C_4)<0\end{array}\right. \quad \textit{then} \quad \mathfrak{S}_2=\left\{\begin{array}{c}(0,0,\alpha_2,1-\alpha_2,0,0):\\ \alpha_2\in\left[0,\frac{2(1-\beta^2)(\psi-\chi)+(\psi+\omega-C_3)\beta^2}{\beta^2(C_4-C_3)}\right)\end{array}\right\}.$$

4. *If*
$$\left\{\begin{array}{l}\omega+\psi-C_3>0\\ \omega-\varkappa>0\\ \psi-\chi>0\\ \left(2-\beta^2\right)(\psi-\chi)+\beta^2(\omega+\chi-C_2)<0\\ \left(2-\beta^2\right)(\psi-\chi)+\beta^2(\omega+\chi-C_4)<0\end{array}\right. \quad \textit{then} \quad \mathfrak{S}_2=\left\{\begin{array}{c}(0,0,\alpha_2,1-\alpha_2,0,0):\\ \alpha_2\in\left[0,\min\left\{\begin{array}{c}\frac{(2-\beta^2)(\psi-\chi)}{\beta^2(C_2-\omega-\chi)},\\ \frac{2(1-\beta^2)(\psi-\chi)+(\psi+\omega-C_3)\beta^2}{\beta^2(C_4-C_3)}\end{array}\right\}\right)\end{array}\right\}.$$

5. *If*
$$\left\{\begin{array}{l}\omega+\psi-C_3>0\\ \omega-\varkappa>0\\ \psi-\chi<0\\ 2(\psi-\chi)\left(1-\beta^2\right)+(\psi+\omega-C_3)\beta^2>0\\ \left(2-\beta^2\right)(\psi-\chi)+\beta^2(\omega+\chi-C_2)>0\\ \left(2-\beta^2\right)(\psi-\chi)+\beta^2(\omega+\chi-C_4)>0\end{array}\right. \quad \textit{then} \quad \mathfrak{S}_2=\left\{\begin{array}{c}(0,0,\alpha_2,1-\alpha_2,0,0):\\ \alpha_2\in\left(\frac{(2-\beta^2)(\psi-\chi)}{\beta^2(C_2-\omega-\chi)},1\right]\end{array}\right\}.$$

6. *If*
$$\left\{\begin{array}{l}\omega+\psi-C_3>0\\ \omega-\varkappa>0\\ \psi-\chi<0\\ 2(\psi-\chi)\left(1-\beta^2\right)+(\psi+\omega-C_3)\beta^2<0\\ \left(2-\beta^2\right)(\psi-\chi)+\beta^2(\omega+\chi-C_2)>0\\ \left(2-\beta^2\right)(\psi-\chi)+\beta^2(\omega+\chi-C_4)>0\end{array}\right. \quad \textit{then} \quad \mathfrak{S}_2=\left\{\begin{array}{c}(0,0,\alpha_2,1-\alpha_2,0,0):\\ \alpha_2\in\left(\max\left\{\begin{array}{c}\frac{(2-\beta^2)(\psi-\chi)}{\beta^2(C_2-\omega-\chi)},\\ \frac{2\left(1-\beta^2\right)(\psi-\chi)+(\psi+\omega-C_3)\beta^2}{\beta^2(C_4-C_3)}\end{array}\right\},1\right]\end{array}\right\}.$$

7. *If*
$$\left\{\begin{array}{l}\omega+\psi-C_3>0\\ \omega-\varkappa>0\\ \psi-\chi<0\\ 2(\psi-\chi)\left(1-\beta^2\right)+(\psi+\omega-C_3)\beta^2>0\\ \left(2-\beta^2\right)(\psi-\chi)+\beta^2(\omega+\chi-C_2)>0\\ \left(2-\beta^2\right)(\psi-\chi)+\beta^2(\omega+\chi-C_4)<0\\ \frac{(2-\beta^2)(\psi-\chi)}{\beta^2(C_2-\omega-\chi)}<\frac{2(1-\beta^2)(\psi-\chi)+(\psi+\omega-C_3)\beta^2}{\beta^2(C_4-C_3)}\end{array}\right. \quad \textit{then} \quad \mathfrak{S}_2=\left\{\begin{array}{c}(0,0,\alpha_2,1-\alpha_2,0,0):\\ \alpha_2\in\left(\frac{(2-\beta^2)(\psi-\chi)}{\beta^2(C_2-\omega-\chi)},\frac{2(1-\beta^2)(\psi-\chi)+(\psi+\omega-C_3)\beta^2}{\beta^2(C_4-C_3)}\right)\end{array}\right\}.$$

Proof. For each point $x^0=(0,0,\alpha_2,1-\alpha_2,0,0)\,,\alpha_2\in[0,1]\,,$ let us consider the function

$$H_{x^0}(x)=\alpha_2\log\frac{\alpha_2}{x_3}+(1-\alpha_2)\log\frac{(1-\alpha_2)}{x_4}.$$

then

$$\dot{H}_{x^0}(x) = \begin{bmatrix} x_1 - x_1^0 \\ \vdots \\ x_6 - x_6^0 \\ x_1^0 - x_1 + \ldots + x_6^0 - x_6 \end{bmatrix}^T A \begin{bmatrix} x_1 \\ \vdots \\ x_6 \\ 1 - x_1 - \ldots - x_6 \end{bmatrix}.$$

To calculate function $\dot{H}_{x^0}(x) = -\left(x^0 - x\right) Ax^T$ for any point $x^0 = \{0, 0, \alpha_2, 1 - \alpha_2, 0, 0\}$, $\alpha_2 \in [0, 1]$, we represent simplex Δ as a collection of planes

$$Pl^2_{\mu,\eta,\nu,\vartheta} : \left\{ \begin{array}{l} x_1 = \mu(1-x_3-x_4) \\ x_2 = \eta(1-x_3-x_4) \\ x_5 = \nu(1-x_3-x_4) \\ x_6 = \vartheta(1-x_3-x_4) \end{array} \right. , \quad \begin{array}{c} \text{where} \\ \mu, \eta, \nu \text{ and } \vartheta \\ \text{belong to} \end{array} \quad Q^2 = \left\{ (\mu, \eta, \nu, \vartheta) : \begin{array}{c} \mu+\eta+\nu+\vartheta \leq 1, \\ 0 \leq \mu,\ 0 \leq \eta, \\ 0 \leq \nu,\ 0 \leq \vartheta. \end{array} \right\}.$$

Then on the plane $Pl^2_{\mu,\eta,\nu,\vartheta}$, there μ, η, ν and ϑ are fixed constants, function $\dot{H}_{x^0}(x)$ can be factorised as follows

$$\dot{H}_{x^0}(x)\Big|_{Pl_{\mu,\eta,\nu,\vartheta}} = -\left(1 - x_3 - x_4\right) L^2_{\mu,\eta,\nu,\vartheta}\left(x_3, x_4\right).$$

Here $L^2_{\mu,\eta,\nu,\vartheta}\left(x_3, x_4\right)$ is a linear function in x_3 and x_4.

Now, let us find points $x^0 = (0, 0, \alpha_2, 1 - \alpha_2, 0, 0)$, $\alpha_2 \in [0, 1]$, such that

$$L^2_{\mu,\eta,\nu,\vartheta}\left(x_3, x_4\right)\big|_{\{x_3 = 1-\alpha_2, x_4 = \alpha_2\}} > 0.$$

Performing direct calculations, we find that

$$\begin{aligned} &L\left(x_3, x_4\right)\big|_{\{x_3=1-\alpha_2, x_4=\alpha_2\}} = L^2_{\mu,\eta,\nu,\vartheta}(\alpha_2, 1-\alpha_2) \\ &= \frac{\beta^2[(\omega+\chi-C_2)\nu+(C_3-C_4)\vartheta]\alpha+(C_3-\varkappa-\psi)[(2-\beta^2)\mu+(2-2\beta^2)\eta]+(C_3-\omega-\chi)[(2-\beta^2)\nu+(2-2\beta^2)\vartheta]+(2-\beta^2)(\omega+\psi-C_3)}{2-\beta^2} \end{aligned} \tag{6.14}$$

Since $L^2_{\mu,\eta,\nu,\vartheta}(\alpha_2, 1 - \alpha_2)$ is a linear function in α_2, it is positive (negative) on the whole interval $[0, 1]$ if its values at the end points $\alpha_2 = 0$ and $\alpha_2 = 1$ are both positive (negative). We also notice that for any fixed α_2 the value of $L^2_{\mu,\eta,\nu,\vartheta}(\alpha_2, 1 - \alpha_2)$ is a linear function in μ, η, ν and ϑ. Therefore we obtain that the value of $L^2_{\mu,\eta,\nu,\vartheta}(\alpha_2, 1 - \alpha_2)$ is positive (negative) for any $(\mu, \eta, \nu, \vartheta) \in Q^2$ if it is positive (negative) at the end points $(0, 0, 0, 0)$, $(1, 0, 0, 0)$,

$(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. In the table below these values are represented.

$L^2_{\mu,\eta,\nu,\vartheta}(\alpha_2, 1-\alpha_2)$	$\alpha_2 = 0$	$\alpha_2 = 1$
$(\mu,\eta,\nu,\vartheta) = (0,0,0,0)$	$\omega + \psi - C_3$	$\omega + \psi - C_3$
$(\mu,\eta,\nu,\vartheta) = (1,0,0,0)$	$\omega - \varkappa$	$\omega - \varkappa$
$(\mu,\eta,\nu,\vartheta) = (0,1,0,0)$	$\frac{2(\omega-\varkappa)(1-\beta^2)+(\psi+\omega-C_3)\beta^2}{2-\beta^2}$	$\frac{2(\omega-\varkappa)(1-\beta^2)+(\psi+\omega-C_3)\beta^2}{2-\beta^2}$
$(\mu,\eta,\nu,\vartheta) = (0,0,1,0)$	$\psi - \chi$	$\frac{(2-\beta^2)(\psi-\chi)+\beta^2(\omega+\chi-C_2)}{2-\beta^2}$
$(\mu,\eta,\nu,\vartheta) = (0,0,0,1)$	$\frac{2(\psi-\chi)(1-\beta^2)+(\psi+\omega-C_3)\beta^2}{2-\beta^2}$	$\frac{(2-\beta^2)(\psi-\chi)+\beta^2(\omega+\chi-C_4)}{2-\beta^2}$

(6.15)

The rest of the proof is analogous to the proof of theorem 6.3 and consists in direct examination of the conditions for $L^2_{0,0,0,0}(\alpha_2, 1-\alpha_2)$, $L^2_{1,0,0,0}(\alpha_2, 1-\alpha_2)$, $L^2_{0,1,0,0}(\alpha_2, 1-\alpha_2)$, $L^2_{0,0,1,0}(\alpha_2, 1-\alpha_2)$ and $L^2_{0,0,0,1}(\alpha_2, 1-\alpha_2)$ all to be greater then zero for any μ, η, ν and ϑ in any of the cases described.

6.3 Stationary points and sets in the dynamics of the multi-state game.

Following the approach discussed in section 4.1.2 of chapter 4, all stationary points for system (6.7) are solutions of the system of equations

$$x_i \left(\sum_{j=1}^{7} a_{ij} x_j - XAX^T \right) = 0, \qquad i = 1, \dots, 7. \tag{6.16}$$

Here A is given by (6.1). To find all solutions of system (6.16) it is necessary to enumerate all possible combinations of m indices from the set $\{1, \dots, 7\}$ where m consecutively takes values from 1 to 7. If a combination $\{j_1, \dots, j_m\}$ is chosen, then we suppose that coordinates $x_{j_1}, \dots, x_{j_m}$ take non-zero values and all remaining coordinates x_i are zero. In this case system (6.16) is equivalent to

$$\begin{cases} \sum\limits_{k=1}^{m} \left(a_{j_1 j_k} - a_{j_l j_k}\right) x_{j_k} = 0, & l = 2, \dots, m; \\ \sum\limits_{k=1}^{m} x_{j_k} = 1. \end{cases} \tag{6.17}$$

If $m = 1$ then the solutions correspond to the vertices of the simplex. They have been discussed in sections 6.2.1 and 6.2.2.

Consider now the case when $m \in \{2, \dots, 7\}$. Denote by

$$A_{j_1,\dots,j_m} = \begin{bmatrix} (a_{j_1j_1} - a_{j_2j_1}) & \dots & (a_{j_1j_m} - a_{j_2j_m}) \\ \vdots & & \vdots \\ (a_{j_1j_1} - a_{j_mj_1}) & \dots & (a_{j_1j_m} - a_{j_mj_m}) \\ 1 & \dots & 1 \end{bmatrix}$$

the matrix of system (6.17). If matrix $A_{j_1,\dots,j_m}$ is non-singular then the system has a unique solution which is obtained by the Cramer' rule as following.

$$x^{j_k} = \frac{\begin{Vmatrix} (a_{j_1j_1}-a_{j_2j_1}) & \cdots & (a_{j_1j_{k-1}}-a_{j_2j_{k-1}}) & 0 & (a_{j_1j_{k+1}}-a_{j_2j_{k+1}}) & \cdots & (a_{j_1j_m}-a_{j_2j_m}) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (a_{j_1j_1}-a_{j_mj_1}) & \cdots & (a_{j_1j_{k-1}}-a_{j_mj_{k-1}}) & 0 & (a_{j_1j_{k+1}}-a_{j_mj_{k+1}}) & \cdots & (a_{j_1j_m}-a_{j_mj_m}) \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 \end{Vmatrix}}{\|A_{j_1,\dots,j_m}\|}$$

If the matrix of system (6.17) is singular then there are two possible cases.

1. The ranks of matrix $A_{j_1,\dots,j_m}$ and matrix

$$\overline{A}_{j_1,\dots,j_m} = \begin{bmatrix} (a_{j_1j_1} - a_{j_2j_1}) & \dots & (a_{j_1j_m} - a_{j_2j_m}) & 0 \\ \vdots & & \vdots & \vdots \\ (a_{j_1j_1} - a_{j_mj_1}) & \dots & (a_{j_1j_m} - a_{j_mj_m}) & 0 \\ 1 & \dots & 1 & 1 \end{bmatrix}$$

are different. Then the equations of system (6.17) are incompatible and there are no solutions for this system. This case takes place if the combinations of indices are chosen as following: $\{1,4\}$, $\{1,3\}$, $\{4,5\}$, $\{1,3,4\}$, $\{1,3,4,5\}$ and $\{1,3,4,6\}$. Since by x_1, x_2, x_3, x_4, x_5, x_6 and x_7 we denote the proportion of the individuals in the population who adopt behaviour S, CP, ATP, AT, P, LFS and U, respectively, this result means that the populations which consist only of the proportions of the individuals who adopt types of behaviour corresponding to the above combinations of indices cannot be stationary. For example since the equations of system (6.17) are incompatible for the combination $\{1,3,4,5\}$ this means that there is no stationary point for the Replicator Dynamics that corresponds to the population consisting of a mixture of the individuals who adopt strategies S, ATP, AT or P.

2. If the ranks of matrix $A_{j_1,\dots,j_m}$ and matrix $\overline{A}_{j_1,\dots,j_m}$ are equal, then system (6.17) has infinitely many solutions. This case is true if the combinations of indices are chosen as following: $\{1,2\}$, $\{3,4\}$, $\{1,2,7\}$, $\{2,3,4\}$, $\{3,4,7\}$, $\{1,3,4,7\}$, $\{1,2,3,4\}$, $\{2,3,4,7\}$, and $\{1,2,3,4,7\}$. This result means that there may be a stationary set of points which correspond to populations consisting of the proportions of the individuals who adopt types of behaviour corresponding to the above combinations of indices. For example, since system (6.17) has infinitely many solutions for the combination $\{1,2,3,4\}$ this means that there may be a stationary set of points which correspond to populations consisting of a mixture of the individuals who adopt strategies S, CP, ATP and AT.

The cases of combinations $\{1,2\}$ (corresponding to the populations which consist of various mixtures of players who use strategy S or CP) and $\{3,4\}$ (corresponding to the populations which consist of various mixtures of players who use strategy ATP or AT) have been considered in sections 6.2.2 and 6.2.3.

There are three combinations of indices for which it is obvious that the corresponding solutions do not belong to the simplex Δ. These are as follows. If combination $\{1,3,4,7\}$ (corresponding to the populations which consist of various mixtures of players who use strategy S, ATP, AT or U)$\left\{\frac{-C_2+C_4}{\varkappa-\psi},0,-\alpha+\frac{C_2-C_4}{\varkappa-\psi},\alpha,0,0,1\right\}$,lution for the system (6.16) is

$$\left\{\frac{-C_2+C_4}{\varkappa-\psi},0,-\alpha+\frac{C_2-C_4}{\varkappa-\psi},\alpha,0,0,1\right\},$$

where α is parameter controlling mixture of strategies ATP or AT. If combination $\{1,2,3,4\}$ (corresponding to the populations which consist of various mixtures of players who use strategy S, CP, ATP or AT) is chosen then the corresponding solution is

$$\left\{\frac{2\varkappa-2\omega+2\beta^2\omega-\beta^2C_2}{\beta^2(\varkappa+\omega-C_2)},\frac{-2\varkappa+\varkappa\beta^2+2\omega-\beta^2\omega}{\beta^2(\varkappa+\omega-C_2)},-\alpha,\alpha,0,0,0\right\}.$$

For combination $\{1,2,3,4,7\}$ (corresponding to the populations which consist of various mixtures of players who use strategy S, CP, ATP, AT or U) there is solution

$$\left\{\frac{(C_2-C_4)(-\beta^2C_2+2\beta^2\omega+2\varkappa-2\omega)}{(-C_4-2\varkappa+C_3+C_2)(-C_2+\varkappa+\omega)\beta^2},\frac{(-2+\beta^2)(C_2-C_4)(\varkappa-\omega)}{\beta^2(-C_2+\varkappa+\omega)(-C_4-2\varkappa+C_3+C_2)},-\alpha,\alpha,0,0,\frac{2\varkappa-C_3}{2\varkappa-C_3-C_2+C_4}\right\},$$

Analysing the form of these solutions, we see that they do not belong to the simplex Δ for any choice of the parameter α. Therefore there are no stable points which represent the corresponding populations.

Analysis of all other solutions of the system (6.16) is quite complex if there are no restrictions on the parameters of the payoff matrix (6.1). In the next chapter we consider some examples of the payoff matrix (6.1) and demonstrate how the analysis can be performed.

6.4 Some invariant subspaces.

It is useful for further analysis to describe some invariant subspaces that are preserved by the dynamics (6.7).

- It can be seen that any subspaces

$$\Delta_{i_1,\dots,i_k} = \{x : x_i = 0, \quad i = 1, \dots, 7, \quad i \neq i_1, \dots, i_k\}$$

  are obviously invariant.

- There is a collection of three-dimensional linear subspaces

$$M_\alpha = \left\{x : x_3 + \frac{\alpha}{\alpha - 1} x_4 = 0, \quad x_5 = 0, \quad x_6 = 0\right\}$$

  which are invariant for any $\alpha \in [0, 1]$. This collection considered as a set is the subspace $\Delta_{1,2,3,4,7}$. Due to this fact it is possible to factorise the four-dimensional subspace $\Delta_{1,2,3,4,7}$ and investigate the dynamics on each three-dimensional subspaces M_α independently, which significantly simplifies the analysis.

- The subspace

$$M = \{x : x_1 - \varrho x_2 + x_3 + x_4 = 0, \quad x_5 = 0, \quad x_6 = 0\}$$

  is also invariant if

$$\varrho = \frac{c_2\beta^2 + 2\omega\left(1 - \beta^2\right) - 2\varkappa}{\beta^2\left(\psi - c_3\right) - \omega\left(2 - \beta^2\right) + 2\varkappa\left(1 - \beta^2\right)}.$$

6.5 Summary.

The significance of the results obtained in this chapter is that they describe the ranges of parameters for the model such that non cooperative populations are unstable and cooperative and partially cooperative populations are stable. Therefore, these results characterise the structure of the model which can be applicable for investigation of the evolution of cooperative behaviour: they allow for the possibility that cooperation may originate in a non-cooperative society. Unfortunately the analysis of the dynamics is very complex for general (not fixed) parameters. These results will help us to investigate the specific examples considered in the next chapter.

Chapter 7

Multi-state games: Some interesting results.

In this chapter we will analyse the Replicator Dynamics corresponding to the two examples considered in section 5.5 (examples 5.1 and 5.2). We will show that populations which consist of various mixtures of cooperative or allocating tasks behaviour can be the end points of the selection process. We will also show that it is possible for such populations to evolve from populations which consist of individuals using non-cooperative types of behaviour.

7.1 Cost of association model.

In this section we give the analysis of the Replicator Dynamics for the multi-state game considered in the example 5.2 of section 5.5. For this game we have

$$ph_1 + t_1\,(1-p) = 3; \quad ph_2 + t_2\,(1-p) = 0; \quad ph_3 + t_3\,(1-p) = 5; \quad ph_4 + t_4\,(1-p) = 1.$$

$c_1 = -\frac{1}{2}$, $c_2 = c_3 = c_4 = 0$ and we chose $z = \frac{2}{3}$ and $\beta = \frac{9}{10}$. In this case strategies CP and ATP are Nash Equilibria (see table 5.9), strategies S, AT, LFS and P are not Nash Equilibria, and strategy U earns the same total value against every strategy and every strategy earns the same total value against U, therefore U is a Nash Equilibrium. For simplicity in this section we do not consider the Replicator Dynamics on the whole simplex (4.3) but restrict ourselves to the case of the populations which consist of various mixtures of the individuals who adopt one of the four strategies: CP, ATP, P or U. Denote the set of such populations by Δ_4. Let x_1, x_2, x_3 and x_4 denote the proportion of the individuals in the population who adopt behaviour CP, ATP, P and U, respectively. Then

$$\Delta_4 = \left\{ x = (x_1, x_2, x_3) : \bigcap_{i=1}^{3} \left(0 \leq x_i \right) \cap \left(\sum_{i=1}^{3} x_i \leq 1 \right) \right\}.$$

We will investigate the possibility of evolving to a cooperative or alternating tasks population from populations of P or U players. In this case the payoffs of the corresponding symmetric two-person game are given by the following bi-matrix.

$P_1 \backslash P_2$	CP	ATP	P	U
CP	$\frac{220}{19}, \frac{220}{19}$	$\frac{656}{119}, \frac{158}{17}$	$\frac{109}{25}, \frac{443}{50}$	$6, 6$
ATP	$\frac{158}{17}, \frac{656}{119}$	$\frac{175}{19}, \frac{175}{19}$	$4, \frac{926}{119}$	$6, 6$
P	$\frac{443}{50}, \frac{109}{25}$	$\frac{926}{119}, 4$	$\frac{40}{19}, \frac{40}{19}$	$6, 6$
U	$6, 6$	$6, 6$	$6, 6$	$6, 6$

(7.1)

Each player uses one of the four possible pure strategies CP, ATP, P and U.

There are four Nash Equilibria for this game: $(\nu_1, \nu_1), (\nu_2, \nu_2), (\nu_4, \nu_4)$ and $(\nu_{1,2}, \nu_{1,2})$. Where $\nu_1 =$ "*choose* CP", $\nu_2 =$ "*choose* ATP", $\nu_4 =$ "*choose* U" and

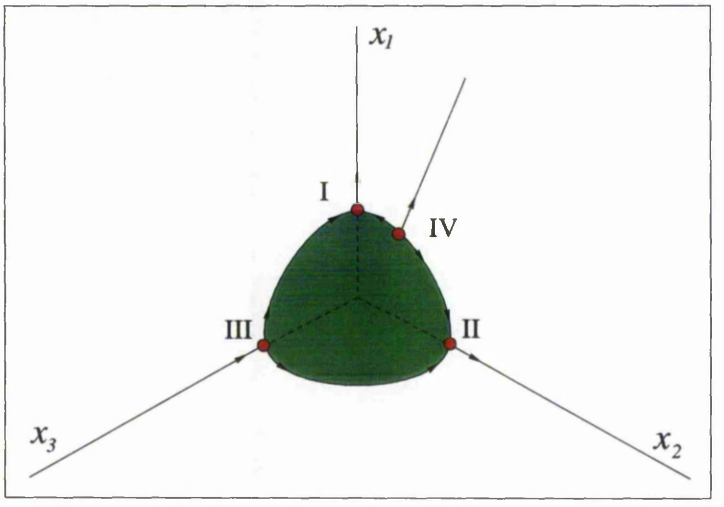
$$\nu_{1,2} = \text{“}\textit{choose } CP \textit{ with probability } \frac{929}{1503} \textit{ and choose } ATP \textit{ with probability } \frac{574}{1503}\text{”}.$$

The qualitative picture of the solution behaviour for the Replicator Dynamics of this system is similar to the one obtained in the section 4.3 for the game determined by bi-matrix (4.15). Here we, therefore, only give the results of the analysis as the proofs are analogous to the ones given for the system in section 4.3.

From the analysis of the eigenvalues of each point we have that point $(1, 0, 0)$ (corresponding to the population in which all members use strategy CP) and $(0, 1, 0)$ (corresponding to the population in which all members use strategy ATP) are attractive nodal points; $(0, 0, 1)$ (corresponding to the population in which all members use strategy P) is a repulsive nodal point; and $(\frac{929}{1503}, \frac{574}{1503}, 0)$ (corresponding to the population in which $\frac{929}{1503}$ proportion of members use strategy CP and $\frac{574}{1503}$ proportion of members use strategy ATP) is a saddle point.

The dynamics in the neighbourhood of the point $(0, 0, 0)$ (corresponding to the population in which all members use strategy U) is as shown in figure 7.1 (figure 4.6 from section 4.3 reproduced below for convenience).

Figure 7.1. The dynamics near the non-hyperbolic point $(0,0,0)$. The point $(0,0,0)$ is "blown up".



There is an invariant line

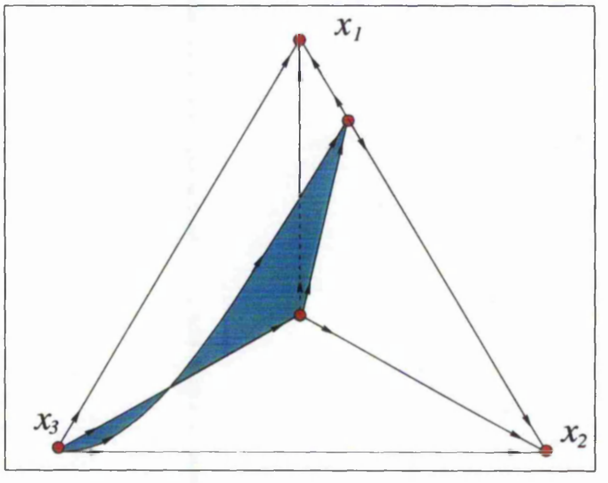
$$\begin{cases} x_1 = \frac{929}{574}x_2 \\ x_3 = 0 \end{cases} \tag{7.2}$$

passing through the origin and the point $\left(\frac{929}{1503}, \frac{574}{1503}, 0\right)$

In the neighbourhood of the point $\left(\frac{929}{1503}, \frac{574}{1503}, 0\right)$ there exists a two-dimensional stable invariant manifold M tangent to the eigenspace generated by the vectors $\left[\frac{929}{574}, 1, 0\right]$ and $\left[1, \frac{75\,660\,998}{144\,454\,855}, -\frac{220\,115\,853}{144\,454\,855}\right]$. If we consider this dynamical system when $t \to -\infty$, we find that all trajectories starting from an interior point of the domain Δ_4 terminate at the point $(0,0,1)$. Taking this into account, we can conclude that the invariant manifold $\mathfrak{M}$ passes through the point $(0,0,1)$. On the other hand, due to the comment made above, the invariant line (4.20) also belongs to the manifold $\mathfrak{M}$. Therefore we obtain a qualitative picture of the invariant manifold $\mathfrak{M}$ as shown in figure 7.2 (compare figure 4.7 from section 4.3).

Manifold $\mathfrak{M}$ separates the domain Δ_4 into two regions with different behaviour of the solutions. Any solution trajectory with the initial condition lying between $\mathfrak{M}$ and $(0,1,0)$ ends at the point $(0,1,0)$ (corresponding to the population in which all members use strategy ATP). Solution trajectories starting on the other side of the manifold end at the point $(1,0,0)$ (corresponding to the population in which all members use strategy CP). Solution trajectories starting exactly in the manifold $\mathfrak{M}$ end at the point $\left(\frac{929}{1503}, \frac{574}{1503}, 0\right)$ (corresponding to the population in which $\frac{929}{1503}$ proportion of members use strategy CP and $\frac{574}{1503}$ proportion of members use strategy ATP), but the probability to start from $\mathfrak{M}$ is zero since its co-dimension is one.

Figure 7.2. The complete picture of the dynamics for the game (7.1). The invariant manifold $\mathfrak{M}$ is shaded.



In this example we, therefore, have found that it is possible to evolve from populations which consist of a majority of P and U players to populations of cooperating or allocating tasks individuals.

7.2 Two Prisoners' Dilemma Games and a Hawk-Dove Game.

In this section we analyse the Replicator Dynamics of the multi-state model considered in example 5.1 of section 5.5. In the example considered context-games G_1 and G_2 (activity context-games) are modelled by a Prisoners' Dilemma Game, which is widely used as a generic model of social interactions. Context-game G_0 is modelled by a Hawk-Dove Game, which is often used when a sharing context of interaction is considered.

We begin with a generic model, with payoffs satisfying the following conditions.

$$\begin{gathered} c_1 = \tfrac{v}{2}; c_2 = 0; c_3 = v; c_4 = \tfrac{v-c}{2}, \qquad \text{where } c > v > 0; \\ h_3 > h_1 > h_4 > h_2, \qquad 2h_1 > h_2 + h_3; \qquad t_3 > t_1 > t_4 > t_2, \qquad 2t_1 > t_2 + t_3. \end{gathered} \tag{7.3}$$

Then for the parameters of the payoff matrix (6.1) we obtain

$$\varkappa = \frac{1}{4}\frac{v+2\beta(ph_1+t_1(1-p))}{1-\beta^2}; \quad \omega = \frac{1}{4}\frac{v+2\beta(ph_3+t_3(1-p))}{1-\beta^2};$$
$$\psi = \frac{1}{4}\frac{v+2\beta(ph_2+t_2(1-p))}{1-\beta^2}; \quad \chi = \frac{1}{4}\frac{v+2\beta(ph_4+t_4(1-p))}{1-\beta^2};$$
$$C_2 = \frac{\beta z}{(1-\beta)}; \qquad C_3 = v + \frac{\beta z}{(1-\beta)}; \qquad C_4 = \frac{v-c}{2} + \frac{\beta z}{(1-\beta)}.$$

We can notice that the following inequalities are satisfied for the parameters of the payoff matrix A (6.1) of chapter 6.

$$\omega > \varkappa > \chi > \psi, \qquad 2\varkappa > \psi + \omega; \qquad C_3 > C_2 > C_4. \tag{7.4}$$

Let us also suppose that conditions (6.6) obtained in Proposition 6.1

$$4\varkappa > 2(\psi + \omega) > C_2 + C_3 > \max\{2C_4, 4\chi\}$$

are satisfied. Recall that these conditions guarantee that S and CP are cooperative strategies, ATP and AT are partially cooperative and P, LFS and U are non cooperative.

Remember that we denote by $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 the proportion of the individuals in the population who adopt behaviour S, CP, ATP, AT, P, LFS and U, respectively. Then we can prove that populations in which everyone adopts one of the P, LFS and U strategies are unstable in the Replicator Dynamics for the parameters considered.

Proposition 7.1 *Points* $(0,0,0,0,0,0)$ *(corresponding to the population in which all members use strategy U),* $(0,0,0,0,1,0)$ *(corresponding to the population in which all members use strategy P) and* $(0,0,0,0,0,1)$ *(corresponding to the population in which all members use strategy LFS) are not asymptotically and, therefore, not evolutionarily stable if the payoffs satisfy conditions (7.3).*

Proof. Since $C_4 - C_2 < 0$ and $2\chi - C_3 < 0$, conditions (6.8) are not satisfied for any of the points $(0,0,0,0,0,0)$, $(0,0,0,0,1,0)$ and $(0,0,0,0,0,1)$. Moreover it then follows from proposition 4.1 that each of these points will have at least one positive eigenvalue and, therefore, is unstable.

7.2.1 Fixing payoffs in the Prisoners Dilemma Games.

To be able to perform further analysis payoff values h_i, t_i, $i = 1,2,3,4$, and p will be fixed in such a way that

$$ph_1 + t_1(1-p) = 3; \quad ph_2 + t_2(1-p) = 0; \quad ph_3 + t_3(1-p) = 5; \quad ph_4 + t_4(1-p) = 1.$$

for example, the payoffs in games G_1 and G_2 can be equal

$$h_1 = t_1 = 3, \quad h_2 = t_2 = 0, \quad h_3 = t_3 = 5, \quad h_4 = t_4 = 1$$

and the probability p can take any value from zero to one. Then

$$\varkappa = \frac{v+6\beta}{4(1-\beta^2)}; \quad \omega = \frac{v+10\beta}{4(1-\beta^2)}; \quad \psi = \frac{v}{4(1-\beta^2)}; \quad \chi = \frac{v+2\beta}{4(1-\beta^2)};$$
$$C_2 = \frac{\beta z}{(1-\beta)}; \qquad C_3 = v + \frac{\beta z}{(1-\beta)}; \qquad C_4 = \frac{v-c}{2} + \frac{\beta z}{(1-\beta)}.$$

In this case conditions (6.6) can be re-written as

$$\frac{v+6\beta}{1-\beta^2} > \frac{v+5\beta}{1-\beta^2} > v + \frac{2\beta z}{(1-\beta)} > \max\left\{v - c + \frac{2\beta z}{(1-\beta)}, \frac{v+2\beta}{1-\beta^2}\right\}$$

Solving these inequalities we obtain the following conditions on the parameters β, v, z and c.

$$\left[\begin{array}{l} \left\{\begin{array}{l} \frac{v\beta^2 - c\beta^2 + 2\beta + c}{2\beta(\beta+1)} > z > \frac{v\beta+2}{2(\beta+1)} \\ \frac{v\beta+5}{2(\beta+1)} > z \end{array}\right. \\ \quad \frac{v\beta+5}{2(\beta+1)} > z > \frac{v\beta^2 - c\beta^2 + 2\beta + c}{2\beta(\beta+1)} \end{array}\right.$$

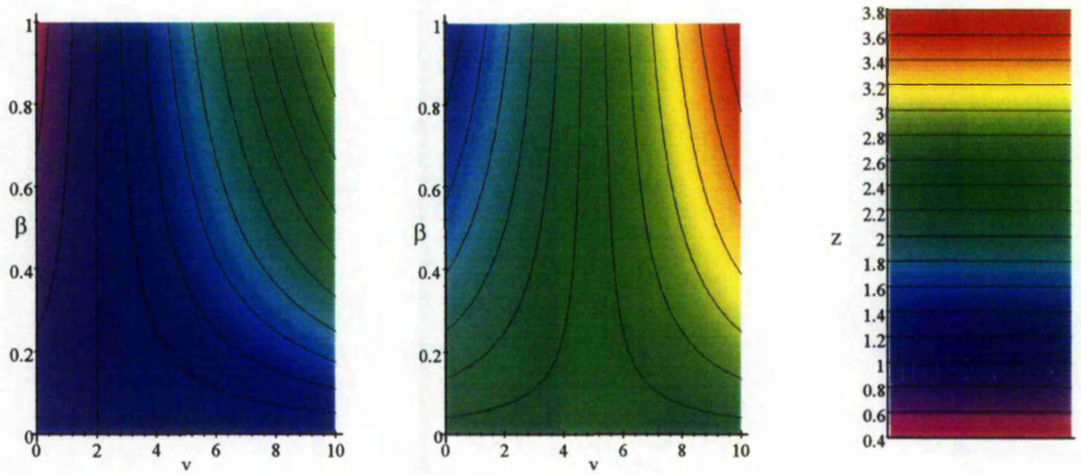
And finally, since for any value of parameters $\frac{v\beta^2 - c\beta^2 + 2\beta + c}{2\beta(\beta+1)} \geq \frac{v\beta+2}{2(\beta+1)}$, we find that conditions (6.6) can in this case be written as

$$\left\{\begin{array}{l} \frac{(5+\beta v)}{2(1+\beta)} > z > \frac{(2+\beta v)}{2(1+\beta)} \\ z \neq \frac{v\beta^2 - c\beta^2 + 2\beta + c}{2\beta(\beta+1)} \end{array}\right. . \tag{7.5}$$

The latter condition necessary to avoid non generic choices of parameters and appears because the inequalities (6.6) are strict.

The plots in figure 7.3 below show the lower and upper boundary for values of the parameter z. On the left-hand plot the colours represent the value of $\frac{(2+\beta v)}{2(1+\beta)}$ for each choice of β and v, that is the lowest value that z can take in order to satisfy condition (7.5). On the middle plot the colours represent the value of $\frac{(5+\beta v)}{2(1+\beta)}$ for each choice of β and v, that is the highest value that z can take in order to satisfy condition (7.5). Correspondence between colours and values of z is shown on the right-hand plot. On the plots below the range of the parameter v is between 0 and 10. It seems to be the most interesting case with respect to the modelling of social behaviours. The further analysis will be restricted to this domain. The analogous plots can be easily constructed for other values of v if necessary.

Figure 7.3. Lower and upper boundaries for z such that conditions (6.6) hold.



Lower boundary for z. Upper boundary for z. Colors and z values.

We will now obtain conditions that guarantee that conditions of theorem 4.2 hold for some subset of the sets I_1 (which correspond to cooperative populations) or I_2 (which correspond to alternating tasks populations).

Let us consider interval $I_1 = \{1-\alpha_1, \alpha_1 0, 0, 0, 0, 0\}$, $\alpha_1 \in [0,1]$. Condition (6.9) has the following form for this example.

$$\begin{cases} \frac{v+6\beta}{2(1-\beta^2)} - \left(v + \frac{\beta z}{(1-\beta)}\right) > 0 \\ \frac{-2\beta}{1-\beta^2} + \beta^2 \left(\frac{v+10\beta}{2(1-\beta^2)} - \frac{\beta z}{(1-\beta)}\right) > 0 \end{cases}.$$

These inequalities can be rewritten as conditions on the parameter z.

$$\begin{cases} \frac{2v\beta^2+6\beta-v}{2\beta(\beta+1)} > z \\ \frac{10\beta^2+v\beta-4}{2\beta^2(\beta+1)} > z \end{cases}.$$

Let us remember that z also satisfies (7.5). Therefore for β and v we have that

$$\frac{2v\beta^2+6\beta-v}{2\beta(\beta+1)} > \frac{(2+\beta v)}{2(1+\beta)}, \text{ which implies that } v < \frac{4\beta}{1-\beta^2},$$

and

$$\frac{10\beta^2+v\beta-4}{2\beta^2(\beta+1)} > \frac{(2+\beta v)}{2(1+\beta)}, \text{ which imples that } v > \frac{4-8\beta^2}{\beta(1-\beta^2)}.$$

Hence if v and β are such that $\frac{4-8\beta^2}{\beta(1-\beta^2)} < v < \frac{4\beta}{1-\beta^2}$ then it is possible to chose z in such a way that conditions of theorem 4.2 holds for the set

$$\mathfrak{S}_1 = \left\{ (1-\alpha_1, \alpha_1 0, 0, 0, 0, 0) : \alpha_1 \in \left(\frac{4}{\beta\left(v + 10\beta - 2z\beta^2 - 2\beta z\right)}, 1 \right] \right\}.$$

The lower boundary for the parameter z is described by function

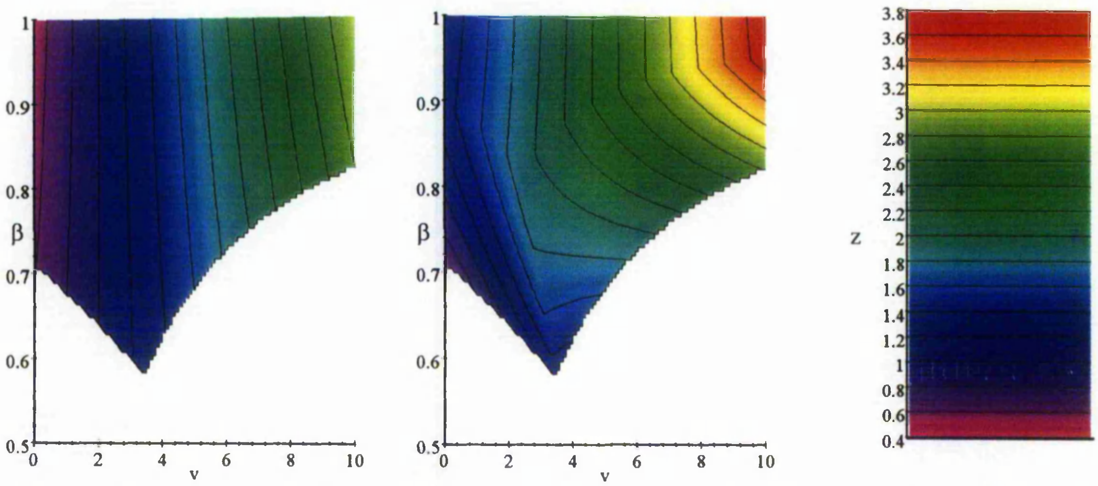
$$g(v,\beta) = \begin{cases} \frac{(2+\beta v)}{2(1+\beta)}, & \text{if } \frac{4-8\beta^2}{\beta(1-\beta^2)} < v < \frac{4\beta}{1-\beta^2}; \\ \text{void} & \text{otherwise}. \end{cases},$$

the upper boundary is defined by function

$$h(v,\beta) = \begin{cases} \min\left\{ \frac{2v\beta^2+6\beta-v}{2\beta(1+\beta)}, \frac{10\beta^2+v\beta-4}{2\beta^2(1+\beta)}, \frac{(5+\beta v)}{2(1+\beta)} \right\}, & \text{if } \frac{4-8\beta^2}{\beta(1-\beta^2)} < v < \frac{4\beta}{1-\beta^2}; \\ \text{void} & \text{otherwise}. \end{cases}$$

On the left-hand plot below the colours represent the value of $g(v,\beta)$ for each choice of β and v. On the middle plot the colours represent the value of $h(v,\beta)$ for each choice of β and v.

Figure 7.4. Lower and upper boundaries for z such that there exists subset $\mathfrak{S}_1$ of the set I_1 for which conditions of theorem 4.2 hold.



$g(v,\beta)-$ lower boundary for z $h(v,\beta)-$ upper boundary for z Colors and z values.

Now consider the interval $I_2 = \{(0,0,\alpha_2,1-\alpha_2,0,0) : \alpha_2 \in [0,1]\}$. We find that conditions (6.13) are as follows in this case.

$$\left\{\begin{array}{l} \frac{1}{2}\frac{2v\beta^2-v+5\beta}{\beta(\beta+1)} > z \\ \frac{1}{2}\frac{7\beta^2+v\beta-2}{\beta^2(1+\beta)} > z \\ \left[\begin{array}{l} z < \frac{1}{2}\frac{v\beta^3-c\beta^3-2+7\beta^2+c\beta}{\beta^2(1+\beta)} \\ \left\{\begin{array}{l} z < \frac{1}{2}\frac{2v\beta^3-2+7\beta^2-v\beta}{\beta^2(1+\beta)} \\ z > \frac{1}{2}\frac{v\beta^3-c\beta^3-2+7\beta^2+c\beta}{\beta^2(1+\beta)} \\ \frac{-2+\beta^2}{\left(2\beta^2 z-6\beta+2\beta z-v\right)} + \frac{-2v\beta^3+2\beta^3 z-7\beta^2+2\beta^2 z+v\beta+2}{(v+c)(-1+\beta)(1+\beta)} < 0 \end{array}\right. \end{array}\right. \end{array}\right.$$

Let us notice that

$$v\beta^3 - c\beta^3 - 2 + 7\beta^2 + c\beta > 2v\beta^3 - 2 + 7\beta^2 - v\beta,$$

because

$$0 > -\beta(1-\beta)(1+\beta)(v+c).$$

Therefore we find that condition (6.13) can be simplified and has the form

$$\left\{\begin{array}{l} \frac{2v\beta^2-v+5\beta}{2\beta^2+2\beta} > z \\ \frac{1}{2}\frac{7\beta^2+v\beta-2}{\beta^2(1+\beta)} > z \\ \frac{v\beta^3-c\beta^3-2+7\beta^2+c\beta}{2\beta^2(1+\beta)} > z \end{array}\right. .$$

Moreover, we notice that $\frac{1}{2}\frac{7\beta^2+v\beta-2}{\beta^2(1+\beta)} < \frac{v\beta^3-c\beta^3-2+7\beta^2+c\beta}{2\beta^2(1+\beta)}$ for any $c > v$ and, therefore, we find that conditions (6.13) are changed into

$$\left\{\begin{array}{l} \frac{2v\beta^2-v+5\beta}{2\beta^2+2\beta} > z \\ \frac{1}{2}\frac{7\beta^2+v\beta-2}{\beta^2(1+\beta)} > z \end{array}\right. .$$

It is also required that z satisfies (7.5). Therefore, for β and v we have

$$\left\{\begin{array}{l} \frac{2v\beta^2-v+5\beta}{2\beta^2+2\beta} > \frac{(2+\beta v)}{2(1+\beta)}, \text{ which implies that } v < 3\frac{\beta}{1-\beta^2}, \\ \frac{1}{2}\frac{7\beta^2+v\beta-2}{\beta^2(1+\beta)} > \frac{(2+\beta v)}{2(1+\beta)}, \text{ which implies that } v > \frac{2-5\beta^2}{\beta\left(1-\beta^2\right)}. \end{array}\right.$$

Hence if v and β are such that $\frac{2-5\beta^2}{\beta\left(1-\beta^2\right)} < v < \frac{3\beta}{1-\beta^2}$ then it is possible to choose such z that there is a subset of the interval $I_2 = \{(0,0,\alpha_2,1-\alpha_2,0,0) : \alpha_2 \in [0,1]\}$ for which conditions of theorem 4.2 hold. The lower boundary for the parameter z is described by the function

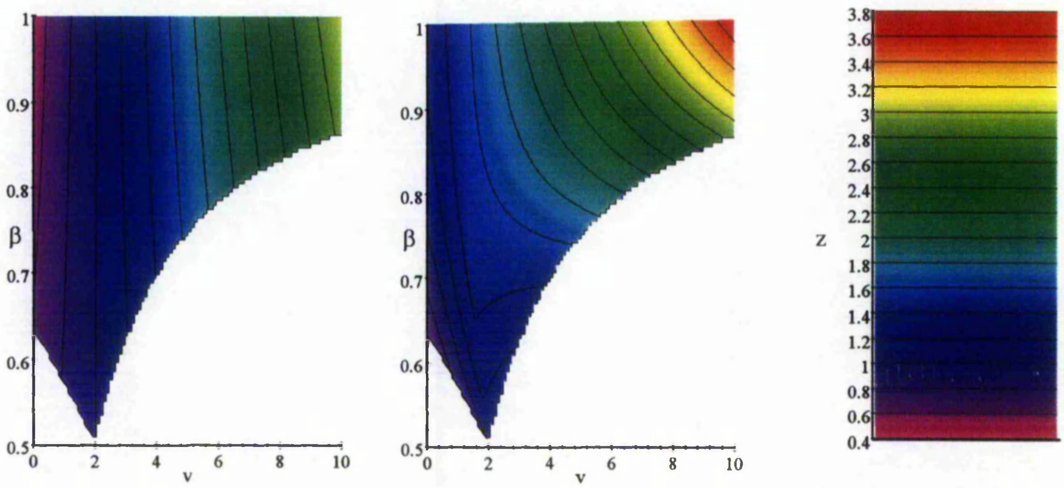
$$l(v,\beta) = \left\{\begin{array}{ll} \frac{(2+\beta v)}{2(1+\beta)}, & \text{if } \frac{2-5\beta^2}{\beta\left(1-\beta^2\right)} < v < 3\frac{\beta}{1-\beta^2}; \\ \text{void} & \text{otherwise}. \end{array}\right.$$

the upper boundary is described by function

$$k(v,\beta)=\begin{cases}\min\left\{\frac{2v\beta^2-v+5\beta}{2\beta(\beta+1)},\frac{7\beta^2+v\beta-2}{2\beta^2(1+\beta)},\frac{(5+\beta v)}{2(1+\beta)}\right\}, & \text{if } \frac{2-5\beta^2}{\beta(1-\beta^2)}<v<3\frac{\beta}{1-\beta^2};\\ \text{void} & \text{otherwise}.\end{cases}$$

On the left-hand plot the colours represent the value of $l(v,\beta)$ for each choice of β and v. On the middle plot the colours represent the value of $k(v,\beta)$ for each choice of β and v.

Figure 7.5. Lower and upper boundaries for z such that there exists subset $\mathfrak{S}_2$ of the set I_2 for which conditions of theorem 4.2 hold.



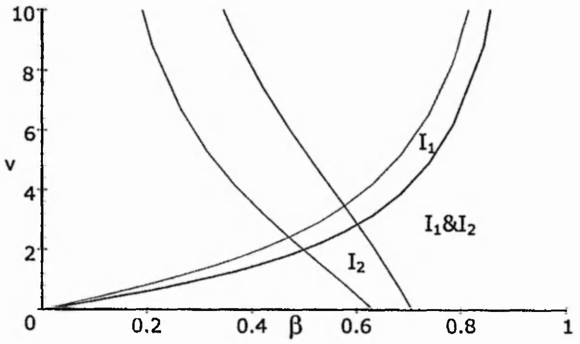
$l(v,\beta)$– lower boundary for z $k(v,\beta)$– upper boundary for z Colors and z values.

Let us finally define values of parameters v,β and z such that there are subsets $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in both intervals I_1 and I_2 for which conditions of theorem 4.2 hold. Parameters v and β therefore must satisfy

$$\begin{cases}\frac{4-8\beta^2}{\beta(1-\beta^2)}<v<\frac{4\beta}{1-\beta^2}\\ \frac{2-5\beta^2}{\beta(1-\beta^2)}<v<\frac{3\beta}{1-\beta^2}\end{cases}. \tag{7.6}$$

In figure 7.6 below $\frac{4-8\beta^2}{\beta(1-\beta^2)}$ is given in blue, $\frac{4\beta}{1-\beta^2}$ is in green, $\frac{2-5\beta^2}{\beta(1-\beta^2)}$ is in red and $\frac{3\beta}{1-\beta^2}$ is given by a black line. Here the label "I_1&I_2" indicates ranges for parameters v and β for which there are subsets $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in both intervals I_1 and I_2 for which conditions of theorem 4.2 hold. The label "I_1" indicates ranges for parameters v and β for which there is subset $\mathfrak{S}_1$ in the interval I_1 but no subset $\mathfrak{S}_2$ in the interval I_2 for which conditions of theorem 4.2 hold. (The label "I_2" represents the reverse.)

Figure 7.6. Ranges for parameters v and β for which there are subsets $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in either one or both intervals I_1 or I_2 for which conditions of theorem 4.2 hold.



It is clear from figure 7.6 that condition (7.6) is equivalent to $\frac{4-8\beta^2}{\beta(1-\beta^2)} < v < \frac{3\beta}{1-\beta^2}$. Now we can define the lower boundary function $n(v,\beta)$ for the parameter z as follows

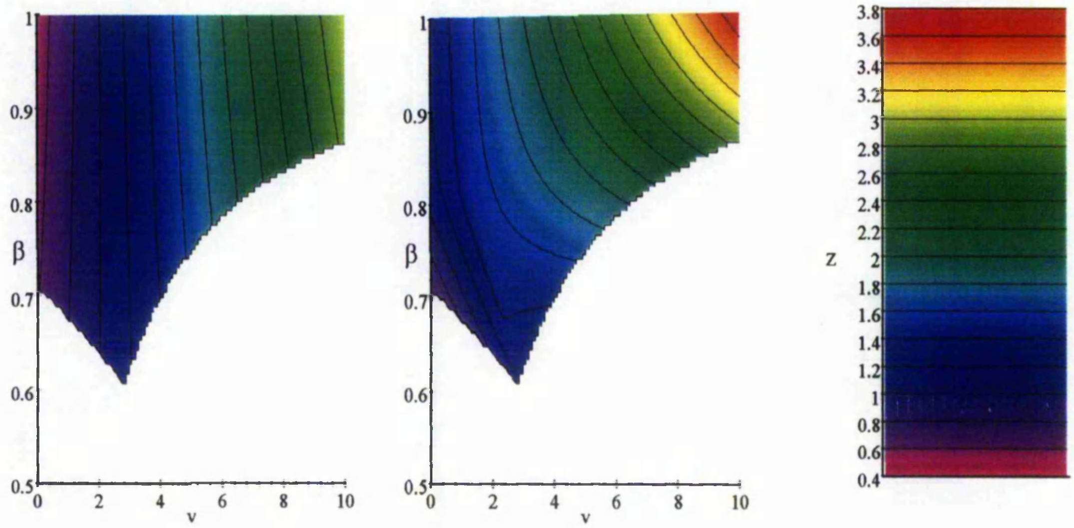
$$n(v,\beta) = \begin{cases} \frac{(2+\beta v)}{2(1+\beta)}, & \text{if } \frac{4-8\beta^2}{\beta(1-\beta^2)} < v < \frac{3\beta}{1-\beta^2}; \\ \text{void}, & \text{if otherwise}. \end{cases} \tag{7.7}$$

The upper boundary function $m(v,\beta)$ for the parameter z is

$$m(v,\beta) = \begin{cases} \min\left\{\frac{2v\beta^2+6\beta-v}{2\beta(1+\beta)}, \frac{10\beta^2+v\beta-4}{2\beta^2(1+\beta)}, \frac{2v\beta^2-v+5\beta}{2\beta(\beta+1)}, \frac{7\beta^2+v\beta-2}{2\beta^2(1+\beta)}, \frac{(5+\beta v)}{2(1+\beta)}\right\}, & \text{if } \frac{4-8\beta^2}{\beta(1-\beta^2)}<v<\frac{3\beta}{1-\beta^2}; \\ \text{void} & \text{otherwise}. \end{cases} \tag{7.8}$$

The plots in figure 7.7 represent these functions. To obtain the conditions for which there is only one interval $\mathfrak{S}_1$ or $\mathfrak{S}_2$ we should refer to figures 7.4 and 7.5, respectively, and consider the lower and the upper boundaries given in these figures restricted to the areas of parameters v and β labeled "I_1" or "I_2" in figure 7.6, respectively.

Figure 7.7. Lower and upper boundaries for z such that there are subsets $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in both intervals I_1 or I_2 for which conditions of theorem 4.2 hold.



$n(v,\beta)-$ lower boundary for z $m(v,\beta)-$ upper boundary for z Colors and z values.

7.2.2 Fixing payoffs in the Hawk-Dove Game and the background payoff.

We now fix parameters v and c and demonstrate how the choice of β affects the stationary points of the dynamical system. We choose v and c be the same as considered in the example 5.1 of section 5.5. That is $v = 6$ and $c = 7$, and in terms of the payoffs for the multi-state game we have

$$c_1 = 3, \quad c_2 = 0, \quad c_3 = 6, \quad c_4 = -\frac{1}{2}. \tag{7.9}$$

Recall that we have already fixed the payoffs in the games G_1 and G_2 as follows.

$$ph_1 + t_1(1-p) = 3; \quad ph_2 + t_2(1-p) = 0; \quad ph_3 + t_3(1-p) = 5; \quad ph_4 + t_4(1-p) = 1.$$

It has been shown in section 5.5 that in this case strategies S, AT, LFS and U are not Nash Equilibria for any values of the parameter z and discount factor $\beta \in [0,1)$. For strategies CP, ATP and P there is a range of values for z and β for which these strategies are Nash

Equilibria. We wish to choose values of the parameter z and discount factor β such that the strategy P is not a Nash Equilibrium and strategies CP and ATP are Nash Equilibria. In this case there are subsets $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in both intervals I_1 and I_2 for which conditions of theorem 4.2 hold. As we have seen in section 5.5 there exist a range of value for z and β then these assumptions are true, in particular, they are true if $z = 2$. This will be the case which we intend to analyse in more detail.

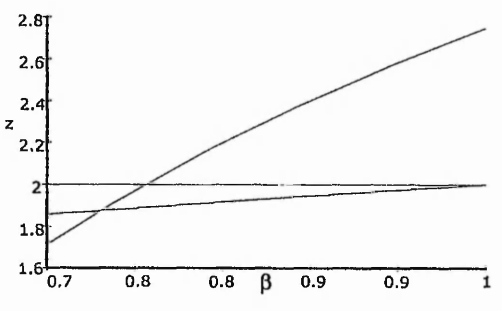
Remark 7.1 *If non-cooperative strategies LFS, P and U are not Nash Equilibria then it may be possible to evolve to cooperative populations from non-cooperative populations. Notice though that this condition is only necessary but not sufficient since there may exist non-cooperative populations which consist of a mixture of individuals using different non-cooperative strategies which can be evolutionarily and asymptotically stable. In the next sections we show that this is not the case for the example where we fix parameter z to be equal 2 and $\beta = .88$.*

From the previous section it is clear that if there are subsets $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in both intervals I_1 and I_2 for which conditions of theorem 4.2 hold then the choice of the values for parameter β such that these conditions are satisfied is determined by the lower boundary function $n(v, \beta)$ (7.7) and the upper boundary function $m(v, \beta)$ (7.8). If $v = 6$ and $z = 2$ then

$$n(6, \beta) = \frac{2+6\beta}{2+2\beta},$$
$$m(6, \beta) = \frac{1}{2}\frac{12\beta^2 - 6 + 5\beta}{\beta(\beta+1)}.$$

On the plot below, the function $n(6, \beta)$ is given by a blue line and the function $m(6, \beta)$ is given by a green line. The red line represents the value of parameter z which is fixed at 2.

Figure 7.8. The exact boundary for the parameter β.



The point of intersection of $m(6, \beta)$ (in green) and $z = 2$ (in red) define the exact boundary for the parameter β. At this point the value of β is

$$\beta_0 = \frac{-1+\sqrt{193}}{16} \approx .805\,78.$$

Let us mention here that the second condition $z \neq \frac{v\beta^2 - c\beta^2 + 2\beta + c}{2\beta(\beta+1)}$ of (7.5) in this case mean that $\beta \neq 1$, which does not add any extra restrictions on β.

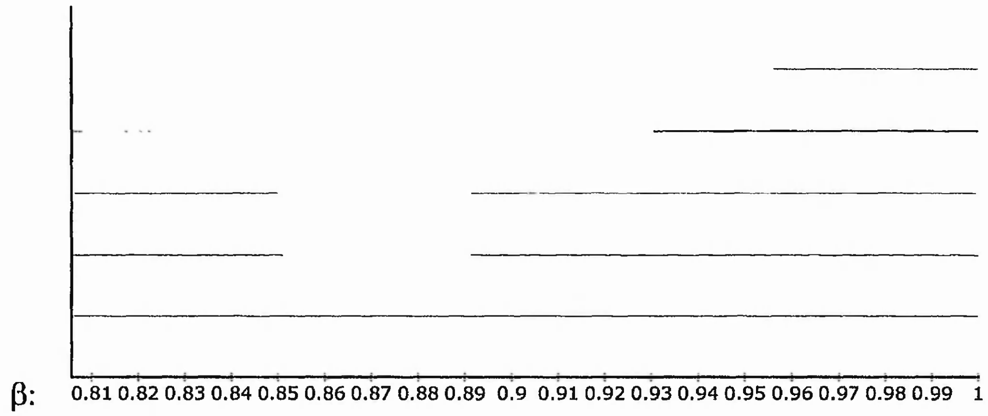
Once all parameters of the model (except β) are fixed it is possible to follow the approach described in section 6.3 and solve system (6.17) using "Maple" software. It is possible to determine all stationary points of the Replicator Dynamics that belong to the simplex Δ. appendix 1 contains all such points for the parameters (7.9) and $\beta \in (\beta_0, 1)$. These results are summarised in table 7.1 and figure 7.9.

Table 7.1. Intervals for β for which different stationary points and sets exist.

Points	Interval for β	Shown in the following colour in figure 7.9.
$\{\overline{1,2}\}$,$\{\overline{3,4}\}$,$\{2,\overline{3,4}\}$, {3,5,6,7},{3,6,7},{3,5,6}, {3,6},{5,6,7},{5,6},{5,7},{6,7}	$\beta \in (\beta_0, 1)$	red
{4,6}	$\beta \in (\beta_0, 0.85118) \cup (0.89147, 1)$	blue
{4,6,7}	$\beta \in (\beta_0, 0.85021) \cup (0.89147, 1)$	cyan
{1,2,3,5},{1,2,3,6},{1,2,4,6}	$\beta \in (0.93078, 1)$	black
{1,2,3,5,6}	$\beta \in (0.95629, 1)$	green
{2,4,6},{2,4,6,7}	$\beta \in (0.82063, 0.84859)$	yellow
$\{2,\overline{3,4},7\}$	$\beta \in (\beta_0, 0.82958)$	grey

Here a combination $\{j_1, \ldots, j_m\}$ stands for the stable point of Replicator Dynamics such that the coordinates $x_{j_1}, \ldots, x_{j_m}$ take non-zero values and all remaining coordinates x_i are equal to zero. Since by x_1, x_2, x_3, x_4, x_5, x_6 and x_7 we denote the proportion of the individuals in the population who adopt behaviour S, CP, ATP, AT, P, LFS and U, respectively, this means that there exists a stationary population which consists only of proportions of individuals who adopt types of behaviour corresponding to the above combinations of indices. For example combination $\{1, 2, 3, 5\}$ which appears in the fourth row of table 7.1 means that if $\beta \in (0.93078, 1)$ then there exists a stationary population which consists of a mixture of individuals who adopt strategies S, CP, ATP, AT or P. The line over indices $\overline{3,4}$ means that this combination represents not a single point but a stationary set of populations in which proportions of individuals using ATP and AT strategies may vary (see appendix 1 for details). For the convenience of comparison the intervals on β for which corresponding points or sets belongs to the simplex Δ are shown in figure 7.9. Last column of table 7.1 indicates colours in which the intervals are shown in figure 7.9. For example interval $(0.93078, 1)$ is shown in black.

Figure 7.9. Intervals for β for which different stationary points and sets exist.



It is clear from the above results that the stationary points and sets appear and disappear depending on what value the parameter β takes.

7.2.3 Fixing β.

At this moment the last parameter β should be fixed in order to proceed with a qualitative analysis of the dynamics. I have chosen $\beta = \frac{22}{25} = .88$, as it appears to be a convenient choice. For example, this value of β belongs to the interval $(0.85118, 0.89147)$, and as it can be seen from figure 7.9 the number of stationary points and sets are the smallest in this case. Therefore it is easier to analyse the dynamics.

For this choice of β there are three stationary intervals

$$\{\overline{1,2}\}, \{\overline{3,4}\}, \{2, \overline{3,4}\}$$

for the corresponding Replicator Dynamics (6.7). The stationary point are as follows.

$$\{3,5,6,7\}, \{3,6,7\}, \{3,5,6\}, \{5,6,7\}, \{3,6\}, \{5,6\}, \{5,7\}, \{6,7\}, \{5\}, \{6\}, \{7\}.$$

The exact values of the components for all these stationary points and sets together with their eigenvectors and eigenvalues are contained in appendix 2. The payoff matrix A (6.1) that corresponds to the choice of the parameters made can also be found in this appendix.

7.3 Dynamics on some invariant manifolds.

Using the information represented in appendix 2 we can analyse the solution's behaviour on some invariant subspaces of Δ.

The analysis of the dynamics on the whole six-dimensional simplex Δ is quite complex since there are few techniques available even in the three-dimensional case if the system of differential equations is not integrable. We will use the main techniques described in section 4.1.2. At the beginning we consider subspaces $\Delta_{1,2,3,4,7}$, $\Delta_{5,6,7}$, $\Delta_{5,6,7,2}$ and $\Delta_{5,6,7,3}$ of the simplex Δ and obtain some interesting results. Then, using the method of separatrix approximation, we comment on the overall picture of the dynamics.

7.3.1 The subspace $\Delta_{1,2,3,4,7}$: S, CP, AT, ATP and U strategies.

Let us consider the four-dimensional subspace $\Delta_{1,2,3,4,7}$. This subspace corresponds to populations which consist of different mixtures of individuals who use S, CP, AT, ATP and U strategies. As has been explained in section 6.4 this subspace can be represented as a collection of three-dimensional invariant subspaces

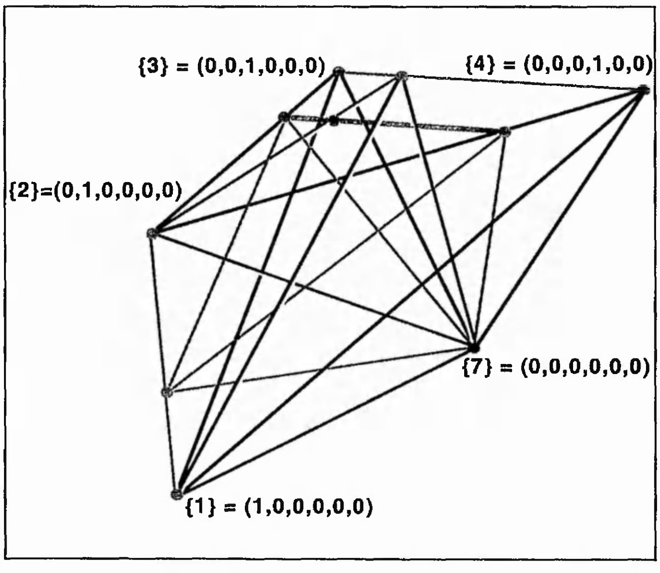
$$M_\alpha = \left\{x : x_3 + \frac{\alpha}{\alpha - 1} x_4 = 0, \quad x_5 = 0, \quad x_6 = 0\right\}, \alpha \in [0,1]. \tag{7.10}$$

Therefore, to describe the dynamics on $\Delta_{1,2,3,4,7}$ it is sufficient to analyse the dynamics on each of the subspaces M_α. If α is fixed then M_α is a three-dimensional linear subspace that passes through the origin point $\{7\}$, points $\{1\}$, $\{2\}$ and the point $\{0,0,\alpha,1-\alpha,0,0\}$ that belongs to the interval $I_2 = \{\overline{3,4}\}$. The second invariant subspace is

$$M = \left\{x : x_1 - \frac{12502}{7221} x_2 + x_3 + x_4 = 0, \quad x_5 = 0, \quad x_6 = 0\right\}. \tag{7.11}$$

As it will be clear from the forthcoming analysis this subspace separates the regions of attraction for the intervals $I_1 = \{\overline{1,2}\}$ and $I_2 = \{\overline{3,4}\}$. Figure 7.10 shows the subspace $\Delta_{1,2,3,4,7}$.

Figure 7.10. The subspace $\Delta_{1,2,3,4,7}$.
The subspace $M_\alpha, \alpha = \frac{3}{4}$, is shown by yellow lines.
The subspace M is shown by green lines.
The stationary points and intervals of the dynamics are shown in red.



Now, let us analyse the dynamics on the subspace M_α, where α is fixed.

The dynamics on the plane $\Delta_{1,2,7}$ is quite simple. All solution trajectories are straight lines that start at the origin (point $\{7\}$) and terminate at some point of the interval $I_2 = \{\overline{1,2}\}$.

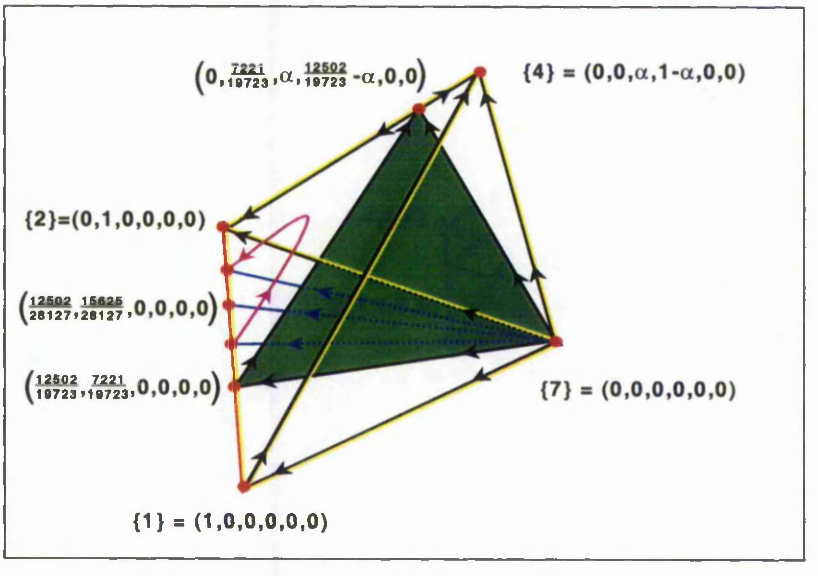
Dynamics on the plane that passes through points $\{1\},\{2\}$ and $(0,0,\alpha,1-\alpha,0,0)$ is similar to the dynamics related to the Iterated Prisoners' Dilemma Game obtained in section 4.1.3. The separatrix line goes from the point $\left(\frac{12502}{19723},\frac{7221}{19723},0,0,0,0\right)$ to the point $\left(0,\frac{7221}{19723},\alpha,\frac{12502}{19723}-\alpha,0,0\right)$. This line divides the plane into two regions. The trajectories from one region (see figure 7.11 below) are all attracted to the point $(0,0,\alpha,1-\alpha,0,0)$. The trajectories from the other region terminate at some point on the interval $I_1 = \{\overline{1,2}\}$.

The plane that passes through points $\{2\}$, $(0,0,\alpha,1-\alpha,0,0)$ and $\{7\}$ also has two distinct regions of solution behaviour. These regions are separated by the trajectory that leaves $\{7\}$ and terminates at $\left(0,\frac{7221}{19723},\alpha,\frac{12502}{19723}-\alpha,0,0\right)$. The solution trajectories from one region are attracted to the point $(0,0,\alpha,1-\alpha,0,0)$ and from the other region to the point $\{2\}$.

The solution trajectories on the plane that passes through points $\{1\}$, $\{(0, 0, \alpha, 1 - \alpha, 0, 0)\}$ and $\{7\}$ are all attracted to the point $(0, 0, \alpha, 1 - \alpha, 0, 0)$.

The intersection of the subspaces M_α (7.10) and M (7.11) is a plane that passes though points $\left(\frac{12502}{19723}, \frac{7221}{19723}, 0, 0, 0, 0\right)$, $\left(0, \frac{7221}{19723}, \alpha, \frac{12502}{19723} - \alpha, 0, 0\right)$ and $\{7\}$. This plane is shown as the green triangle in figure 7.11 This plane divides the subspace M_α: solution trajectories that start above this plane terminate at the point $(0, 0, \alpha, 1 - \alpha, 0, 0)$ (corresponding to a population in which the proportion of ATP players and AT players are as α to $1-\alpha$), solution trajectories that start below this plane terminate at some point of the interval $I_1 = \{\overline{1, 2}\}$ (corresponding to the populations which consist of mixture of players who use strategy S or CP).

Figure 7.11. The subspace $\mathbf{M}_\alpha$.
The intersection of $\mathbf{M}_\alpha$ and M is shown in green.



Since the point $\{(0, 0, \alpha, 1 - \alpha, 0, 0)\}$ belongs to the interval $I_2 = \{\overline{3, 4}\}$ it is clear now that the subspace M is the separatrix subspace for the dynamics on $\Delta_{1,2,3,4,7}$.

7.3.2 The subspace $\Delta_{5,6,7}$: P, LFS and U strategies.

The dynamics on the subspace $\Delta_{5,6,7}$ is considered in this section. This subspace corresponds to populations which consist only of non cooperative types of behaviour P, LFS and U.

To analyse the dynamics on the plane $\Delta_{5,6,7}$ note that the line $x_5 - \frac{47}{326}x_6 = 0$ is invariant. This line passes through the point $\{5, 6, 7\}$ (see appendix 2 for exact values of the coordinates for this point). This point is the only point that has two negative eigenvalues (see appendix 2 where all eigenvalues are given) if the dynamics are restricted to the plane $\Delta_{5,6,7}$. Therefore, there are no limit cycles on the plane $\Delta_{5,6,7}$ and all trajectories that start from the interior of $\Delta_{5,6,7}$ are attracted to the point $\{5,6,7\}$. The qualitative pictures of the dynamics are given in figures 7.12 and 7.13 below.

Figure 7.12. The subspace $\Delta_{5,6,7}$.

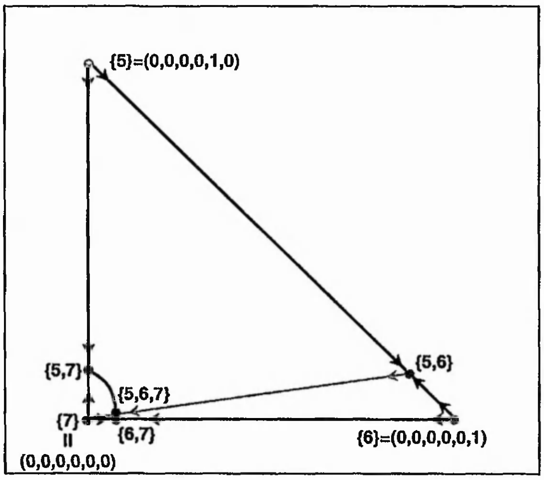
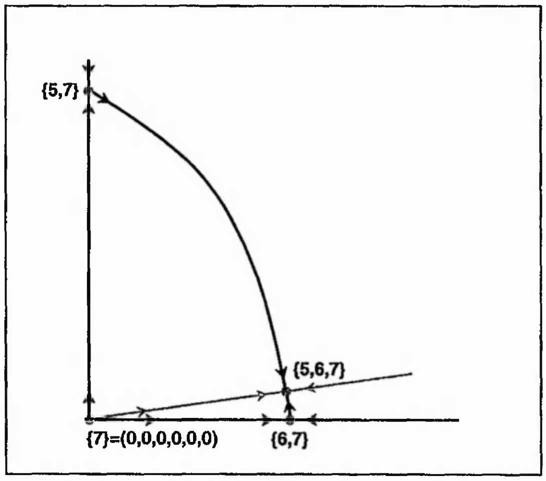


Figure 7.13. The subspace $\Delta_{5,6,7}$. Neighborhood of the point $\{7\}$.



The invariant line $x_5 - \frac{47}{326}x_6 = 0$ is shown in pink.

Red points indicate the stationary points of the dynamical system.

It will be shown in section 7.3.4 that point $\{5,6,7\}$ becomes unstable when the population contains a proportion of *CP* players.

7.3.3 The subspace $\Delta_{3,5,6,7}$: ATP, P, LFS, and U strategies.

Now, let us consider the subspace $\Delta_{3,5,6,7}$. This subspace corresponds to populations which consist of different mixtures of individuals who use *ATP*, *P*, *LFS* and *U* strategies.

Dynamics on the plane $\Delta_{5,6,7}$ has been described in the previous section.

All the solutions trajectories starting from the interior of the plane $\Delta_{3,5,7}$ (corresponding to populations which consist of different mixtures of *ATP*, *P* and *U* players) are attracted to the point $\{3\}$ (corresponding to the population which consists of *ATP* players).

The plane $\Delta_{3,5,6}$ (corresponding to populations which consist of different mixtures of *ATP*, *P* and *LFS* players) has an interior saddle point $\{3,5,6\}$. A non integrable invariant curve (shown in blue in figures 7.13 and 7.14 below) passes through this point splitting $\Delta_{3,5,6}$ into two parts: the trajectories starting above this curve are attracted to the point $\{3\}$, the trajectories starting below this curve are attracted to the point $\{5,6\}$ (corresponding to a population which consists of a mixture of P and LFS players).

There are two more invariant lines (shown in blue) for the subspace $\Delta_{3,5,6,7}$.

The first line

$$\begin{cases} x_5 = 0 \\ x_6 = \frac{500\,625}{29\,234} x_3 \end{cases}$$

belongs to the plane $\Delta_{3,6,7}$ (corresponding to populations which consist of different mixtures of *ATP*, LFS and U players). It passes through points $\{7\}$, $\{3,6,7\}$ and $\{3,6\}$ and divides $\Delta_{3,6,7}$ into two regions. Trajectories starting above this line are attracted to the point $\{3\}$. Trajectories starting below this line are attracted to the point $\{6,7\}$ (corresponding to a population which consists of a mixture of LFS and U players).

The second line

$$\begin{cases} x_5 = \frac{2008\,750\,000}{1251\,537\,143} x_3 \\ x_6 = \frac{1213\,715\,695\,625}{117\,644\,491\,442} x_3 \end{cases}$$

belongs to the interior of $\Delta_{3,5,6,7}$. It passes through points $\{7\}$, $\{3,5,6,7\}$ and $\{3,5,6\}$.

The analysis of the dynamics indicates that there is a non integrable separatrix surface (shown in cyan in figures 7.14 and 7.15 below) in the subspace $\Delta_{3,5,6,7}$. It contains all invariant lines mentioned above together with the line

$$\begin{cases} x_3 = 0 \\ x_6 = 0 \end{cases}.$$

Any trajectory starting from an interior point of $\Delta_{3,5,6,7}$ that lies above this surface is attracted to the point $\{3\}$. Trajectories starting from interior points of $\Delta_{3,5,6,7}$ that lie below this surface are attracted to the point $\{5,6,7\}$.

Figure 7.14. The subspace $\Delta_{3,5,6,7}$.

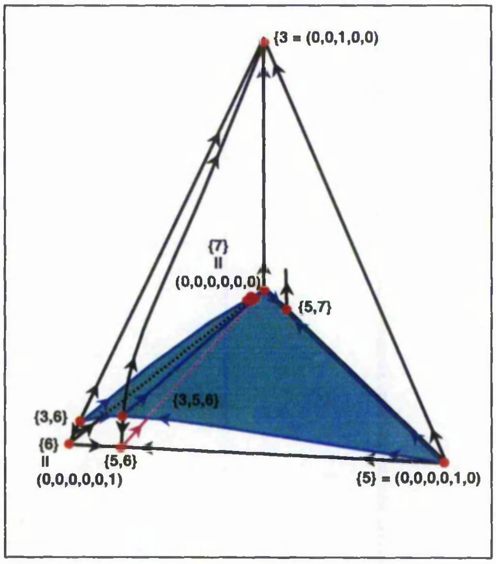
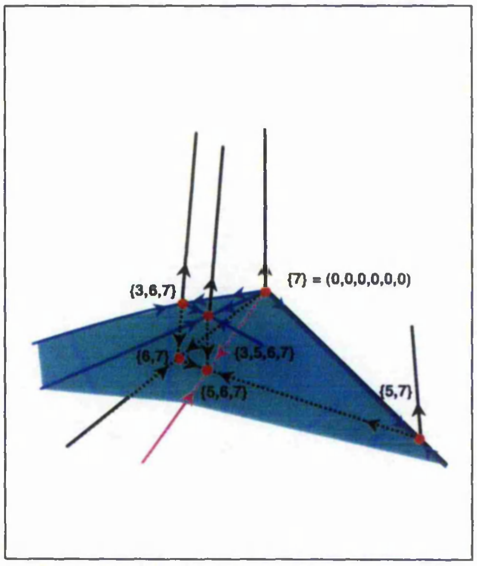


Figure 7.15. The subspace $\Delta_{3,5,6,7}$. Neighborhood of the point $\{7\}$.



The non integrable separatrix surface is shown in cyan.
Red points indicate the stationary points of the dynamical system.

Although the separatrix surface is non integrable it is possible to estimate it in a small neighbourhood of a stationary point using computer simulations of the corresponding Discrete Replicator Dynamics. The results of such simulations can be found in Appendix 3.

7.3.4 The subspace $\Delta_{2,5,6,7}$: CP, P, LFS and U strategies.

Now, consider the subspace $\Delta_{2,5,6,7}$. This subspace corresponds to populations which consist of different mixtures of individuals who use *CP*, *P*, *LFS* and *U* strategies.

Solutions starting at interior points of the planes $\Delta_{2,5,6}$, $\Delta_{2,5,7}$ and $\Delta_{2,6,7}$ all terminate at the point $\{2\}$. The same is true for solutions on the whole subspace $\Delta_{2,5,6,7}$: all solutions starting at interior points are attracted to the point $\{2\}$.

Figure 7.16. The subspace $\Delta_{2,5,6,7}$.

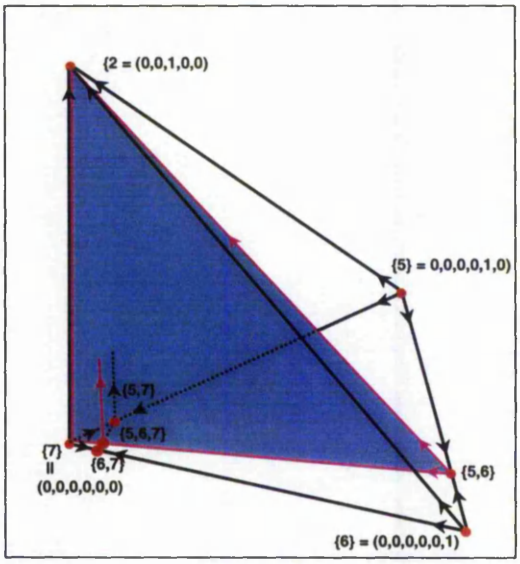
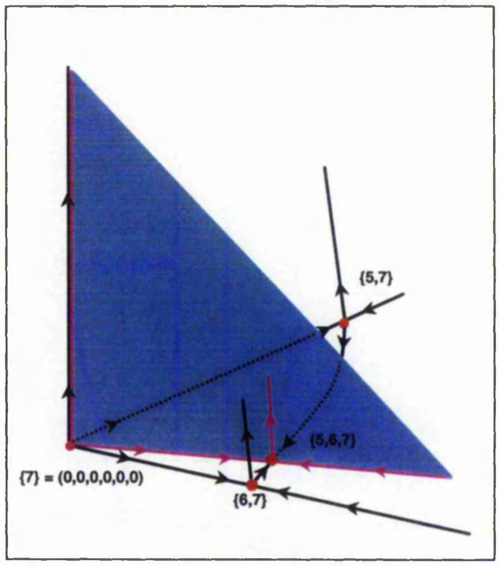


Figure 7.17. The subspace $\Delta_{2,5,6,7}$. Neighborhood of the point $\{7\}$.



The invariant plane $\mathbf{x_5 = \frac{47}{326} x_6}$ is shown in lilac.

Red points indicate the stationary points of the dynamical system.

There is an invariant plane $x_5 = \frac{47}{326}x_6$ passing through the points $\{2\}$, $\{7\}$ and $\{5,6\}$. The point $\{5,6,7\}$ is also in this plane. There is a repulsive eigenvector for this point that belongs to the plane $x_5 = \frac{47}{326}x_6$. Therefore there is a solution trajectory that goes from the point $\{5,6,7\}$ to the point $\{2\}$.

7.3.5 Overall picture.

In general, the following conclusions can be drawn from the analysis of the Replicator Dynamics.

1. Every point of the intervals $\{\overline{1,2}\}_{att} = \{(\alpha, 1-\alpha, 0, 0, 0, 0) : \alpha \in [0, \frac{12\,502}{28\,127}]\}$ (corresponding to the populations which consist of various mixtures of players who use strategy S or CP) and $\{\overline{3,4}\}_{att} = \{(0, 0, 1-\alpha, \alpha, 0, 0) : \alpha \in [0, \frac{22\,479}{32\,054}]\}$(corresponding to the population which consists of various mixtures of players who use strategy S or CP) has non-positive eigenvalues. It has been shown in section 6.2 that the conditions of

theorem 4.2 hold for every point of the intervals

$$\mathfrak{S}_1 = \left\{(\alpha, 1-\alpha, 0, 0, 0, 0) : \alpha \in \left[0, \tfrac{12\,502}{28\,127}\right)\right\}$$

and

$$\mathfrak{S}_2 = \left\{(0, 0, 1-\alpha, \alpha, 0, 0) : \alpha \in \left[0, \tfrac{22\,479}{32\,054}\right)\right\}.$$

Although this property has not been proved analytically to hold for the end points $\left(\frac{12\,502}{28\,127}, 1-\frac{12\,502}{28\,127}, 0, 0, 0, 0\right)$ and $\left(0, 0, 1-\frac{22\,479}{32\,054}, \frac{22\,479}{32\,054}, 0, 0\right)$, the results of computer simulations performed indicate that conditions of the theorem 4.2 hold for the end points as well. (The simulations were performed using the software "Mathematica". The discrete version of the Replicator Dynamics (1.10) was used.) Using these results, we can claim that sets $\{\overline{1,2}\}_{att}$ and $\{\overline{3,4}\}_{att}$ are evolutionarily attractive (see definition 4.1 in section 4.2.1). These sets have the following property: solution trajectories starting at a point from sufficiently small neighbourhood of the set lead back to the set with probability one.

2. All stationary states that do not belong to $\{\overline{1,2}\}_{att}$ or $\{\overline{3,4}\}_{att}$ possess a positive eigenvalue, which means that there is an outgoing solution trajectory that leaves the state.

3. For any stationary state in the dynamics there exists a small deviation from this state such that the solution trajectory leads towards either $\{\overline{1,2}\}_{att}$ or $\{\overline{3,4}\}_{att}$.

   (a) This is a remarkable fact, since it means that a population state that consists of cooperative or partially cooperative types of behaviour can be reached from a population state that consists of non cooperative types of behaviour. Note that such result cannot be obtained in the Iterated Prisoners' Dilemma. From the analysis of Replicator Dynamics for the Iterated Prisoners' Dilemma game considered in section 4.1.3 we can see that, although cooperative populations can evolve, there is a barrier (a separatrix line) which does not allow the cooperative behaviour to originate in a non-cooperative society. In this instance, it is also true for the multi-state model considered that populations consisting of any proportions of non cooperative types of behaviour has a possibility to evolve towards a cooperative or partially cooperative (allocating tasks) population.

   (b) Table 7.2 below shows the changes to the population state resulting from deviations from particular stationary states. In the left-hand column a particular stationary state is given. These states are the vertices of the simplex Δ

$$\begin{aligned}
\{1\} &= (1,0,0,0,0,0) = \{\mathrm{S}\}\,,\\
\{2\} &= (0,1,0,0,0,0) = \{\mathrm{CP}\}\,,\\
\{3\} &= (0,0,1,0,0,0) = \{\mathrm{ATP}\}\,,\\
\{4\} &= (0,0,0,1,0,0) = \{\mathrm{AT}\}\,,\\
\{5\} &= (0,0,0,0,1,0) = \{\mathrm{P}\}\,,\\
\{6\} &= (0,0,0,0,0,1) = \{\mathrm{LFS}\}\,,\\
\{7\} &= (0,0,0,0,0,0) = \{\mathrm{U}\}
\end{aligned}$$

and stationary points that belong to the plane of non cooperative behaviours $\Delta_{5,6,7}$

$$\begin{aligned}
\{5{,}6\} &= \left(0,0,0,0,\tfrac{47}{373},\tfrac{326}{373}\right) = \{\mathrm{P,\ LFS}\}\,,\\
\{5{,}7\} &= \left(0,0,0,0,\tfrac{47}{373},0\right) = \{\mathrm{P,\ U}\}\,,\\
\{6{,}7\} &= \left(0,0,0,0,0,\tfrac{125}{1579}\right) = \{\mathrm{LFS,\ U}\}\,,\\
\{5{,}6{,}7\} &= \left(0,0,0,0,\tfrac{29\,375}{2649\,111},\tfrac{203\,750}{2649\,111}\right) = \{\mathrm{P,\ LFS,\ U}\}\,.
\end{aligned}$$

In the first row of table 7.2 the direction of deviation is shown. This means that a small proportion of mutants (who use the corresponding strategy) is introduced in to the population. Then the solution trajectory $\xi(\tilde{x}^0, t)$ that originates from the new population state $\tilde{x}^0$ goes towards the stationary state that is given in the body of the table. For example, if a small proportion of CP players is introduced into population of P players (which correspond to the point $\{5\} = (0,0,0,0,1,0) = \{\mathrm{P}\}$), the population state will evolve towards the population of CP players (which correspond to the point $\{2\} = (0,1,0,0,0,0) = \{\mathrm{CP}\}$). If the proportion of U players has increased slightly (as a result of mutation) in a population that corresponds to the stationary point $\{5{,}6{,}7\} = \left(0,0,0,0,\frac{29\,375}{2649\,111},\frac{203\,750}{2649\,111}\right) = \{\mathrm{P,\ LFS,\ U}\}$), the new population state evolves back towards the $\{5{,}6{,}7\}$ population.

We can see from table 7.2 that for every non-cooperative stationary state there exists a mutation such that the new population will evolve towards stationary state $\{2\}$ or $\{3\}$. The only possibility of diverting from these states appears if mutations in direction $\{\mathrm{S}\}$ or $\{\mathrm{AT}\}$ occur, respectively. The resulting population will then be changed to some population that belongs to the evolutionary attractive sets $\overline{\{1{,}2\}}_{att}$ or $\overline{\{3{,}4\}}_{att}$, for which the observed behaviour is still cooperative or allocating tasks.

Table 7.2. The changes to the population state resulting from deviations from stationary states.

stationary point \ direction of deviation	{S}	{CP}	{ATP}	{AT}	{P}	{LFS}	{U}
{1} = {S}	■	$\overline{\{1,2\}}_{att}$	{3}	{4}	{5}	{6}	{1}
{2} = {CP}	$\overline{\{1,2\}}_{att}$	■	{2}	{2}	{2}	{2}	{2}
{3} = {ATP}	{3}	{3}	■	$\overline{\{3,4\}}_{att}$	{3}	{3}	{3}
{4} = {AT}	{4}	{4}	$\overline{\{3,4\}}_{att}$	■	{5}	{4}	{4}
{5} = {P}	{5}	{2}	{3}	{5}	■	{5,6}	{5,7}
{6} = {LFS}	{6}	{2}	{6}	{4}	{5,6}	■	{6,7}
{7} = {U}	{1}	{2}	{3}	{4}	{5,7}	{6,7}	■
{5,6} = {P, LFS}	{5,6}	{2}	{5,6}	{5,6}	{5,6}	{5,6}	{5,6,7}
{5,7} = {P, U}	{5,7}	{2}	{3}	{5,7}	{5,7}	{5,6,7}	{5,7}
{6,7} = {LFS, U}	{6,7}	{2}	{6,7}	{4}	{5,6,7}	{6,7}	{6,7}
{5,6,7} = {P, LFS, U}	{5,6,7}	{2}	{5,6,7}	{5,6,7}	{5,6,7}	{5,6,7}	{5,6,7}

7.4 Summary.

It has been shown in this chapter that, by considering games that allow for the possibility of individuals interacting in more than one context, it is possible to demonstrate that apparently altruistic and cooperative behaviour can be the outcome of an evolutionary process.

By analysing the Replicator Dynamics corresponding to this model, it is possible to obtain the following results:

- Cooperative and "allocating tasks" types of behaviour can be the end points of the evolutionary process for this model and the probability of mutations leading to trajectories which diverge from such states is zero.

- A population state that consists of cooperative or "allocating tasks" types of behaviour can be reached from a population state that consists of non cooperative types of behaviour.

- A population that consists of any proportions of non cooperative types of behaviour can evolve towards either the cooperative or "allocating tasks" population.

This results, of course, depend on the values of parameters of the game. If we change the parameters of the model then non-cooperative states may appear which are asymptotically stable or, although non cooperative vertices may be all unstable there can exist periodic solutions for which there is no mutation that forces them leave the subspace of non cooperative populations.

Chapter 8

Conclusions.

In this thesis an approach for modelling social interactions in a context of long-term relationships was developed in order to investigate the apparently altruistic reciprocal and non reciprocal behaviour. The common model of social interactions based on Prisoners' Dilemma game was generalised in a few different ways. The idea, used in the Iterated Prisoners' Dilemma model, that considering the interaction of the two players not in isolation but in the context of conditions in which such an interaction takes place (for example the long-term repeated interaction) allows us to conclude that cooperative behaviour may be rational or evolved in this context. In this thesis this idea was taken further to consider different contexts of interaction.

For example a three player model was introduced in which the third player interacts with two other players engaged in a single interaction Prisoners' Dilemma. The existence of the third player in the interaction changed the payoffs in such a way that the two players were induced to cooperate. If the third player had not been taken into account the first two players would be considered to be playing the Prisoners' Dilemma game and cooperative behaviour would be inexplicable.

Another way of generalising the standard approach was to introduce additional games in the model. The existence of additional states allowed, in particular, for the possibility of introducing completely new types of strategies such as allocating tasks strategies. These strategies are relevant to the explanations of apparently altruistic behaviour since the observed behaviour for them (in a given state) is such that one player is cooperating while the other is defecting.

A number of techniques were used to analyse the multi-state games. These techniques include using competitive Markov decision processes to check that a particular strategy is a Nash Equilibrium and to obtain the total payoffs for the strategies. The techniques of the qualitative analysis of dynamical systems and the "blowing up" technique (which has not been applied before in the context of analysing the Replicator Dynamics) were used to investigate the corresponding dynamical system and to find the end points of the evolutionary process.

The main results obtained from the analysis of the multi-state games are the following.

1. A strategy "allocating tasks with punishment" can be a Nash Equilibrium and populations which consist of different mixtures of "allocating tasks without punishment" and "allocating tasks with punishment" players can be the end points of the evolutionary process. Such a strategy does not have an analogue in the Iterated Prisoners' Dilemma game. Using this strategy one player cooperates in game G_1 while the other player defects and the first player defects in game G_2 while the other cooperates. Therefore, considering similar strategies may provide a framework for the explanation of reciprocal altruism.

2. There exists a range of parameters in the model for which non-cooperative strategies such as "looking for a sucker", "pathological" or "unsociable" are not Nash Equilibria, nor are they evolutionarily or asymptotically stable. For example, this is the case if the association game is modeled by Hawk-Dove game and the parameters of the model are such that these strategies are indeed non cooperative (this means that Definition 6.2 holds). This result is also new compared to the Iterated Prisoners' Dilemma model, for which defection is a Nash Equilibrium and is also evolutionarily and asymptotically stable.

3. As was obtained for repeated Prisoners' Dilemma models, it was shown that a cooperative punishing strategy is a Nash Equilibrium strategy for multi-state models. Populations which consist of different mixtures of "cooperating without punishment" and "cooperating with punishment" player can be the end points of the evolutionary process. If the association game models the sharing context of interaction and payoffs in activity games G_1 and G_2 do not vary significantly then the ranges of parameters for which there are stable cooperative populations and stable "allocating tasks" popu-

lations are quite similar (see sections 5.5 and 7.2.2). If the association game includes the cost of association and the payoffs in activity games G_1 and G_2 are different then the ranges depend on the value of the probability p with which the activity games are played. In this case cooperative behaviour is stable for a wider range of p than the "allocating tasks" behaviour.

Therefore, we can conclude that "cooperative" and "alternating tasks" populations can evolve under the influence of the natural selection and it is possible to evolve to "cooperative" or "alternating tasks" types of populations from populations initially composed of a majority of uncooperative individuals.

There are still questions that can not be answered by using this model. Although it is possible to explain the existence of a cooperative population, it is not clear how to explain the fact that some cooperative populations can include a proportion of individuals who use a non cooperative behaviour. As with Iterated Prisoners' Dilemma models, it does not seem to be possible to obtain a population that contains a mixture of "cooperative" or "alternating tasks" and at least one "non-cooperative" type of behaviour as a result of modeled selection process. This restriction can be partially overcome by considering models with equal payoffs in context-games G_1 and G_2 which correspond to some activities modeled by Prisoners' Dilemma Games. The Nash Equilibrium conditions then do not depend on value of the probability p with which context game G_1 is played. The outcome of the evolutionary process obtained as a result of the analysis of the Replicator Dynamics also does not depend on the probability p. Therefore, if once the "alternating tasks" type of behaviour has appeared and the population has been driven towards a stable state that represents such behaviour (the condition $0 < p < 1$ is required at this moment so that the division of labor is possible), then during later times the value of p can vary and eventually may become equal zero or one. Then only one of the games G_1 or G_2 will be played. If in this situation the "alternating tasks" type of behaviour is the end point of the evolutionary process (as is the case, for example, if the association game is modeled by the Hawk-Dove game) then one player engaged in the association will always cooperate and the other will always defect. This example seems to be very interesting as it models the possible mechanism of an evolutionary process resulting in unreciprocated cooperation between unrelated individuals. Although this example is quite important, the question of explaining evolutionary mechanisms which can produce a population consisting of a mixtures cooperating, conditionally cooperating and defecting individuals still needs to be investigated.

There are a few directions for further research in this area. For example, in this thesis I have mostly concentrated on the case in which games G_1 or G_2 are modeled by Prisoners' Dilemmas. It would be interesting to investigate the consequences of changing one or both Prisoners Dilemma context-games for another type.

In this work I have only considered seven different strategies for the multi-state model. These strategies represented the main types of behaviour and if "punishment" was used by a strategy it was to discontinue the association if the other player did not cooperate at some particular state in the past. It is possible to consider strategies which use a different type of "punishment". Various possibilities are open here: for example, a strategy may prescribe discontinuing an association unless both players cooperate at some particular state in the past. These types of strategy are similar to the strategy Grim in the Iterated Prisoners' Dilemma. It is also possible to enlarge the state space of the corresponding Markov process to allow the "punishment" to be placed not only on the association game but also on the activity games G_1 or G_2. This gives the possibility of continuing association while accepting the "punishment" in the form of non-cooperation in an activity game.

Another approach may consist of considering a model in which the players are allowed to be engaged in an interaction with another player after the first association breaks up. The analysis of such a model is quite involved. For example, the Markov decision process approach is not applicable in this situation and if a player is allowed to be engaged in association with two different players sequentially it is quite hard to calculate the total payoffs for the strategies. However such models seem to be very interesting.

Another way of extending the approach developed here is to include the possibility that the behaviours adopted by the two individuals may lead to an involuntary termination of the association. This may be modelled by allowing the discount factor β to depend on the behaviours adopted: $\beta_i(\sigma_1, \sigma_2), i = 1, 2$. The discount factors may, therefore, be different for each player in the game.

Hopefully, by considering such models, it will be possible to shed light on the reasons for the existence of populations of animals which consist of cooperating, conditionally cooperating and defecting individuals (see [5]).

Appendix 1.

This appendix contains all stationary points and sets of the Replicator Dynamics that belong to the simplex Δ if the parameters of the model are as follows

$$ph_1 + t_1(1-p) = 3; \quad ph_3 + t_3(1-p) = 5;$$
$$ph_2 + t_2(1-p) = 0; \quad ph_4 + t_4(1-p) = 1.$$
$$c_1 = 3; \quad c_2 = 0; \quad c_3 = 6; \quad c_4 = -\tfrac{1}{2};$$
$$z = 2; \quad \beta \in (\beta_0, 1);$$

where

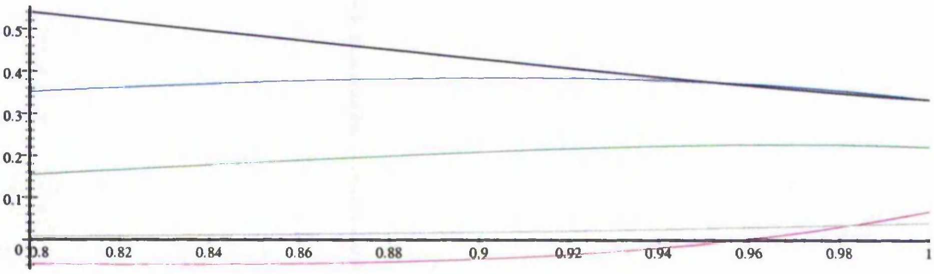
$$\beta_0 = \frac{-1+\sqrt{193}}{16} \approx .805\,78.$$

For each point (set) only the nonzero components are given. The exact formulae of the components are accompanied by a plot (or several plots). Each component is given in the prescribed colour. The plot helps to visualise the interval for parameter β such that the point (set) belongs to the simplex Δ (which means that values of all components are between zero and one). The boundary values for the intervals are estimated up to five significant figures.

$$\left\{\begin{array}{l} x_1=\dfrac{-(\beta-2)(2\beta-1)(7+9\beta)(\beta+1)\left(4\beta^2-2\beta-3\right)\left(2\beta^4-5\beta^3-3\beta^2+7\beta+2\right)}{\beta\left(360\beta^8-368\beta^7-1794\beta^6+378\beta^5+2730\beta^4+665\beta^3-1320\beta^2-813\beta-126\right)} \\ x_2=\dfrac{-\left(156\beta^7+450\beta^6-5\beta^5-984\beta^4-555\beta^3+492\beta^2+458\beta+84\right)}{\beta\left(360\beta^8-368\beta^7-1794\beta^6+378\beta^5+2730\beta^4+665\beta^3-1320\beta^2-813\beta-126\right)} \\ x_3=\dfrac{2(\beta-2)(2\beta-1)(7+9\beta)(\beta+1)\left(\beta^2-2\right)\left(4\beta^2-2\beta-3\right)\beta}{360\beta^8-368\beta^7-1794\beta^6+378\beta^5+2730\beta^4+665\beta^3-1320\beta^2-813\beta-126} \\ x_5=\dfrac{3(2\beta-1)(\beta-2)(\beta+1)^2\beta^2}{360\beta^8-368\beta^7-1794\beta^6+378\beta^5+2730\beta^4+665\beta^3-1320\beta^2-813\beta-126} \\ x_6=\dfrac{2(\beta-2)(2\beta-1)(\beta+1)\left(36\beta^3+22\beta^2-32\beta-21\right)\beta}{360\beta^8-368\beta^7-1794\beta^6+378\beta^5+2730\beta^4+665\beta^3-1320\beta^2-813\beta-126} \end{array}\right\}$$

x_1*−blue*,
x_2*−black*,
x_3*−green*,
x_5*−grey*,
x_6*−magenta*

This point belongs to the simplex Δ if $\beta \in (0.95629, 1)$

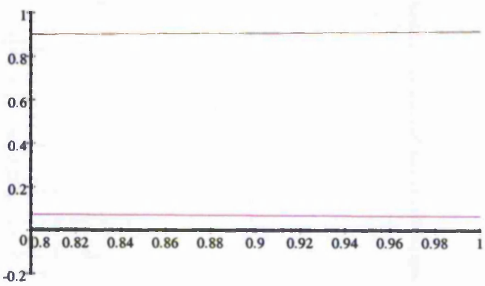


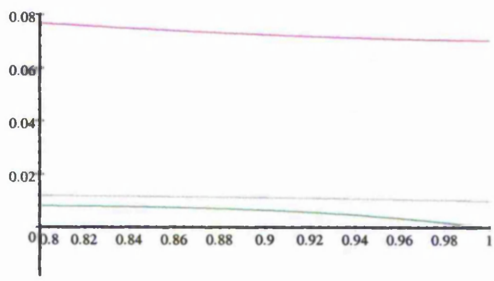
==

$$\left\{\begin{array}{l} x_3=\dfrac{2(\beta-1)(\beta+1)\left(\beta^2-2\right)\left(16\beta^4+20\beta^3-18\beta^2-9\beta+7\right)}{\left(288\beta^7+392\beta^6-956\beta^5-1539\beta^4+492\beta^3+1455\beta^2+483\beta-21\right)\beta} \\ x_5=\dfrac{-(\beta+1)\left(5\beta^4+5\beta^3-8\beta^2-7\beta+2\right)}{\left(288\beta^7+392\beta^6-956\beta^5-1539\beta^4+492\beta^3+1455\beta^2+483\beta-21\right)\beta} \\ x_6=2\dfrac{16\beta^6-69\beta^4-12\beta^3+77\beta^2+22\beta-13}{\left(288\beta^7+392\beta^6-956\beta^5-1539\beta^4+492\beta^3+1455\beta^2+483\beta-21\right)\beta} \\ x_7=2\dfrac{128\beta^7+176\beta^6-428\beta^5-698\beta^4+227\beta^3+671\beta^2+214\beta-17}{288\beta^7+392\beta^6-956\beta^5-1539\beta^4+492\beta^3+1455\beta^2+483\beta-21} \end{array}\right\}$$

x_3*−green*
x_5*−grey*,
x_6*−magenta*,
x_7*−sienna*

This point belongs to the simplex Δ if $\beta \in (\beta_0, 1)$



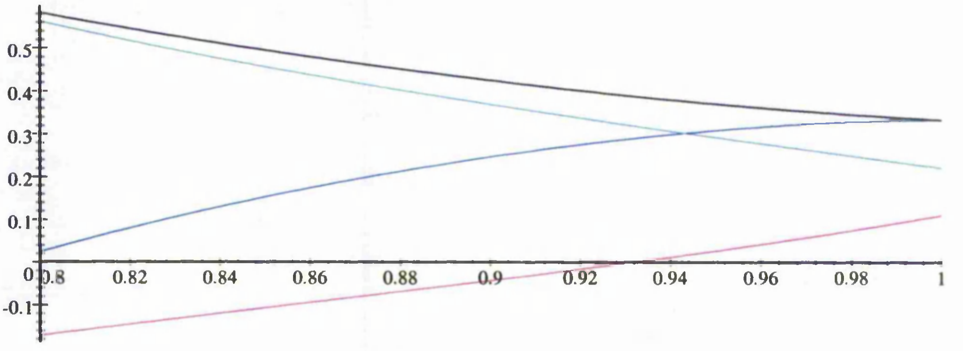


==

$$\left\{\begin{aligned} x_1&=\frac{(\beta-2)(2\beta-1)(\beta+1)\left(16\beta^6+64\beta^5-69\beta^4-98\beta^3+59\beta^2+34\beta-12\right)}{\beta\left(144\beta^7-194\beta^6-330\beta^5+365\beta^4+276\beta^3-195\beta^2-66\beta+36\right)}\\ x_2&=\frac{104\beta^7-13\beta^6-332\beta^5+95\beta^4+296\beta^3-94\beta^2-68\beta+24}{\beta\left(144\beta^7-194\beta^6-330\beta^5+365\beta^4+276\beta^3-195\beta^2-66\beta+36\right)}\\ x_4&=\frac{-4\left(4\beta^2-2\beta-3\right)\left(\beta^2-2\right)\beta(\beta-2)(2\beta-1)(\beta+1)}{144\beta^7-194\beta^6-330\beta^5+365\beta^4+276\beta^3-195\beta^2-66\beta+36}\\ x_6&=\frac{-2(\beta-2)(2\beta-1)(\beta+1)\left(8\beta^2-\beta-6\right)\beta}{144\beta^7-194\beta^6-330\beta^5+365\beta^4+276\beta^3-195\beta^2-66\beta+36}\end{aligned}\right\}$$

x_1 – *blue*,
x_2 – *black*,
x_4 – *cyan*,
x_6 – *magenta*

This point belongs to the simplex Δ if $\beta \in (0.93078,1)$

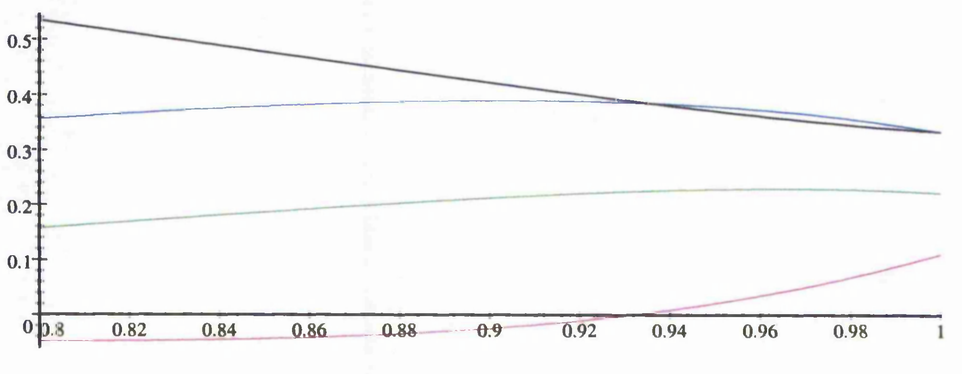


==

$$\left\{\begin{aligned} x_1&=\frac{-2(\beta-2)(2\beta-1)(\beta+1)\left(4\beta^2-2\beta-3\right)\left(2\beta^4-5\beta^3-3\beta^2+7\beta+2\right)}{\beta\left(80\beta^7-144\beta^6-291\beta^5+306\beta^4+373\beta^3-138\beta^2-186\beta-36\right)}\\ x_2&=-\frac{39\beta^6+74\beta^5-63\beta^4-174\beta^3+12\beta^2+100\beta+24}{\beta\left(80\beta^7-144\beta^6-291\beta^5+306\beta^4+373\beta^3-138\beta^2-186\beta-36\right)}\\ x_3&=\frac{4\left(4\beta^2-2\beta-3\right)\left(\beta^2-2\right)\beta(\beta-2)(2\beta-1)(\beta+1)}{80\beta^7-144\beta^6-291\beta^5+306\beta^4+373\beta^3-138\beta^2-186\beta-36}\\ x_6&=\frac{2(\beta-2)(2\beta-1)(\beta+1)\left(8\beta^2-\beta-6\right)\beta}{80\beta^7-144\beta^6-291\beta^5+306\beta^4+373\beta^3-138\beta^2-186\beta-36}\end{aligned}\right\}$$

x_1 – *blue*,
x_2 – *black*,
x_3 – *green*,
x_6 – *magenta*

This point belongs to the simplex Δ if $\beta \in (0.93078,1)$



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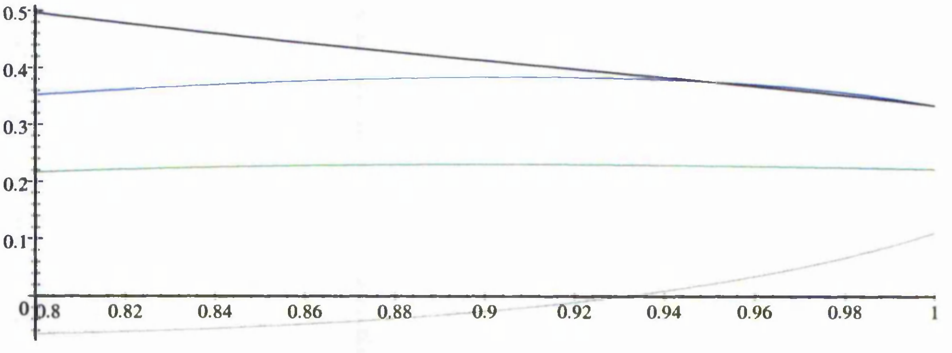
$$\left\{\begin{array}{l} x_1=\dfrac{(\beta-2)(2\beta-1)(\beta+1)\left(8\beta^6+12\beta^5-22\beta^4-5\beta^3+14\beta^2-4\beta-6\right)}{\beta\left(32\beta^7-80\beta^6-36\beta^5+110\beta^4-11\beta^3-45\beta^2+30\beta+18\right)} \\ x_2=-\dfrac{16\beta^7-12\beta^6-2\beta^5-\beta^4-3\beta^3+16\beta^2-8\beta-12}{\beta\left(32\beta^7-80\beta^6-36\beta^5+110\beta^4-11\beta^3-45\beta^2+30\beta+18\right)} \\ x_3=\dfrac{-2\left(\beta^2-2\right)\left(4\beta^2-2\beta-3\right)(\beta-2)(2\beta-1)(\beta+1)\beta}{32\beta^7-80\beta^6-36\beta^5+110\beta^4-11\beta^3-45\beta^2+30\beta+18} \\ x_5=\dfrac{-\beta(\beta-2)(2\beta-1)(\beta+1)\left(8\beta^2-\beta-6\right)}{32\beta^7-80\beta^6-36\beta^5+110\beta^4-11\beta^3-45\beta^2+30\beta+18} \end{array}\right\}$$

x_1 – blue,
x_2 – black,
x_3 – green,
x_5 – grey

This point belongs to the simplex Δ if $\beta \in (0.93078,1)$

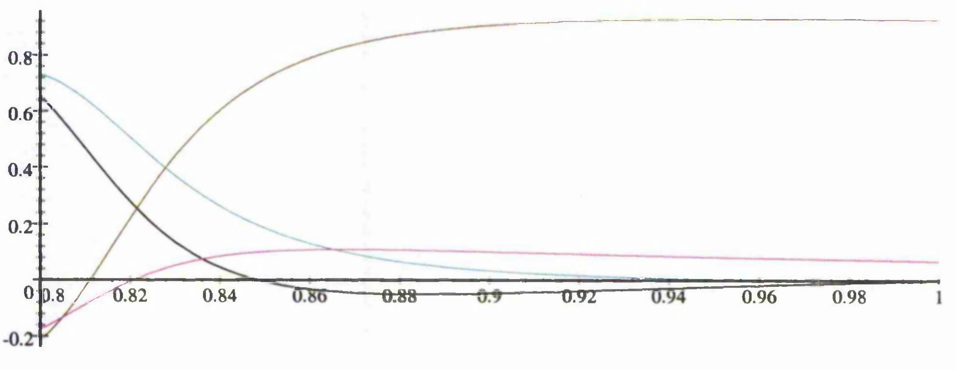


==

$$\left\{\begin{array}{l} x_2=\dfrac{-(\beta-1)\left(104\beta^6-7\beta^5-304\beta^4+18\beta^3+172\beta^2-20\beta+4\right)}{288\beta^9-832\beta^8-788\beta^7+1765\beta^6+2277\beta^5-2400\beta^4-1558\beta^3+1372\beta^2-12\beta-28} \\ x_4=\dfrac{2(\beta-1)(\beta-2)(2\beta-1)(\beta+1)\left(\beta^2-2\right)\left(8\beta^2-4\beta-5\right)\beta}{288\beta^9-832\beta^8-788\beta^7+1765\beta^6+2277\beta^5-2400\beta^4-1558\beta^3+1372\beta^2-12\beta-28} \\ x_6=\dfrac{2(\beta-2)(2\beta-1)\left(8\beta^5-3\beta^4-22\beta^3+4\beta^2+12\beta-2\right)}{288\beta^9-832\beta^8-788\beta^7+1765\beta^6+2277\beta^5-2400\beta^4-1558\beta^3+1372\beta^2-12\beta-28} \\ x_7=\dfrac{2\left(-12-8\beta-744\beta^3+612\beta^2-1004\beta^4+1059\beta^5+712\beta^6-368\beta^8-336\beta^7+128\beta^9\right)}{288\beta^9-832\beta^8-788\beta^7+1765\beta^6+2277\beta^5-2400\beta^4-1558\beta^3+1372\beta^2-12\beta-28} \end{array}\right\}$$

x_2 – black,
x_4 – cyan,
x_6 – magenta,
x_7 – sienna

This point belongs to the simplex Δ if $\beta \in (0.82063,0.84859)$



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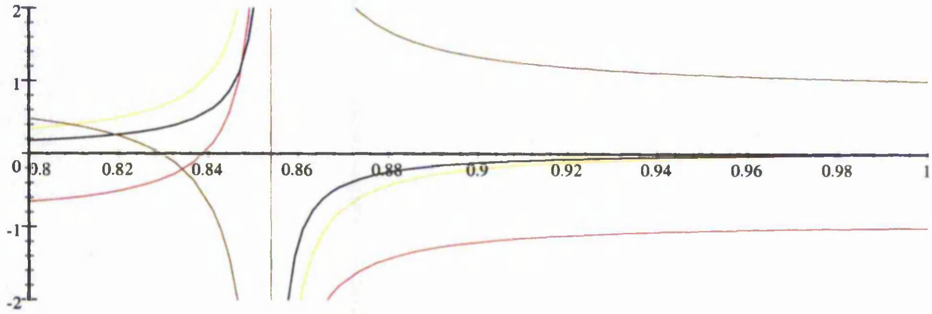
$$\left\{\begin{array}{l} x_2=\frac{(\beta-1)(\beta^2-2)(8\beta^3-3\beta^2-6\beta+4)}{\beta(-66\beta^2+126\beta-24-141\beta^3+95\beta^4+4\beta^5)} \\ x_3=\frac{-2(\beta-1)(\beta-2)(2\beta-1)(\beta+1)(\beta^2-2)}{\beta(-66\beta^2+126\beta-24-141\beta^3+95\beta^4+4\beta^5)}-x_4 \\ x_4=x_4 \\ x_7=\frac{2(-12+60\beta-34\beta^2-65\beta^3+48\beta^4)}{-66\beta^2+126\beta-24-141\beta^3+95\beta^4+4\beta^5} \\ \left\{\begin{array}{l} 0<x_4<1 \\ x_4<\frac{-2(\beta-1)(\beta-2)(2\beta-1)(\beta+1)(\beta^2-2)}{\beta(-66\beta^2+126\beta-24-141\beta^3+95\beta^4+4\beta^5)}\ (yellow) \\ x_4>\frac{-(8\beta^6-149\beta^4+85\beta^5-36\beta^3-44\beta+122\beta^2+8)}{\beta(-66\beta^2+126\beta-24-141\beta^3+95\beta^4+4\beta^5)}\ (red) \end{array}\right. \end{array}\right\}$$

x_2 – black,
x_7 – sienna

This point belongs to the simplex Δ if $\beta \in (\beta_0, 0.82958)$



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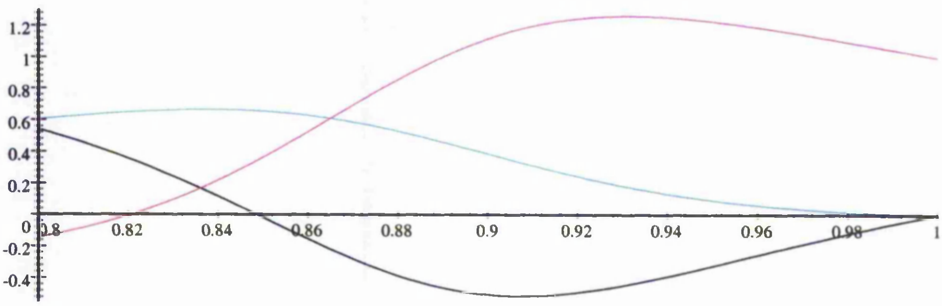
$$\left\{\begin{array}{l} x_2=\frac{-(\beta-1)(104\beta^6-7\beta^5-304\beta^4+18\beta^3+172\beta^2-20\beta+4)}{-4-70\beta^3-392\beta^4+159\beta^5+148\beta^2+4\beta-116\beta^7-96\beta^8+341\beta^6+32\beta^9} \\ x_4=\frac{2\beta(\beta-1)(\beta-2)(2\beta-1)(\beta+1)(8\beta^2-4\beta-5)(\beta^2-2)}{-4-70\beta^3-392\beta^4+159\beta^5+148\beta^2+4\beta-116\beta^7-96\beta^8+341\beta^6+32\beta^9} \\ x_6=\frac{2(\beta-2)(2\beta-1)(8\beta^5-3\beta^4-22\beta^3+4\beta^2+12\beta-2)}{-4-70\beta^3-392\beta^4+159\beta^5+148\beta^2+4\beta-116\beta^7-96\beta^8+341\beta^6+32\beta^9} \end{array}\right\}$$

x_2 – black,
x_4 – cyan,
x_6 – magenta,

This point belongs to the simplex Δ if $\beta \in (0.82063, 0.84859)$

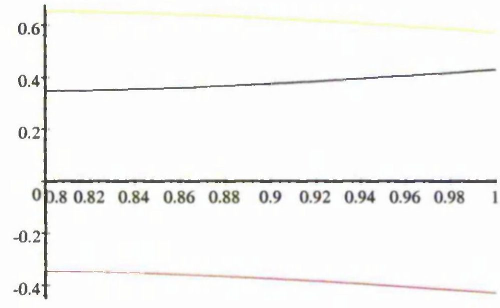


==

x_2 – black

$$\left\{ \begin{array}{l} x_2 = \dfrac{-3\beta^2+8\beta^3-6\beta+4}{\beta^2(3+4\beta)} \\ x_3 = x_3 \\ x_4 = \dfrac{-2(\beta-2)(2\beta-1)(\beta+1)}{\beta^2(3+4\beta)} - x_3 \\ \left\{ \begin{array}{l} 0 < x_3 < 1 \\ x_3 < \dfrac{-2(\beta-2)(2\beta-1)(\beta+1)}{\beta^2(3+4\beta)} \, (yellow) \\ x_3 > \dfrac{6\beta+3\beta^2-8\beta^3-4}{\beta^2(3+4\beta)} \, (red) \end{array} \right. \end{array} \right\}$$

This point belongs to the simplex Δ if $\beta \in (\beta_0, 1)$

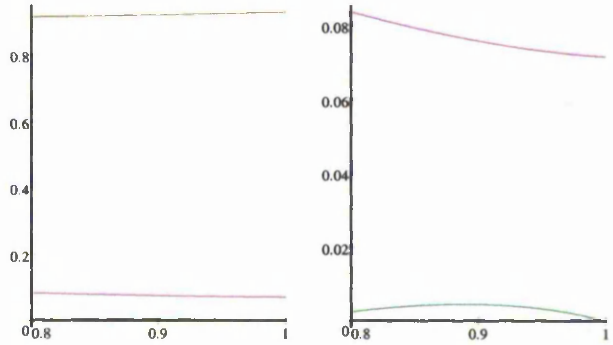


==

x_3 – green, x_6 – magenta, x_7 – sienna

$$\left\{ \begin{array}{l} x_3 = \dfrac{(\beta-1)(\beta+1)(5\beta^3+2\beta^2-7\beta+2)(\beta^2-2)}{(45\beta^6+23\beta^5-166\beta^4-91\beta^3+163\beta^2+84\beta-16)\beta} \\ x_6 = \dfrac{(\beta^2-2)(5\beta^3-3\beta^2-7\beta+2)}{(45\beta^6+23\beta^5-166\beta^4-91\beta^3+163\beta^2+84\beta-16)\beta} \\ x_7 = \dfrac{-149\beta^4+149\beta^2+21\beta^5-84\beta^3+40\beta^6-16+78\beta}{45\beta^6+23\beta^5-166\beta^4-91\beta^3+163\beta^2+84\beta-16} \end{array} \right\}$$

This point belongs to the simplex Δ if $\beta \in (\beta_0, 1)$



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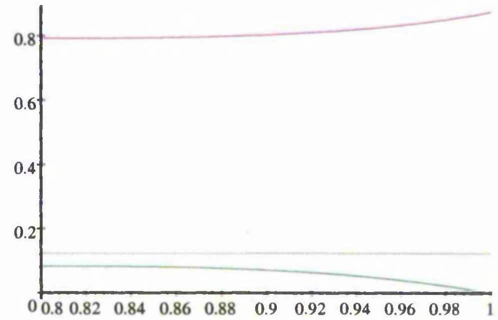
x_3 – green, x_5 – grey, x_6 – magenta

$$\left\{ \begin{array}{l} x_3 = \dfrac{2(\beta-1)(\beta+1)(\beta^2-2)(16\beta^4+20\beta^3-18\beta^2-9\beta+7)}{(32\beta^7+40\beta^6-100\beta^5-143\beta^4+38\beta^3+113\beta^2+55\beta+13)\beta} \\ x_5 = \dfrac{-(\beta+1)(5\beta^4+5\beta^3-8\beta^2-7\beta+2)}{\beta(32\beta^7+40\beta^6-100\beta^5-143\beta^4+38\beta^3+113\beta^2+55\beta+13)} \\ x_6 = 2\dfrac{16\beta^6-69\beta^4-12\beta^3+77\beta^2+22\beta-13}{(32\beta^7+40\beta^6-100\beta^5-143\beta^4+38\beta^3+113\beta^2+55\beta+13)\beta} \end{array} \right\}$$

This point belongs to the simplex Δ if $\beta \in (\beta_0, 1)$



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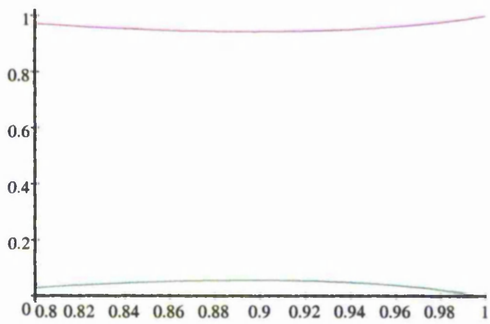
x_3 – green, x_6 – magenta

$$\left\{ \begin{array}{l} x_3 = \dfrac{(\beta-1)(\beta+1)(5\beta^3+2\beta^2-7\beta+2)}{\beta^2(-7\beta-3+2\beta^2+5\beta^3)} \\ x_6 = \dfrac{5\beta^3-3\beta^2-7\beta+2}{\beta^2(-7\beta-3+2\beta^2+5\beta^3)} \end{array} \right\}$$

This point belongs to the simplex Δ if $\beta \in (\beta_0, 1)$



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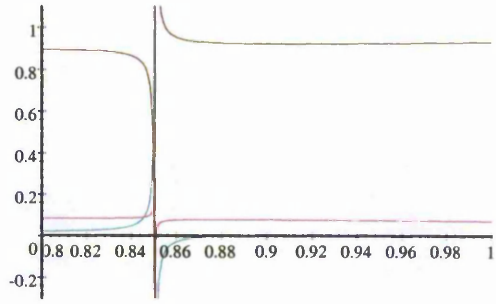
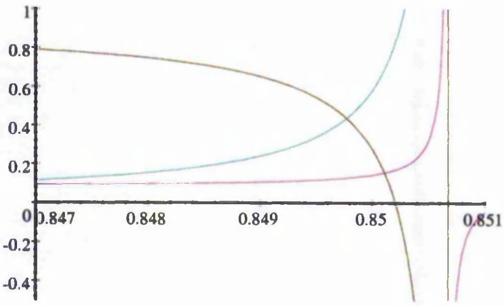
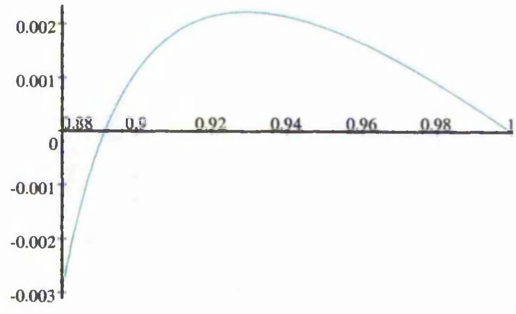
$x_4-cyan,\ x_6-magenta,\ x_7-sienna$

$$\left\{ \begin{array}{l} x_4=\dfrac{(-2+\beta^2)(\beta-1)(\beta+1)(5\beta^3+2\beta^2-8\beta+2)}{(\beta+2)(45\beta^5-67\beta^4-101\beta^3+185\beta^2+58\beta-106)\beta} \\ x_6=\dfrac{-(-2+\beta^2)(8\beta^3+3\beta^2-6\beta-2)}{(\beta+2)(45\beta^5-67\beta^4-101\beta^3+185\beta^2+58\beta-106)\beta} \\ x_7=\dfrac{40\beta^5-59\beta^4-86\beta^3+162\beta^2+48\beta-92}{45\beta^5-67\beta^4-101\beta^3+185\beta^2+58\beta-106} \end{array} \right\}$$

This point belongs to the simplex Δ if $\beta\in(\beta_0,0.85021)\cup(0.89147,1)$







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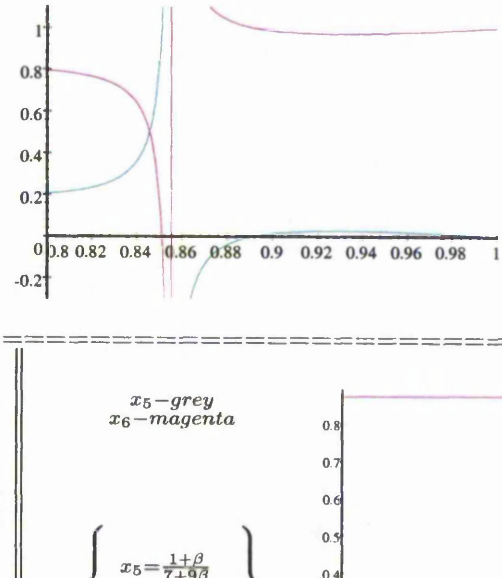
$x_4-cyan,\ x_6-magenta$

$$\left\{ \begin{array}{l} x_4=\dfrac{(\beta-1)(\beta+1)(5\beta^3+2\beta^2-8\beta+2)}{\beta(\beta+2)(5\beta^3-8\beta^2-5\beta+7)} \\ x_6=-\dfrac{8\beta^3-2-6\beta+3\beta^2}{\beta(\beta+2)(5\beta^3-8\beta^2-5\beta+7)} \end{array} \right\}$$

This point belongs to the simplex Δ if $\beta\in(\beta_0,0.85118)\cup(0.89147,1)$



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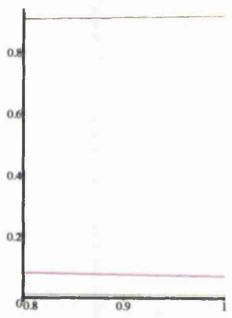
x_5-grey
$x_6-magenta$
$x_7-sienna$

$$\left\{ \begin{array}{l} x_5=\dfrac{(1+\beta)}{32\beta^3+40\beta^2+77\beta+49} \\ x_6=\dfrac{2(3+4\beta)}{32\beta^3+40\beta^2+77\beta+49} \\ x_7=\dfrac{2(3+4\beta)(4\beta^2+2\beta+7)}{32\beta^3+40\beta^2+77\beta+49} \end{array} \right\}$$

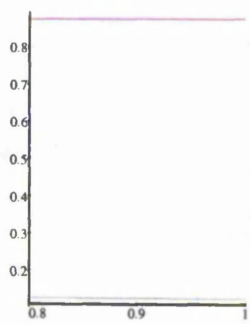
This point belongs to the simplex Δ if $\beta\in(\beta_0,1)$



x_5-grey
$x_6-magenta$

$$\left\{ \begin{array}{l} x_5=\dfrac{1+\beta}{7+9\beta} \\ x_6=2\dfrac{3+4\beta}{7+9\beta} \end{array} \right\}$$

This point belongs to the simplex Δ if $\beta\in(\beta_0,1)$

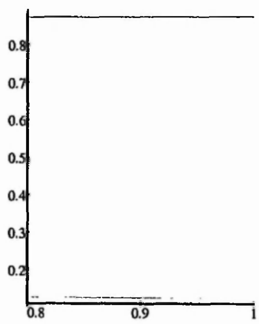


==

x_5—grey
x_7—sienna

$$\left\{ \begin{array}{l} x_5=\frac{1+\beta}{7+9\beta} \\ x_7=2\frac{3+4\beta}{7+9\beta} \end{array} \right\}$$

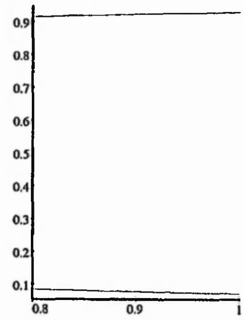
This point belongs to the simplex Δ if $\beta\in(\beta_0,1)$



x_6—magenta
x_7—sienna

$$\left\{ \begin{array}{l} x_6=\frac{1}{5\beta^2+2\beta+7} \\ x_7=\frac{6+5\beta^2+2\beta}{5\beta^2+2\beta+7} \end{array} \right\}$$

This point belongs to the simplex Δ if $\beta\in(\beta_0,1)$



Appendix 2.

This appendix contains all stationary points and sets of the Replicator dynamics that belong to the simplex Δ together with their eigenvectors and eigenvalues. The parameters of the model satisfy the following equations in the case considered.

$$ph_1 + t_1(1-p) = 3; \quad ph_3 + t_3(1-p) = 5;$$
$$ph_2 + t_2(1-p) = 0; \quad ph_4 + t_4(1-p) = 1.$$
$$c_1 = 3; \quad c_2 = 0; \quad c_3 = 6; \quad c_4 = -\tfrac{1}{2};$$
$$z = 2; \quad \beta = \tfrac{22}{25}.$$

Then the parameters of the payoff matrix A (6.1) are as follows.

$$\varkappa = \tfrac{25}{2}; \quad \omega = \tfrac{4625}{282}; \psi = \tfrac{625}{94}; \quad \chi = \tfrac{2425}{282};$$
$$C_2 = \tfrac{44}{3}; \quad C_3 = \tfrac{62}{3}; \quad C_4 = \tfrac{85}{6}.$$

The matrix A in this case is given by

$$A = \begin{bmatrix} 25 & 25 & \frac{900}{47} & \frac{900}{47} & \frac{625}{47} & \frac{625}{47} & \frac{44}{3} \\ 25 & 25 & \frac{23\,104}{1149} & \frac{23\,104}{1149} & \frac{35\,633}{1875} & \frac{35\,633}{1875} & \frac{44}{3} \\ \frac{4075}{141} & \frac{22\,873}{1149} & \frac{3250}{141} & \frac{3250}{141} & \frac{21\,454}{1149} & \frac{16\,735}{1149} & \frac{44}{3} \\ \frac{4075}{141} & \frac{22\,873}{1149} & \frac{3250}{141} & \frac{3250}{141} & \frac{2150}{141} & \frac{17\,098}{1149} & \frac{44}{3} \\ \frac{4625}{141} & \frac{35\,171}{1875} & \frac{21\,223}{1149} & 25 & \frac{2425}{141} & \frac{28\,571}{1875} & \frac{44}{3} \\ \frac{4625}{141} & \frac{35\,171}{1875} & \frac{20\,860}{1149} & \frac{25\,579}{1149} & \frac{37\,283}{1875} & \frac{5569}{375} & \frac{44}{3} \\ \frac{62}{3} & \frac{62}{3} & \frac{62}{3} & \frac{62}{3} & \frac{62}{3} & \frac{62}{3} & \frac{85}{6} \end{bmatrix} \tag{8.1}$$

$$\approx \begin{bmatrix} 25.0 & 25.0 & 19.149 & 19.149 & 13.298 & 13.298 & 14.667 \\ 25.0 & 25.0 & 20.108 & 20.108 & 19.004 & 19.004 & 14.667 \\ 28.901 & 19.907 & 23.05 & 23.05 & 18.672 & 14.565 & 14.667 \\ 28.901 & 19.907 & 23.05 & 23.05 & 15.248 & 14.881 & 14.667 \\ 32.801 & 18.758 & 18.471 & 25.0 & 17.199 & 15.238 & 14.667 \\ 32.801 & 18.758 & 18.155 & 22.262 & 19.884 & 14.851 & 14.667 \\ 20.667 & 20.667 & 20.667 & 20.667 & 20.667 & 20.667 & 14.167 \end{bmatrix}$$

==

point: $\{7\} = \{0,0,0,0,0,0\}$

eigenvectors: eigenvalue:

$$\left\{ \begin{array}{l} [1,0,0,0,0,0], \\ [0,1,0,0,0,0], \\ [0,0,1,0,0,0], \\ [0,0,0,1,0,0], \\ [0,0,0,0,1,0], \\ [0,0,0,0,0,1] \end{array} \right\} \leftrightarrow \frac{1}{2}$$

==

point: $\{6\} = \{0,0,0,0,0,1\}$

eigenvectors: eigenvalues:

$$[0,0,0,-1,0,1] \leftrightarrow \frac{1441}{47875} \approx 3.0099 \times 10^{-2}$$
$$[0,0,-1,0,0,1] \leftrightarrow -\frac{13684}{47875} \approx -.28583$$
$$[0,0,0,0,0,1] \leftrightarrow \frac{727}{125} \approx 5.816$$
$$[0,1,0,0,0,-1] \leftrightarrow \frac{2596}{625} \approx 4.1536$$
$$[0,0,0,0,-1,1] \leftrightarrow \frac{242}{625} \approx .3872$$
$$[1,0,0,0,0,-1] \leftrightarrow -\frac{27368}{17625} \approx -1.5528$$

==

point: $\{5\} = \{0,0,0,0,1,0\}$

eigenvectors: eigenvalues:

$$[0,0,0,0,-1,1] \leftrightarrow \frac{78892}{29375} \approx 2.6857$$
$$[0,1,0,0,-1,0] \leftrightarrow \frac{53042}{29375} \approx 1.8057$$
$$[1,0,0,0,-1,0] \leftrightarrow -\frac{550}{141} \approx -3.9007$$
$$[0,0,0,0,1,0] \leftrightarrow \frac{163}{47} \approx 3.4681$$
$$[0,0,0,-1,1,0] \leftrightarrow -\frac{275}{141} \approx -1.9504$$
$$[0,0,1,0,-1,0] \leftrightarrow \frac{26521}{18001} \approx 1.4733$$

---

interval of points: $\{\overline{3,4}\} = \{\{0,0,1-\alpha,\alpha,0,0\} : \alpha \in [0,1]\}$

eigenvectors:		eigenvalues:		
$\left[0,0,\frac{2(\alpha-1)(3619\alpha-4005)}{6721\alpha-8010},\frac{-\alpha(7238\alpha-8527)}{6721\alpha-8010},0,1\right]$	$\leftrightarrow$	$\frac{1573}{383}\alpha-\frac{88\,110}{18\,001}$	$\approx$	$4.107\alpha-4.8947$
$\left[0,0,\frac{3(\alpha-1)(5082\alpha-7493)}{32\,054\alpha-22\,479},\frac{-\alpha(15\,246\alpha-5671)}{32\,054\alpha-22\,479},1,0\right]$	$\leftrightarrow$	$\frac{7502}{1149}\alpha-\frac{82\,423}{18\,001}$	$\approx$	$6.5292\alpha-4.5788$
$\left[-\frac{1}{\alpha},0,\frac{(1-\alpha)}{\alpha},1,0,0\right]$	$\leftrightarrow$	$-\frac{550}{141}$	$\approx$	-3.9007
$\left[0,-\frac{1}{\alpha},\frac{(1-\alpha)}{\alpha},1,0,0\right]$	$\leftrightarrow$	$-\frac{52\,954}{18\,001}$	$\approx$	-2.9417
$\left[0,0,\frac{(1-\alpha)}{\alpha},1,0,0\right]$	$\leftrightarrow$	$-\frac{112}{47}$	$\approx$	-2.383
[0,0,1,−1,0,0]	$\leftrightarrow$	0		

---

point: $\{3\} = \{0,0,1,0,0,0\}$

eigenvectors:		eigenvalues:		
[0, 0, −1, 0, 1, 0]	$\leftrightarrow$	$-\frac{82\,423}{18\,001}$	$\approx$	-4.5788
[0, 0, −1, 0, 0, 1]	$\leftrightarrow$	$-\frac{88\,110}{18\,001}$	$\approx$	-4.8947
[−1, 0, 1, 0, 0, 0]	$\leftrightarrow$	$-\frac{550}{141}$	$\approx$	-3.9007
[0, −1, 1, 0, 0, 0]	$\leftrightarrow$	$-\frac{52\,954}{18\,001}$	$\approx$	-2.9417
[0, 0, 1, 0, 0, 0]	$\leftrightarrow$	$-\frac{112}{47}$	$\approx$	-2.383
[0, 0, −1, 1, 0, 0]	$\leftrightarrow$	0		

---

point: $\{0,0,\frac{9575}{32\,054},\frac{22\,479}{32\,054},0,0\} \approx \{0,0,.29871,.70129,0,0\}$

eigenvectors:		eigenvalues:		
[0, 0, 1.0, 2.7613, 0, −3.7613]	$\leftrightarrow$	$-\frac{2248\,323}{1116\,062}$	$\approx$	-2.0145
[−3.3477, 0, 1, 2.3477, 0, 0]	$\leftrightarrow$	$-\frac{550}{141}$	$\approx$	-3.9007
[0, −3.3477, 1, 2.3477, 0, 0]	$\leftrightarrow$	$-\frac{52\,954}{18\,001}$	$\approx$	-2.9417
[0, 0, 1, 2.3477, 0, 0]	$\leftrightarrow$	$-\frac{112}{47}$	$\approx$	-2.383
[0, 0, 1, −1, 0, 0]	$\leftrightarrow$	{0, 0}		

---

point: $\{4\} = \{0,0,0,1,0,0\}$

eigenvectors:		eigenvalues:		
$[0,0,0,-1,1,0]$	$\leftrightarrow$	$\frac{275}{141}$	$\approx$	1.9504
$[0,0,0,-1,0,1]$	$\leftrightarrow$	$-\frac{14179}{18001}$	$\approx$	$-.78768$
$[-1,0,0,1,0,0]$	$\leftrightarrow$	$-\frac{550}{141}$	$\approx$	-3.9007
$[0,-1,0,1,0,0]$	$\leftrightarrow$	$-\frac{52954}{18001}$	$\approx$	-2.9417
$[0,0,0,1,0,0]$	$\leftrightarrow$	$-\frac{112}{47}$	$\approx$	-2.383
$[0,0,1,-1,0,0]$	$\leftrightarrow$	0		

==

interval of points: $\{\overline{1,2}\} = \{\{\alpha, 1-\alpha, 0,0,0,0\} : \alpha \in [0,1]\}$

eigenvectors:		eigenvalues:		
$\left\{ \left[\frac{-\alpha(19723\alpha-10148)}{22077\alpha-12502}, \frac{(\alpha-1)(19723\alpha-12502)}{22077\alpha-12502},1,0,0,0\right], \left[\frac{-\alpha(19723\alpha-10148)}{22077\alpha-12502}, \frac{(\alpha-1)(19723\alpha-12502)}{22077\alpha-12502},0,1,0,0\right] \right\}$	$\leftrightarrow$	$\frac{161898}{18001}\alpha-\frac{5852}{1149}$	$\approx$	$8.9938\alpha-5.0931$
$\left\{ \left[\frac{-\alpha(16698\alpha-1073)}{28127\alpha-12502}, \frac{2(\alpha-1)(8349\alpha-6251)}{28127\alpha-12502},0,0,0,1\right], \left[-\alpha\frac{16698\alpha-1073}{28127\alpha-12502}, \frac{2(\alpha-1)(8349\alpha-6251)}{28127\alpha-12502},0,0,1,0\right] \right\}$	$\leftrightarrow$	$\frac{1237588}{88125}\alpha-\frac{11704}{1875}$	$\approx$	$14.044\alpha-6.2421$
$[1,-\frac{-1+\alpha}{\alpha},0,0,0,0]$	$\leftrightarrow$	$-\frac{13}{3}$	$\approx$	-4.3333
$[-1,1,0,0,0,0]$	$\leftrightarrow$	0		

==

point: $\{2\} = \{0,1,0,0,0,0\}$

eigenvectors:		eigenvalues:		
$\left\{ [0,-1,0,1,0,0], [0,-1,1,0,0,0] \right\}$	$\leftrightarrow$	$-\frac{5852}{1149}$	$\approx$	-5.0931
$\left\{ [0,-1,0,0,1,0], [0,-1,0,0,0,1] \right\}$	$\leftrightarrow$	$-\frac{11704}{1875}$	$\approx$	-6.2421
$[0,1,0,0,0,0]$	$\leftrightarrow$	$-\frac{13}{3}$	$\approx$	-4.3333
$[-1,1,0,0,0,0]$	$\leftrightarrow$	0		

==

point: $\left\{\frac{12\,502}{28\,127}, \frac{15\,625}{28\,127}, 0,0,0,0\right\} \approx \{.444\,48, .555\,52, 0,0,0,0\}$

$$\begin{array}{lcccc} \text{eigenvectors:} & & \text{eigenvalues:} & & \\ \left\{\begin{array}{l} [0,0,1,-1,0,0] \\ [0.295\,9, 1, 0, -1.295\,9, 0, 0] \end{array}\right\} & \leftrightarrow & -\frac{3218\,600}{2937\,993} & \approx & -1.095\,5 \\ [0,0,0,0,-1,1] & \leftrightarrow & \{0,0\} & & \\ [1, 1.249\,8, 0,0,0,0] & \leftrightarrow & -\frac{13}{3} & \approx & -4.333\,3 \\ [-1,1,0,0,0,0] & \leftrightarrow & 0 & & \end{array}$$

==

point: $\left\{\frac{12\,502}{22\,077}, \frac{9575}{22\,077}, 0,0,0,0\right\} \approx \{.566\,29, .433\,71, 0,0,0,0\}$

$$\begin{array}{lcccc} \text{eigenvectors:} & & \text{eigenvalues:} & & \\ [0,0,-1,1,0,0] & \leftrightarrow & \{0,0\} & & \\ \left\{\begin{array}{l} [-1.385\,6, .385\,61, 0,0,1,0] \\ [-1.385\,6, .385\,61, 0,0,0,1] \end{array}\right\} & \leftrightarrow & \frac{257\,488}{150\,525} & \approx & 1.710\,6 \\ [1.305\,7, 1.0, 0,0,0,0] & \leftrightarrow & -\frac{13}{3} & \approx & -4.333\,3 \\ [-1,1,0,0,0,0] & \leftrightarrow & 0 & & \end{array}$$

==

point: $\{1\} = \{1,0,0,0,0,0\}$

$$\begin{array}{lcccc} \text{eigenvectors:} & & \text{eigenvalues:} & & \\ \left\{\begin{array}{l} [-1,0,0,1,0,0] \\ [-1,0,1,0,0,0] \end{array}\right\} & \leftrightarrow & \frac{550}{141} & \approx & 3.900\,7 \\ \left\{\begin{array}{l} [-1,0,0,0,0,1] \\ [-1,0,0,0,1,0] \end{array}\right\} & \leftrightarrow & \frac{1100}{141} & \approx & 7.801\,4 \\ [1,0,0,0,0,0] & \leftrightarrow & -\frac{13}{3} & \approx & -4.333\,3 \\ [-1,1,0,0,0,0] & \leftrightarrow & 0 & & \end{array}$$

==

interval of points: $\{2, \overline{3,4}\} = \left\{\left\{0, \frac{7221}{19\,723}, \alpha, \frac{12\,502}{19\,723} - \alpha, 0, 0\right\} : 0 \le \alpha \le \frac{12\,502}{19\,723}\right\}$

$\left\{0, \frac{7221}{19\,723}, \alpha, \frac{12\,502}{19\,723} - \alpha, 0, 0\right\} \approx \{0, .366\,12, \alpha, .633\,88 - \alpha, 0, 0\}$

$$\begin{array}{lcccc} \text{eigenvectors:} & & \text{eigenvalues:} & & \\ \left[0, -\frac{12\,502}{19\,723\alpha}, 1, -\frac{-12\,502+19\,723\alpha}{19\,723\alpha}, 0, 0\right] & \leftrightarrow & \frac{1280\,524}{686\,719} & \approx & 1.864\,7 \\ \left[-\frac{12\,502}{19\,723\alpha}, 0, 1, -\frac{1}{19\,723}\frac{-12\,502+19\,723\alpha}{\alpha}, 0, 0\right] & \leftrightarrow & -\frac{113\,848}{187\,287} & \approx & -.607\,88 \\ \left[0, 1, \frac{19\,723}{7221}\alpha, \frac{12\,502}{7221} - \frac{19\,723}{7221}\alpha, 0, 0\right] & \leftrightarrow & -\frac{846\,275}{686\,719} & \approx & -1.232\,3 \\ e_1 & \leftrightarrow & -\frac{1573}{383}\alpha - \frac{394\,850\,134}{429\,199\,375} & \approx & -4.107\alpha - .919\,97 \\ e_2 & \leftrightarrow & -\frac{7502}{1149}\alpha + \frac{1050\,182\,098}{1287\,598\,125} & \approx & -6.529\,2\alpha + .815\,61 \\ [0,0,1,-1,0,0] & \leftrightarrow & 0 & & \end{array}$$

$$e_1 = \begin{bmatrix} 0 \\ 1 \\ \frac{19\,723}{59\,392\,725}\alpha\frac{1269\,171\,806\,326\,124\,134+38\,689\,011\,143\,637\,091\,875\alpha+78\,637\,664\,141\,364\,453\,125\alpha^2}{(1762\,743\,125\alpha+394\,850\,134)(5423\,825\alpha+6620\,056)} \\ -\frac{(-12\,502+19\,723\alpha)}{118\,785\,450}\frac{157\,275\,328\,282\,728\,906\,250\alpha^2+88\,611\,974\,307\,469\,105\,625\alpha+10\,155\,203\,951\,260\,057\,018}{(1762\,743\,125\alpha+394\,850\,134)(5423\,825\alpha+6620\,056)} \\ 0 \\ -\frac{19\,723}{2527\,350}\frac{1195\,177\,634+1762\,743\,125\alpha}{5423\,825\alpha+6620\,056} \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ \frac{19\,723}{118\,785\,450}\alpha\frac{4651\,297\,879\,218\,727\,818\,125\alpha+2354\,669\,417\,765\,556\,508\,519+12\,376\,358\,525\,633\,205\,468\,750\alpha^2}{(-525\,091\,049+4203\,464\,375\alpha)(178\,986\,225\alpha+197\,686\,982)} \\ -\frac{(-12\,502+19\,723\alpha)}{356\,356\,350}\frac{37\,129\,075\,576\,899\,616\,406\,250\alpha^2-26\,979\,170\,923\,312\,513\,420\,625\alpha+487\,004\,723\,434\,653\,120\,557}{(-525\,091\,049+4203\,464\,375\alpha)(178\,986\,225\alpha+197\,686\,982)} \\ -\frac{926\,981}{3791\,025}\frac{675\,400\,201+4203\,464\,375\alpha}{178\,986\,225\alpha+197\,686\,982} \\ 0 \end{bmatrix}$$

---

point: $\{0, \frac{7221}{19\,723}, 0, \frac{12\,502}{19\,723}, 0, 0\} \approx \{0, .366\,12, 0, .633\,88, 0, 0\}$

eigenvectors:		eigenvalues:		
$[0,-1,0,1,0,0]$	$\leftrightarrow$	$\frac{1280\,524}{686\,719}$	$\approx$	$1.864\,7$
$[-1,0,0,1,0,0]$	$\leftrightarrow$	$-\frac{113\,848}{187\,287}$	$\approx$	$-.607\,88$
$[0,1,0,1.731\,3,0,0]$	$\leftrightarrow$	$-\frac{846\,275}{686\,719}$	$\approx$	$-1.232\,3$
$[0,1,0,.408\,89,0,-1.408\,9]$	$\leftrightarrow$	$-\frac{394\,850\,134}{429\,199\,375}$	$\approx$	$-.919\,97$
$[0,-1.197,0,.197\,02,1,0]$	$\leftrightarrow$	$\frac{1050\,182\,098}{1287\,598\,125}$	$\approx$	$.815\,61$
$[0,0,1,-1,0,0]$	$\leftrightarrow$	0		

---

point: $\{0, \frac{7221}{19\,723}, \frac{525\,091\,049}{4203\,464\,375}, \frac{2139\,397\,701}{4203\,464\,375}, 0, 0\} \approx \{0, .366\,12, .124\,92, .508\,96, 0, 0\}$

eigenvectors:		eigenvalues:		
$[0,-5.074\,3,1,4.074\,3,0,0]$	$\leftrightarrow$	$\frac{1280\,524}{686\,719}$	$\approx$	$1.864\,7$
$[-5.074\,3,0,1,4.074\,3,0,0]$	$\leftrightarrow$	$-\frac{113\,848}{187\,287}$	$\approx$	$-.607\,88$
$[0,1,.341\,2,1.390\,1,0,0]$	$\leftrightarrow$	$-\frac{846\,275}{686\,719}$	$\approx$	$-1.232\,3$
$[0,14.763,1,6.581\,5,0,-22.344]$	$\leftrightarrow$	$-\frac{19\,066\,537\,791}{13\,305\,180\,625}$	$\approx$	-1.433
$[0,0,1,-1,0,0]$	$\leftrightarrow$	$\{0,0\}$		

---

point: $\{0, \frac{7221}{19\,723}, \frac{12\,502}{19\,723}, 0, 0, 0\} \approx \{0, .366\,12, .633\,88, 0, 0, 0\}$

eigenvectors:		eigenvalues:		
$[0, -1, 1, 0, 0, 0]$	$\leftrightarrow$	$\frac{1280\,524}{686\,719}$	$\approx$	$1.864\,7$
$[-1, 0, 1, 0, 0, 0]$	$\leftrightarrow$	$-\frac{113\,848}{187\,287}$	$\approx$	$-.607\,88$
$[0, 1, 1.731\,3, 0, 0, 0]$	$\leftrightarrow$	$-\frac{846\,275}{686\,719}$	$\approx$	$-1.232\,3$
$[0, 1.259\,1, 1, 0, 0, -2.259\,1]$	$\leftrightarrow$	$-\frac{1512\,216\,384}{429\,199\,375}$	$\approx$	$-3.523\,3$
$[0, 1, 1.624\,7, 0, -2.624\,7, 0]$	$\leftrightarrow$	$-\frac{1426\,265\,134}{429\,199\,375}$	$\approx$	$-3.323\,1$
$[0, 0, 1, -1, 0, 0]$	$\leftrightarrow$	0		

===

point: $\{3, 5, 6, 7\} = \{0, 0, \frac{117\,644\,491\,442}{16\,524\,580\,055\,753}, 0, \frac{188\,822\,500\,000}{16\,524\,580\,055\,753}, \frac{1213\,715\,695\,625}{16\,524\,580\,055\,753}\}$
$\approx \{0, 0, .007\,119, 0, .011\,427, .073\,449\}$

$x_7 = \frac{1364\,036\,124\,426}{1502\,234\,550\,523} \approx .908$

eigenvectors:		eigenvalues:		
$[0, 0, 1, 4.451, 9.454\,7, -1.644\,3]$	$\leftrightarrow$		$\approx$	$-1.591\,7 \times 10^{-2}$
$[2.494\,6, 0, 1, 0, 1.085\,7, 7.027\,6]$	$\leftrightarrow$		$\approx$	$-.182\,24$
$[0, 0, 1, 0, -1.132\,4 \times 10^{-2}, -.444\,85]$	$\leftrightarrow$		$\approx$	$.104\,19$
$[0, 0, 1, 0, 8.920\,2, -2.405]$	$\leftrightarrow$		$\approx$	$-.0\,302\,1$
$[0, 175.7, 1, 0, -6.753\,2, -43.767]$	$\leftrightarrow$		$\approx$	$.308\,93$
$[0, 0, 1, 0, 1.605, 10.317]$	$\leftrightarrow$		$\approx$	$-.454$

===

point: $\{3, 6, 7\} = \{0, 0, \frac{11\,196\,622}{2489\,494\,623}, 0, 0, \frac{63\,913\,125}{829\,831\,541}\} \approx \{0, 0, .004\,497, 0, 0, .077\,01\,9\}$

$x_7 = \frac{207\,868\,966}{226\,317\,693} \approx .918\,48$

eigenvectors:		eigenvalues:		
$[0, 488.25, 1, 0, 0, -123.77]$	$\leftrightarrow$	$\frac{74\,388\,700}{226\,317\,693}$	$\approx$	$.328\,69$
$[3.778\,3, 0, 1, 0, 0, 8.847\,5]$	$\leftrightarrow$	$-\frac{3673\,718\,600}{31\,910\,794\,713}$	$\approx$	$-.115\,12$
$[0, 0, -8.42, 0, 14.349, 1]$	$\leftrightarrow$	$\frac{7070\,800}{226\,317\,693}$	$\approx$	$3.124\,3 \times 10^{-2}$
$[0, 0, 1, 0, 0, 17.125]$	$\leftrightarrow$	$-\frac{103\,934\,483}{226\,317\,693}$	$\approx$	$-.459\,24$
$[0, 0, -2.864\,4, -1.912\,6, 0, 1]$	$\leftrightarrow$	$\frac{1835\,625}{75\,439\,231}$	$\approx$	$2.433\,2 \times 10^{-2}$
$[0, 0, -2.182\,6, 0, 0, 1]$	$\leftrightarrow$	$\frac{1660\,740}{75\,439\,231}$	$\approx$	$2.201\,4 \times 10^{-2}$

===

point: $\{3,5,6\} = \{0, 0, \frac{117\,644\,491\,442}{1520\,182\,687\,067}, 0, \frac{188\,822\,500\,000}{1520\,182\,687\,067}, \frac{1213\,715\,695\,625}{1520\,182\,687\,067}\}$
$\approx \{0, 0, .077\,388, 0, .124\,21, .798\,4\}$

eigenvectors:		eigenvalues:		
$[24.533, 0, 1, 0, -3.502\,1, -22.031]$	$\leftrightarrow$	$-\frac{38\,600\,281\,084\,600}{19\,485\,978\,079\,677}$	$\approx$	$-1.980\,9$
$[0, -20.046, 1, 0, 2.558\,6, 16.487]$	$\leftrightarrow$	$\frac{464\,080\,519\,300}{138\,198\,426\,097}$	$\approx$	$3.358\,1$
$[0, 0, 1, 0, -.082\,338, -.917\,66]$	$\leftrightarrow$		$\approx$	$.373\,5$
$[0, 0, 1, 0, 19.088, -20.088]$	$\leftrightarrow$		$\approx$	$-.328\,38$
$[0, 0, 1, -169.38, -297.11, 465.48]$	$\leftrightarrow$	$-\frac{27\,473\,522\,044\,375}{158\,789\,991\,585\,453}$	$\approx$	$-.173\,02$
$[0, 0, 1, 0, 1.605, 10.317]$	$\leftrightarrow$	$\frac{682\,018\,062\,213}{138\,198\,426\,097}$	$\approx$	$4.935\,1$

---

point: $\{3,6\} = \{0, 0, \frac{29\,234}{529\,859}, 0, 0, \frac{500\,625}{529\,859}\} \approx \{0, 0, .055\,173, 0, 0, .944\,83\}$

eigenvectors:		eigenvalues:		
$[0, 0, -1, 0, 0, 1]$	$\leftrightarrow$	$\frac{4982\,220}{18\,448\,727}$	$\approx$	$.270\,06$
$[0, 0, 1, .720\,11, 0, -1.720\,1]$	$\leftrightarrow$	$\frac{500\,625}{1677\,157}$	$\approx$	$.298\,5$
$[15.222, 0, 1, 0, 0, -16.222]$	$\leftrightarrow$	$-\frac{3673\,718\,600}{2601\,270\,507}$	$\approx$	$-1.412\,3$
$[0, 0, 1, 0, 0, 17.125]$	$\leftrightarrow$	$\frac{103\,934\,483}{18\,448\,727}$	$\approx$	$5.633\,7$
$[0, -25.476, 1, 0, 0, 24.476]$	$\leftrightarrow$	$\frac{74\,388\,700}{18\,448\,727}$	$\approx$	$4.032\,2$
$[0, 0, 1.586\,8, 0, -2.586\,8, 1]$	$\leftrightarrow$	$\frac{642\,800}{1677\,157}$	$\approx$	$.383\,27$

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point: $\{5,6,7\} = \{0, 0, 0, 0, \frac{29\,375}{2649\,111}, \frac{203\,750}{2649\,111}\} \approx \{0, 0, 0, 0, .011\,089, .076\,913\}$
$x_7 = \frac{2415\,986}{2649\,111} \approx .912$

eigenvectors:		eigenvalues:		
$[2.199\,9, 0, 0, 0, 1, 6.936\,2]$	$\leftrightarrow$	$-\frac{71\,889\,862}{373\,524\,651}$	$\approx$	$-.192\,46$
$[0, 0, -2.723\,6, 0, 1.833\,9, 1]$	$\leftrightarrow$	$-\frac{35\,944\,931}{1014\,609\,513}$	$\approx$	$-3.542\,7 \times 10^{-2}$
$[0, 0, 0, 0, 1, 6.936\,2]$	$\leftrightarrow$	$-\frac{1207\,993}{2649\,111}$	$\approx$	$-.456$
$[0, -27.549, 0, 0, 1, 6.936\,2]$	$\leftrightarrow$	$\frac{273\,482}{883\,037}$	$\approx$	$.309\,71$
$[0, 0, 0, 1.883\,1, -2.816\,3, 1]$	$\leftrightarrow$	$-\frac{149\,428\,543}{3043\,828\,539}$	$\approx$	$-4.909\,2 \times 10^{-2}$
$[0, 0, 0, 0, -4.717\,9, 1]$	$\leftrightarrow$	$-\frac{78\,892}{2649\,111}$	$\approx$	$-2.978\,1 \times 10^{-2}$

---

point: $\{5,6\} = \{0,0,0,0,\frac{47}{373},\frac{326}{373}\} \approx \{0,0,0,0,.12601,.87399\}$

eigenvectors:		eigenvalues:		
$[0,-7.9362,0,0,1,6.9362]$	$\leftrightarrow$	$\frac{820446}{233125}$	$\approx$	3.5193
$[0,0,0,1,-1.5001,.50006]$	$\leftrightarrow$	$-\frac{149428543}{267860625}$	$\approx$	$-.55786$
$[0,0,0,0,-1,1]$	$\leftrightarrow$	$-\frac{78892}{233125}$	$\approx$	$-.33841$
$[0,0,-1.4965,0,1,.49651]$	$\leftrightarrow$	$-\frac{35944931}{89286875}$	$\approx$	$-.40258$
$[0,0,0,0,1,6.9362]$	$\leftrightarrow$	$\frac{1207993}{233125}$	$\approx$	5.1817
$[-7.9362,0,0,0,1.0,6.9362]$	$\leftrightarrow$	$-\frac{71889862}{32870625}$	$\approx$	-2.1871

---

point: $\{5,7\} = \{0,0,0,0,\frac{47}{373},0\} \approx \{0,0,0,0,.12601,0\}$

$x_7 = \frac{326}{373} \approx .87399$

eigenvectors:		eigenvalues:		
$[0,0,-1.9441,0,1,0]$	$\leftrightarrow$	$\frac{26521}{142859}$	$\approx$	$.18564$
$[0,-2.2607,0,0,1,0]$	$\leftrightarrow$	$\frac{53042}{233125}$	$\approx$	$.22753$
$[0,0,0,1,2.3694,0]$	$\leftrightarrow$	$-\frac{275}{1119}$	$\approx$	$-.24576$
$[0,0,0,0,1,-1.1148]$	$\leftrightarrow$	$\frac{78892}{233125}$	$\approx$	$.33841$
$[0,0,0,0,1,0]$	$\leftrightarrow$	$-\frac{163}{373}$	$\approx$	$-.437$
$[1,0,0,0,-24.641,0]$	$\leftrightarrow$	$-\frac{550}{1119}$	$\approx$	$-.49151$

---

point: $\{6,7\} = \{0,0,0,0,0,\frac{125}{1579}\} \approx \{0,0,0,0,0,.079164\}$

$x_7 = \frac{1454}{1579} \approx .92084$

eigenvectors:		eigenvalues:		
$[0,0,0,0,-5.1201,1]$	$\leftrightarrow$	$\frac{242}{7895}$	$\approx$	$.030652$
$[0,0,-2.0105,0,0,1]$	$\leftrightarrow$	$-\frac{13684}{604757}$	$\approx$	$-.022627$
$[0,-3.9144,0,0,0,1]$	$\leftrightarrow$	$\frac{2596}{7895}$	$\approx$	$.32882$
$[0,0,0,5.81,0,1]$	$\leftrightarrow$	$\frac{1441}{604757}$	$\approx$	$.0023828$
$[0,0,0,0,0,1]$	$\leftrightarrow$	$-\frac{727}{1579}$	$\approx$	$-.46042$
$[1,0,0,0,0,2.5419]$	$\leftrightarrow$	$-\frac{27368}{222639}$	$\approx$	$-.12293$

Appendix 3.

This appendix contains the results of computer simulations of the Discrete Replicator Dynamics performed to estimate the separatrix surface of the subspace $\Delta_{3,5,6,7}$ in a small neighbourhood of the stationary points.

The Discrete Replicator Dynamics has been obtain in the following way. Firstly, the time substitution $t = 2\left(1-\beta^2\right)\left(2-\beta^2\right)\tau_1$ has been made, so the dynamical system (6.7) is transformed into the system

$$\frac{dx_i}{d\tau_1} = \left(\left\{\sum_{j=1}^{6}\left(\widetilde{a}_{ij} - \widetilde{a}_{i6}\right)x_j + \widetilde{a}_{i6}\right\} - x\widetilde{A}x^T\right)x_i, \qquad i = 1, \dots, 6,$$

Here $\widetilde{A} = 2\left(1-\beta^2\right)\left(2-\beta^2\right)A$, where A is the payoff matrix (8.1). For the example considered $\beta = \frac{22}{25}$, $\widetilde{A} = \frac{216\,012}{390\,625}A$. Then the time substitution $\tau_1 = \frac{\tau_2}{x\widetilde{A}x^T}$ has been made which transforms the dynamical system (6.7) into

$$\frac{dx_i}{d\tau_2} = \frac{x_i}{x\widetilde{A}x^T}\left\{\sum_{j=1}^{6}\left(\widetilde{a}_{ij} - \widetilde{a}_{i7}\right)x_j + \widetilde{a}_{i7}\right\} - x_i, \qquad i = 1, \dots, 6.$$

Finally, replacing $\frac{dx_i}{d\tau_2}$ by $\frac{\Delta x_i}{\Delta\tau_2}$, where $\Delta\tau_2 = h$ is fixed and $\Delta x_i = x_i^{k+1} - x_i^k$ we obtain

$$\frac{x_i^{k+1} - x_i^k}{h} = \frac{x_i^k}{x^k\widetilde{A}\left(x^k\right)^T}\left\{\sum_{j=1}^{6}\left(\widetilde{a}_{ij} - \widetilde{a}_{i7}\right)x_j^k + \widetilde{a}_{i7}\right\} - x_i^k, \qquad i = 1, \dots 6.$$

Choosing $h = 1$ in the above formula, the following discrete time analogue of the Replicator Dynamics (6.7) is obtained

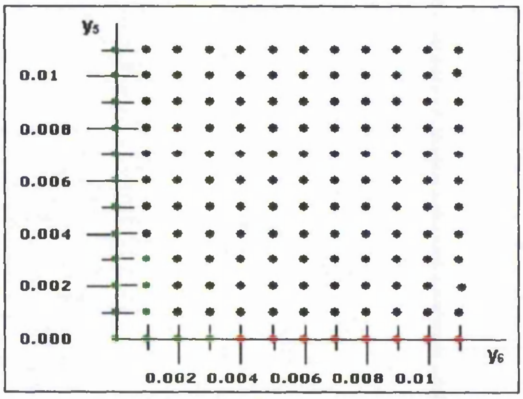
$$x_i^{k+1} = \frac{x_i^k}{x^k\widetilde{A}\left(x^k\right)^T}\left\{\sum_{j=1}^{6}\left(\widetilde{a}_{ij} - \widetilde{a}_{i7}\right)x_j^k + \widetilde{a}_{i7}\right\}, \qquad i = 1, \dots 6. \tag{8.2}$$

This procedure represents a one step Euler method for the solution of the continuous Replicator Dynamics. Therefore, analysis of the linearisation of the system (8.2) gives qualitatively similar results to those that have been obtained for the dynamics (6.7). Therefore it is possible to estimate the form of an invariant manifold of the dynamics (6.7) in a small neighbourhood of a stationary point using results obtained for the dynamics (8.2).

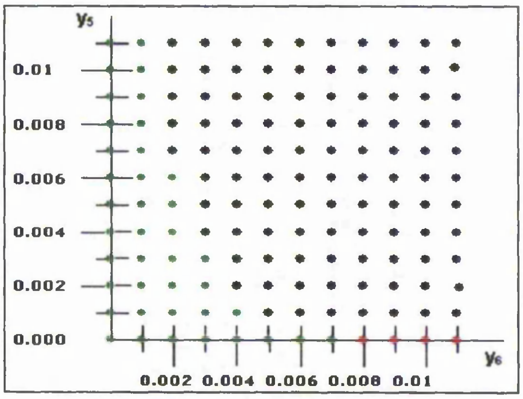
The separatrix surface that belongs to the subspace $\Delta_{3,5,6,7}$ has been estimated in a small neighbourhood of the stationary points $\{7\}$, $\{3,6,7\}$ and $\{3,5,6,7\}$. Different initial points x^1 have been chosen and iterated to obtain x^k using the formulae (8.2). As x^{1500} is calculated, the distances $\left|x^{1500} - \{3\}\right|$ and $\left|x^{1500} - \{5,6,7\}\right|$ are estimated.

- If $\left|x^{1500} - \{3\}\right| < 0.01$ the initial point x^1 is shown in green at the plots below.
- If $\left|x^{1500} - \{5,6,7\}\right| < 0.01$ the initial point x^1 is shown in black.
- If neither of the above conditions are satisfied the point x^1 is shown in red.

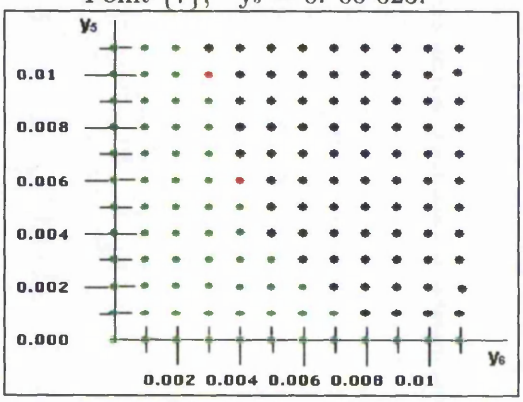
The following plots represent the results obtained.



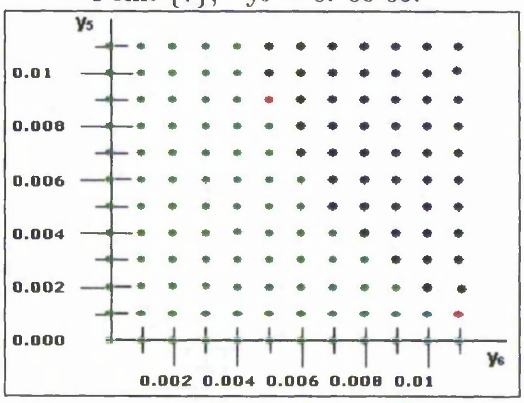
Point $\{7\}$, $y_3 = 0.\ 00\ 025$.



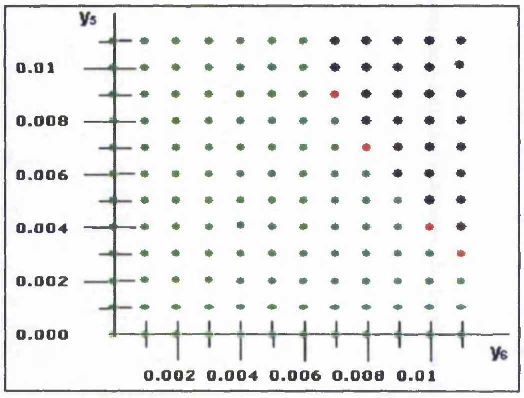
Point $\{7\}$, $y_3 = 0.\ 00\ 05$.

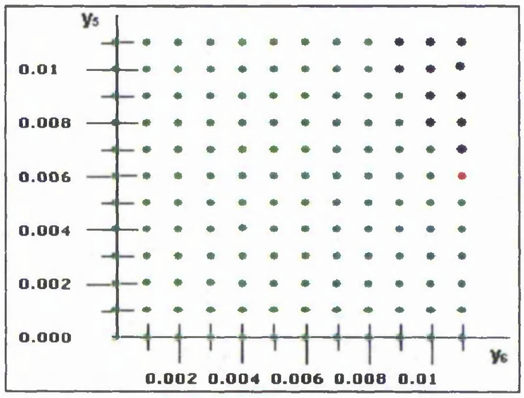


Point $\{7\}$, $y_3 = 0.\ 00\ 075$.



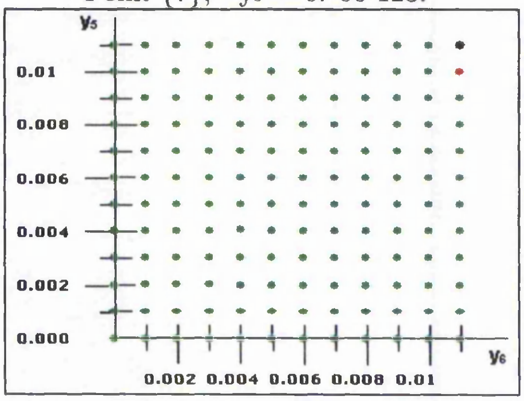
Point $\{7\}$, $y_3 = 0.\ 00\ 1$.

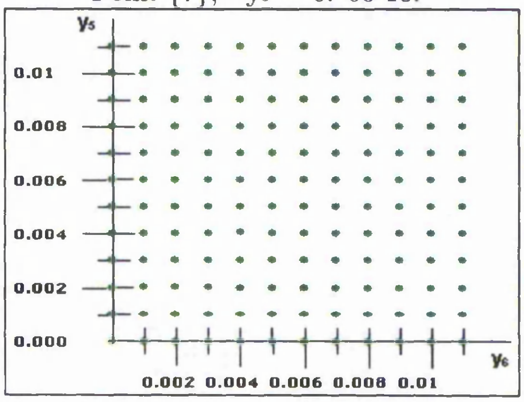




Point {7}, $y_3 = 0.\ 00\ 125$.

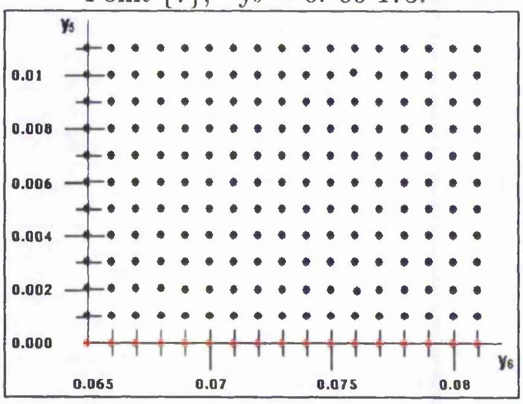
Point {7}, $y_3 = 0.\ 00\ 15$.

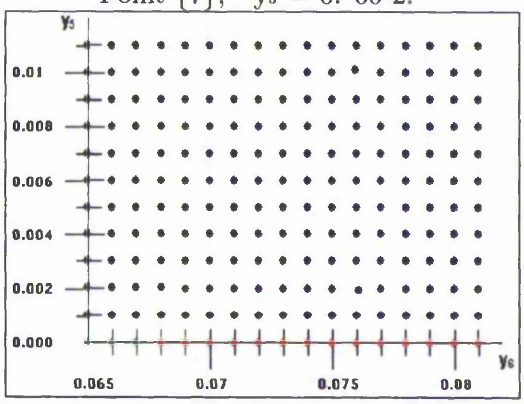




Point {7}, $y_3 = 0.\ 00\ 175$.

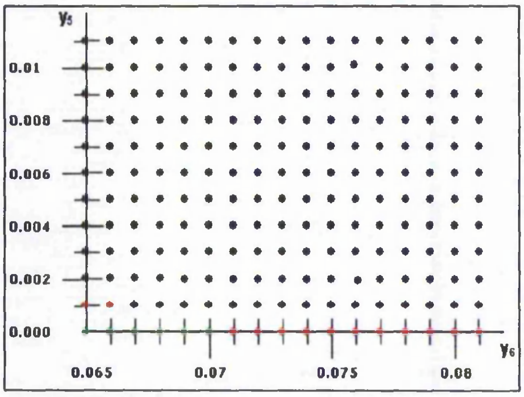
Point {7}, $y_3 = 0.\ 00\ 2$.



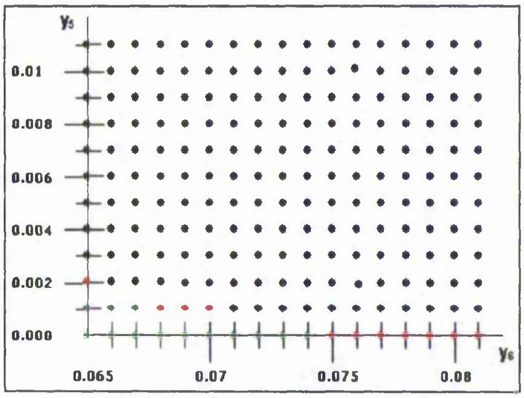


Point {3,6,7}, $y3 = \underset{k=0,\ldots 17.}{\left(25 * 10^{-5}\right) k}$,

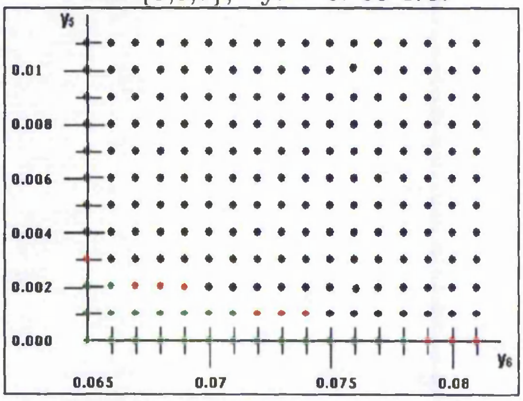
Point {3,6,7}, $y_3 = 0.\ 00\ 4\ 5$.



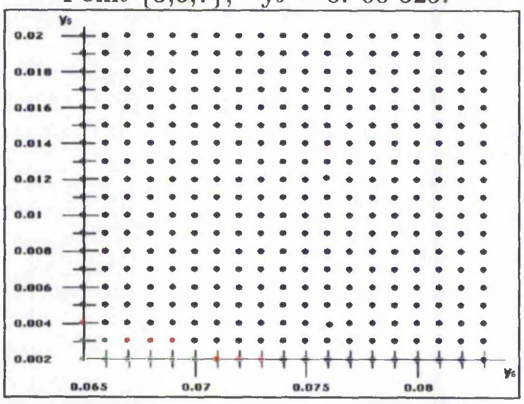
Point {3,6,7}, y_3 = 0. 00 475.



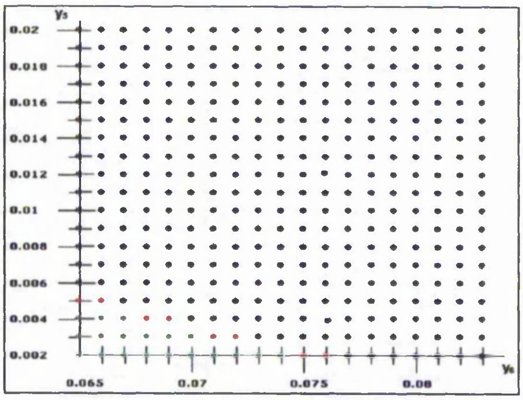
Point {3,6,7}, y_3 = 0. 00 5.



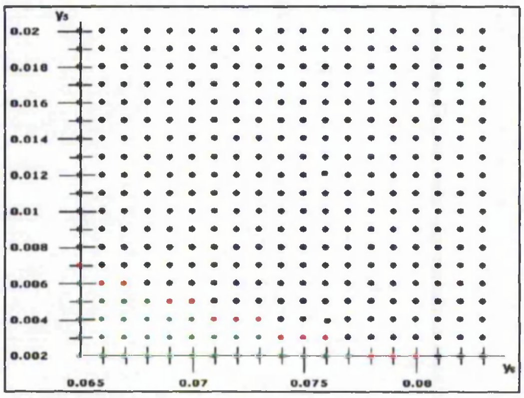
Point {3,6,7}, y_3 = 0. 00 525.



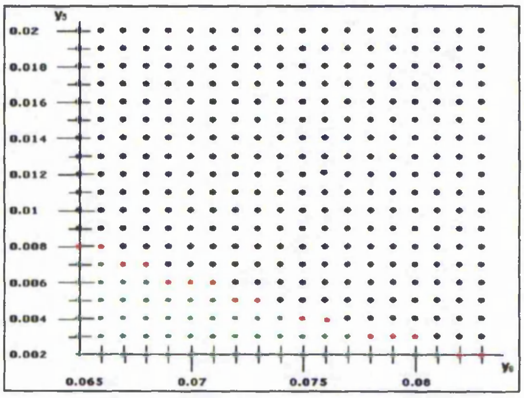
Point {3,5,6,7}, y_3 = 0. 00 55.



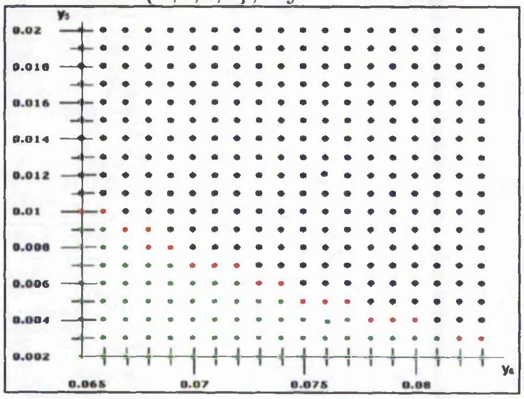
Point {3,5,6,7}, y_3 = 0. 00 575.



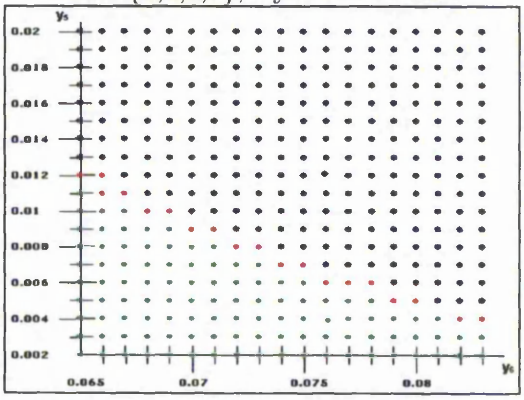
Point {3,5,6,7}, $y_3 = 0.006$.



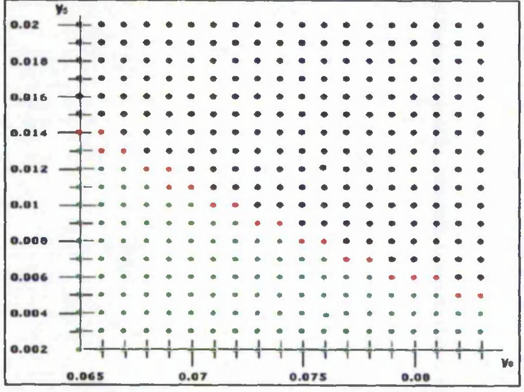
Point {3,5,6,7}, $y_3 = 0.00625$.



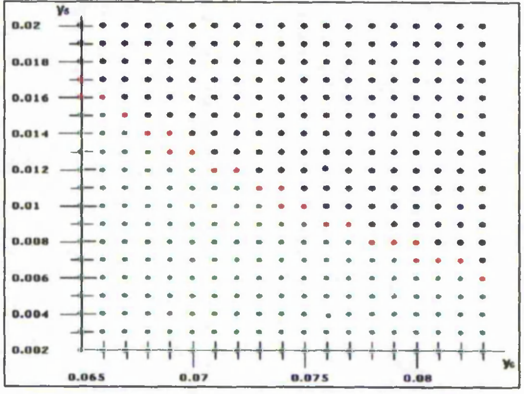
Point {3,5,6,7}, $y_3 = 0.0065$.



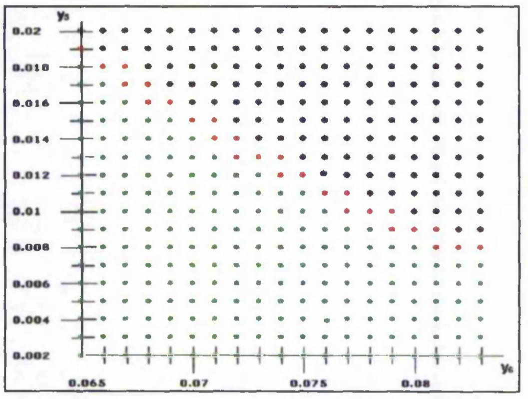
Point {3,5,6,7}, $y_3 = 0.00675$.



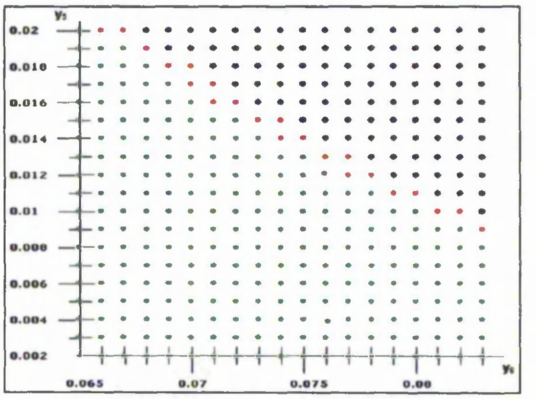
Point {3,5,6,7}, $y_3 = 0.007$.



Point {3,5,6,7}, $y_3 = 0.00725$.



Point {3,5,6,7}, y_3 = 0. 00 75.



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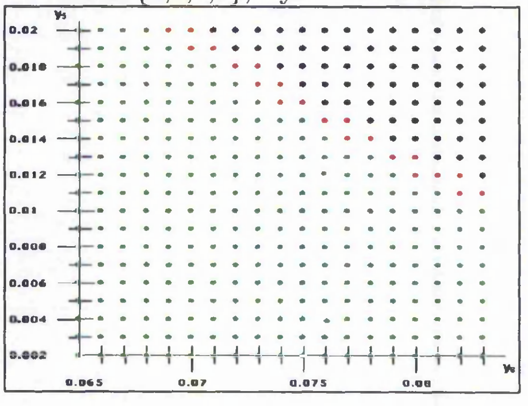
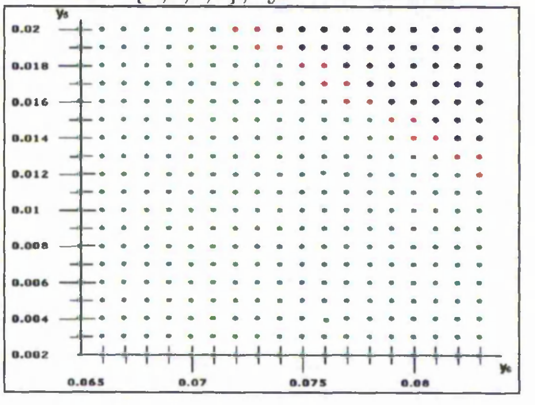
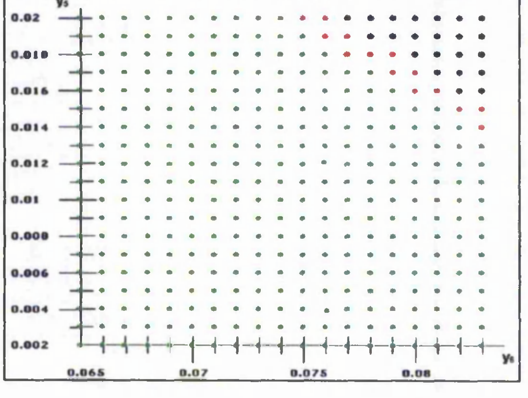


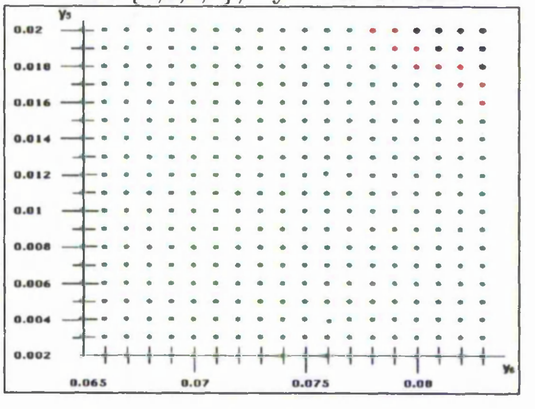
Point {3,5,6,7}, y_3 = 0. 00 8.



Point {3,5,6,7}, y_3 = 0. 00 825.



Point {3,5,6,7}, y_3 = 0. 00 85.



Point {3,5,6,7}, y_3 = 0. 00 857.

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