INVESTIGATION OF THE ACCELERATING SUSPENDED GYROSCOPE AS APPLIED TO GYROTHEODOLITE AZIMUTH DETERMINATION

by

Yousef Kebbeih

A thesis submitted in partial fulfilment of the requirements of
The Nottingham Trent University
for the degree of Doctor of Philosophy (PhD)

May 1999

ABSTRACT

The Wild GAK1 is a surveying gyroscope coupled with a theodolite. This instrument can be used to orientate an underground survey baseline relative to true North. During the operation of the gyro, the spinner is run up to approximately 22,000 r.p.m. which causes the gyro, when released, to seek true North and oscillate about the meridian. These oscillations can be observed through an eyepiece. The movement of oscillations is seen in the form of a moving mark against a background of scale divisions. From an analysis of this movement, an orientation with respect to North can be established. Jeudy established the theory of the motion of the suspended gyroscope. However, he assumed that the motor drives the gyroscope spinner at a constant angular velocity, which in practice is not true. In this research, the movement equations of the suspended Gyrotheodolite are derived taking account of all significant terms. These terms reflect the changes in the physical environment within the suspended gyroscope taking into account the fact that the angular velocity of the spinner is not constant. These equations are linearised to get oscillation equations. Having resolved these equations, there are two differential equations, which are, in turn, resolved to get a new mathematical model concerned with the motion of the moving mark. This model deals with the general and practical cases. For example, when the gyro is used in a tunnel where the battery, which runs down with time, is the main source of power.

A new method of time capture and "data" processing is described. This method requires a video camera, video imagery, frame analysis and a computer. The method may be used in many practical applications but it must be acceptable for mine safety for electric equipment if used in a mine. The method leads to a great increase in the quantity and precision of time observations. The observations have a precision five times better than those observed by manual methods. After processing, the time data is used in a rigorous mathematical model and processed by least squares techniques. This leads to high quality solutions and statistical assessments. Least squares adjustments showed that the computed values of the midpoint of swing might be determined to standard deviations of less than one second of arc.

ACKNOWLEDGEMENTS

Foremost, I am grateful to my supervisor, Mr. M.C. Breach for his guidance and valuable assistance in my investigation of this topic.

I am very grateful to Professor R. Hawkins, Dean of the Faculty of Construction and the Environment, for his encouragement and assistance since I came to study my PhD in UK three years ago.

Sincere gratitude and thanks are due to Dr. I. Sholji and Mr. J. Abbiss, second and third supervisors of this research for their advice and assistance.

I would like also to thank staff in the Department of Civil and Structural Engineering, especially the Survey Subject Group for providing this research with survey equipment, computing and workshop facilities.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

I. INTRODUCTORY AND BACKGROUND

1.1 Introduction:

A Wild GAK1 Gyrotheodolite is a surveying instrument used to find the direction of North. One of the major applications of this instrument is to orientate an underground survey baseline relative to true North. The Wild GAK1 can be mounted on a bridge device over a conventional theodolite. When it is levelled the spinner hangs like a plumb bob and is constrained in the horizontal plane. During operation the spinner is run up to approximately 22,000 r.p.m. which causes the gyro, when released, to seek true North and oscillate about the meridian. The oscillations can be observed through an eyepiece as a light moving mark against a background of scale divisions.

Jeudy (1981 and 1982) established the theory of the motion of the suspended gyroscope at a new and higher level. However, he assumed that the motor drives the gyroscope spinner at a constant angular velocity. This could be true if the power source for the motor maintains a constant output. If the gyroscope is used in the field, for example, in a tunnel, then the power source is a battery, which with time runs down. Consequently, there is a decelerating force applied to the spinner. In this research all terms, which reflect the changes in the physical environment within the suspended gyroscope are taken into account. Jeudy considered the system of suspended gyroscope as a system with five degrees of freedom, two co-ordinates of the suspended point and three Euler angles of the orientation of the carriage with respect to space. His solution of the system, uniquely, is in the form of a sum of sine and cosine waves of differing periods:

$$\sum_{i=1}^{5}(A_i' \cos\theta_i t + A_i'' \sin\theta_i t)$$

Where:

θ_i are frequencies of oscillations.

A_i' and A_i'' are magnitudes of oscillations.

t is the time.

In this research the complete equation of motion of the moving mark for the suspended gyroscope is given. In this equation no force affecting the gyroscope is neglected. There are insignificant errors due to neglecting the earth's rotation centrifugal forces. Also, the deflection of the vertical is not considered. The equation is:

$$\sum_{i=1}^{10}[A_i e^{-\lambda_i t}\cos((\theta_i+\theta_i' t)(t-B_i))]+\boldsymbol{Q}-\Delta=0$$

Where:

A_i are the amplitudes (magnitudes of the oscillations).

θ_i' are rates of change in the frequencies of the oscillations.

λ_i are the coefficients of damping forces.

B_i are the times at a positive turning point.

$\boldsymbol{Q}$ is the mid-point of swing.

Δ is the scale reading.

A semi-empirical assumption that agrees with practical observations is that there is a linear change in the rate of the frequency with time. Comparison of this mathematical model with the previous one showed that the inclusion of the term $\theta' t$ in the observation equation has an effect on the computed value of the standard deviation of the midpoint of swing, σ_Q. The precision of σ_Q has improved by 0.4″ (arc seconds).

In conventional methods, the observer takes a few observations of time, scale divisions and turning points during the period of one or two oscillations of the moving mark. These observations are put in an appropriate mathematical model to give a unique determination of the midpoint of swing, without any degrees of freedom. The midpoint of swing is found from the minimum use of the available data. Therefore, the quality of individual values of the midpoint of swing and the azimuth determined in such methods cannot be assessed since the standard deviations cannot be assessed from the observations.

This research describes a new method of capturing and processing Gyrotheodolite data. The method requires a video camera, frame analysis and a computer. The midpoint of swing, in this method, is found from the maximum data practically available. The method is to observe time at each instant the moving mark crosses a scale division during two or three hours of observations. The least squares adjustment techniques showed that the position of the midpoint of swing can be determined with standard deviation of ± 0.2″ (arc seconds) and the azimuth may be determined with standard deviations of ± 3″ to ± 6″. This assumes knowledge of the instrument constant.

The time recording device, in conventional methods, is often a stopwatch that is a very inaccurate and highly personal. According to Gregerson, the quality of this manual method of timing is 0.2 seconds at best (Gregerson et al, 1974). In this research, time may be observed with a precision of one frame, 0.04 seconds, or less by using video frames. Thus, the accuracy of timing increased 5 times, and the quantity of observations increased up to 10 to 15 times by automated data capture. The time data is used in a rigorous mathematical model and processed by least squares techniques. The values of most parameters and their standard deviations up to the tenth term of differing periods of oscillations are found. However, only three terms, apart from the main period term, may be considered for practical applications because the other terms of oscillations have no significant effect on the computed values of the midpoints of swing and their standard deviations.

The method used in this research resulted in a better understanding of the motion of the suspended gyroscope and it can be used safely in industrial applications, for example, railway tunnels. However, the method must be acceptable for mine safety for electrical equipment if used in a mine. The importance of this method is that it used the instrument of the Department without the need for modification and without using an electronic registration device. The video camera used may be any standard one with sufficient resolution. As a result, the method is faster and more cost effective for "capturing" and processing Gyrotheodolite data.

Although a new automated Gyrotheodolite from Bochum University was used on the Channel Tunnel it was not used on the Jubilee Line extension because of cost. The purchase price is understood to be £50,000-70,000. The Wild GAK1 is 1970's technology but existing instruments could be used according to the method proposed in this research, to approach comparable precision but with less cost compared with the Bochum instrument.

1.2 Aims and objectives of the project:

The aims of this research are:

1. To investigate a new mathematical model as it relates to the Wild GAK1 suspended gyroscope, taking account of all potentially significant terms, which reflect the changes in the physical environment within the suspended gyroscope, that is, to account for gyroscopic accelerations.

2. To investigate an effective method of obtaining and processing the maximum data available from the Wild GAK1 Gyrotheodolite and to improve the azimuth determination.

The objectives are:

- In this research a new mathematical model concerned with the motion of the gyroscope moving mark is to be derived. The model deals with general and practical cases, for example, when the Gyrotheodolite is used in a tunnel. In that case the battery, which runs down with time, is the main source of power. The model is to take account of the fact that the angular velocity of the spinner is not constant. The same mathematical model also is to take into account all terms of oscillations produced by the precession torque due to the couple, applied to the spinner axis. Most of the previous work in this area considered only one oscillation term. The effect of including the other terms in the observation equation on the determined values of the midpoints of swing and on their standard deviations is to be investigated.

- In this research use is to be made of the maximum data practically available from the Gyrotheodolite by observing time at each instant the moving mark crosses a scale division. Data is to be captured on videotape. The quality of timing will improve from approximately 0.2 seconds, at best by manual methods to 0.04 seconds with video frames. The method is expected to lead to a great increase in the quantity and precision of time observations. The precision of position of the midpoint of the swing is expected to be improved. This may be achieved by processing the time data by least squares techniques applied to a rigorous mathematical model. A precise determination of the midpoint of swing with more accurately measured data will lead to a higher precision of azimuth determination.

1.3 Previous work in this area:

Many authors, including Rellensmann (Rellensmann 1959-1960) have reported the technique of using a suspended gyroscope. At that time, there was only one existing theoretical approach. The instrument was used in marine navigation, and was developed by Deimel (Deimel, 1950). There was some difference in experimental results because of small wobbles on the main gyroscope axis, which oscillates about the direction of North. Later, this theory was used again by Vanicek (Vanicek, 1972), who takes account of this wobble by giving the expression of a damped sine function but it does not result in a reduction of the number of necessary observations. There is no theoretical justification for this approach. In 1975, a much more precise theory was developed (Schultz, 1975) effectively taking into account the fact that the gyroscope is suspended. Nevertheless, the Coriolis forces were neglected so that the theory is not precise enough for geodetic applications. However, the improvement in precision of measurements, gained by using photoelectric cells and a chronograph, allows a series of times measured to the 1/100 second (Halmos, 1977) and even to 1/1000 second (EMR, 1975). Such an increase in precision requires the development of a new theory.

This thesis develops the theory for a suspended gyroscopic system in which no force is neglected. The gyroscopic system under consideration is a system with five degrees of freedom (Schultz, 1975) which, two of them are $\bar{x}_I$ and $\bar{y}_I$, the co-ordinates of point I in figure 4 where the carriage is fixed. The other three degrees of freedom can be as Euler angles of the orientation of the carriage in space, for example, by Leimanis (Leimanis, 1965). They may be expressed also as Cardan angles, for example, by Schultz (Schultz, 1975). The application of the laws and principles of classical mechanics (Goldstein, 1959) allows general movement equations to be established. As well as the three Euler equations, it is necessary to derive new equations because there are five unknown functions corresponding to the five degrees of freedom.

Jeudy (1981 and 1982) in a pair of papers established the theory of the motion of the suspended gyroscope. However, he assumed in his work that the gyroscope spinner runs at a constant angular velocity. This could be true if the power source for the motor driving the spinner maintains a constant output. If the gyroscope is used in the field, for example, in a tunnel, then the main power source is a battery, which with time runs down. Consequently, there is a decelerating force applied to the spinner. With the large volume and greater precision of observations possible from considerations of the above paragraphs, current theory is no longer adequate. A major part of this work has been to develop the theory of the suspended gyroscope to take account of accelerating forces, which alter the angular velocity of the spinner and to make use of the maximum data available practically from the Gyrotheodolite data. The equations derived in chapters two and three, which are based upon the work of Jeudy and his notation and method are referenced to him.

With particular reference to the Wild GAK1, there are two basic classical methods of azimuth determination using a gyroscopic attachment, the Turning Point and Transit methods. Many researchers have investigated and developed techniques to improve the precision and speed of these methods of azimuth determination. These two methods were described by Strasser and Schwendener

(1964) and involve modification of the gyro instrument. Modified Wild GAK1 was used and tested by Smith (1977). The techniques to improve the precision and reduce the time required to obtain a determination of the direction of true North from gyroscopic devices are described by Bennett (1970), Thomas (1982) and Williams (1986). King (1987) described a tracking method based on the Wild T2000 theodolite. In the UK the Wild GAK1 may be fitted to a specially adapted T2 or T16 theodolite. Work on semi-automated data capture by non-video means was carried out by Breach (1983) (project for MSc in Geodesy). The current project follows on from that earlier project but with 1990's technology.

Martusewicz (1993) used two Gyrotheodolites for measurements of extra gyroscopic azimuths to check underground traverses. He derived a mathematical model for the optimal positions for Gyrotheodolites for azimuth determination. Hodges (Hodges et al. 1994 and 1996) made a precise calibration of a Gyrotheodolite on a field base line. He used a Wild GAK1 and a Gyromat-2000 and obtained a precision of ± 3″ (arc seconds). Plakhtienko (Plakhtienko and Dmitriev, 1996) investigated the effect of the anomaly of the earth's gravity on Gyrotheodolite readings. A detailed comparison between the Wild GAK1 and the Gyromat-2000 was made by Eyre (Eyre and Wetherelt, 1995).

There are many papers published every year on the theory of the gyroscope, for example, (Lin et al. 1995), (Bencze et al. 1996), (Brown and Xu, 1996), (Heiberg et al. 1997), (Tanaka and Wakatsuki, 1998) and (Jaroszewicz and Szelmanowski, 1998). However, the type and applications of these gyroscopes are different. They are electronic, fibre optic and piezo electric gyroscopes. The applications are for military use, spacecraft, missiles and navigational purposes and also for robots. They are less precise than mechanical gyroscopes. The gyroscope, which is the subject of this investigation, is a suspended mechanical one, the major application of which is to determine azimuths in underground environments.

1.4 Description of the suspended gyroscope:

The Wild GAK1 is a surveying instrument used to find directions toward North. The instrument can be mounted on a bridge device over a conventional theodolite. It consists of oscillating and supporting systems, figures1 and 2.

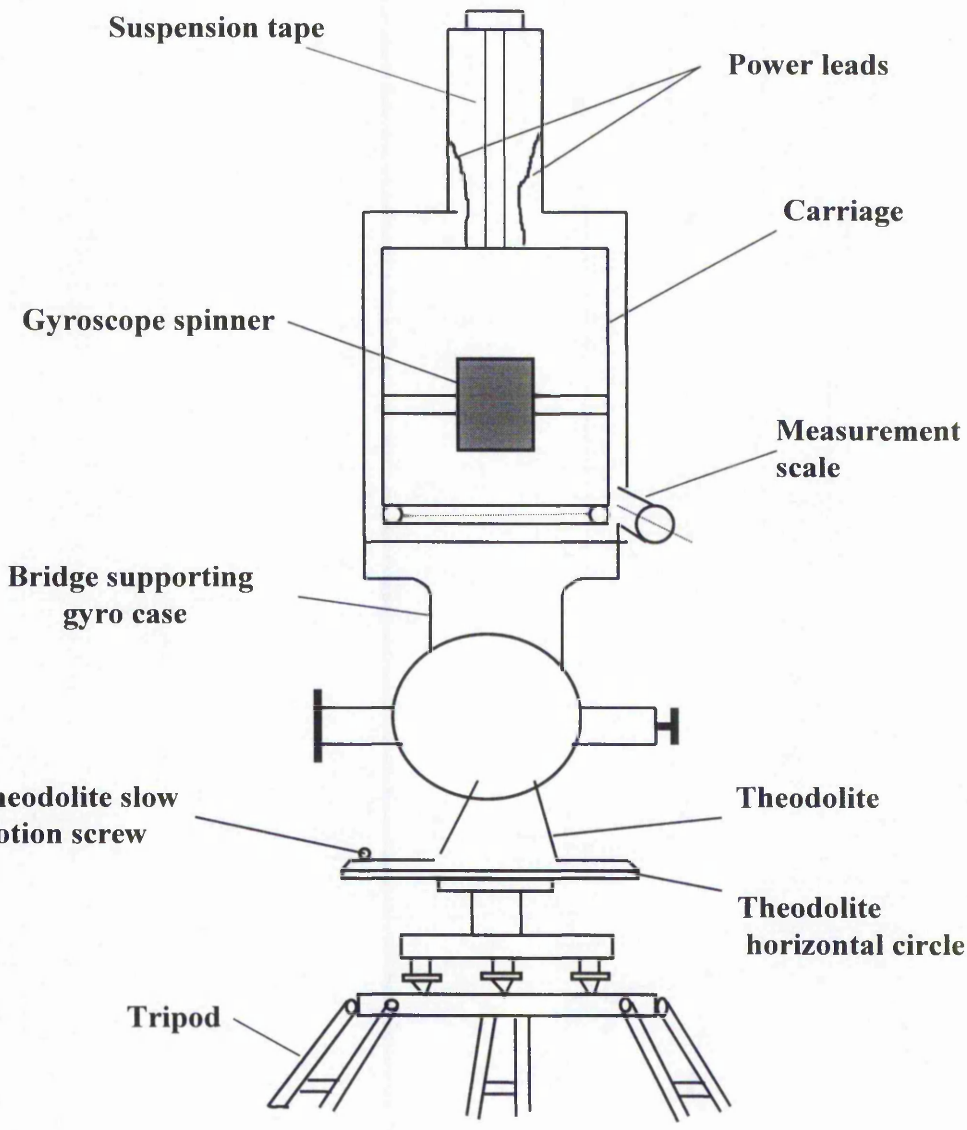
Figure 1

Gyrotheodolite Wild GAK1 mounted over theodolite set on a pillar

The oscillating system consists of a rotating solid body – the spinner of the gyroscope turning at high velocity, 22,000 r.p.m. approximately, about an axis fixed within a rigid covering - the carriage. The carriage is attached at its upper end by a suspension fine tape (point I in figure 4) where its co-ordinates in the system $O_{\overline{xyz}}$ are $\overline{x}_I$, $\overline{y}_I$ and $\overline{z}_I$. The upper end of the tape, point A in figure 4, is attached to a fixed frame with respect to the earth's surface. The acceleration of the spinner is not taken as a constant value due to accelerating and decelerating forces applied to the spinner. The supporting system consists of three columns and contains a clamping device. There are three plates, which act as springs.

They dampen the gyro oscillations by friction on the damping plate, with the clamping device half-open.

Figure 2
Diagram of suspended gyroscope



1.5 Basic definitions and reference systems:

Described here are the different reference systems and the main parameters of movement used in this thesis, see also Appendix A for symbols and notations.

$O_{\overline{xyz}}$ is a system of reference fixed with respect to the earth's surface, figure 3.

Where:

$O_{\overline{z}}$ has the inverse direction to the direction of gravity.

$O_{\overline{x}}$ is perpendicular to $O_{\overline{z}}$ and is chosen such that the plane $O_{\overline{xz}}$ contains $\vec{\varpi}$ the angular velocity vector of the earth with respect to inertial space.

$O_{\overline{xz}}$ is the astronomical meridian plane through the point O.

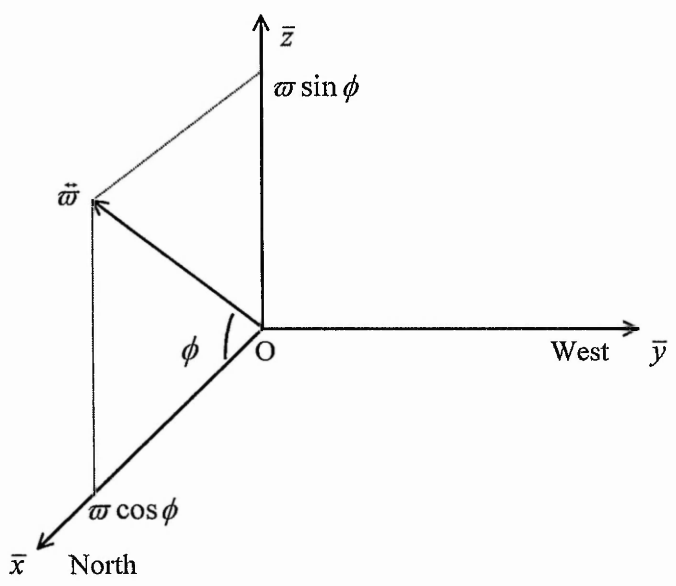
$O_{\overline{y}}$ is directed towards the west.

$O_{\overline{xyz}}$ is a right handed or direct system.

ϕ is astronomical latitude of the point O.

<u>Figure 3</u>

Description of the reference system $O_{\overline{xyz}}$



Other reference systems related to the gyroscope and the carriage are defined as follows:

$G_{X'Y'Z'}$ is a reference system having its axes parallel to those of $O_{\overline{xyz}}$. The origin of $G_{X'Y'Z'}$ is in **G**, the centre of gravity of the gyroscope.

G_{xyz} is a system of reference fixed with respect to the carriage, G_x is the axis of rotation of the gyroscope.

Both co-ordinate systems G_{xyz} and $G_{X'Y'Z'}$ are right-handed and the planes G_{xy} and $G_{X'Y'}$ intersect along the line G_N which is perpendicular to the plane through the axes $G_{Z'}$ and G_z (see figures 4 and 11).

α, β and γ are the Euler angles of the system G_{xyz} with respect to the system $G_{X'Y'Z'}$ (see figures 4 and 11).

In figure 4:

Λ and φ are the angles allowing the marking of the position of point I.

$\vec{\omega}$ is the angular velocity of the gyroscope with respect to the carriage.

I is a fixing point of the tape attached to the carriage.

$\overline{x}_I$, $\overline{y}_I$ and $\overline{z}_I$ are the co-ordinates of point I in the system $O_{\overline{xyz}}$.

A is a fixed end of the tape in the system $O_{\overline{xyz}}$.

ℓ is the length of the tape.

Figure 4

Plan of basic geometric elements of the suspended gyroscope

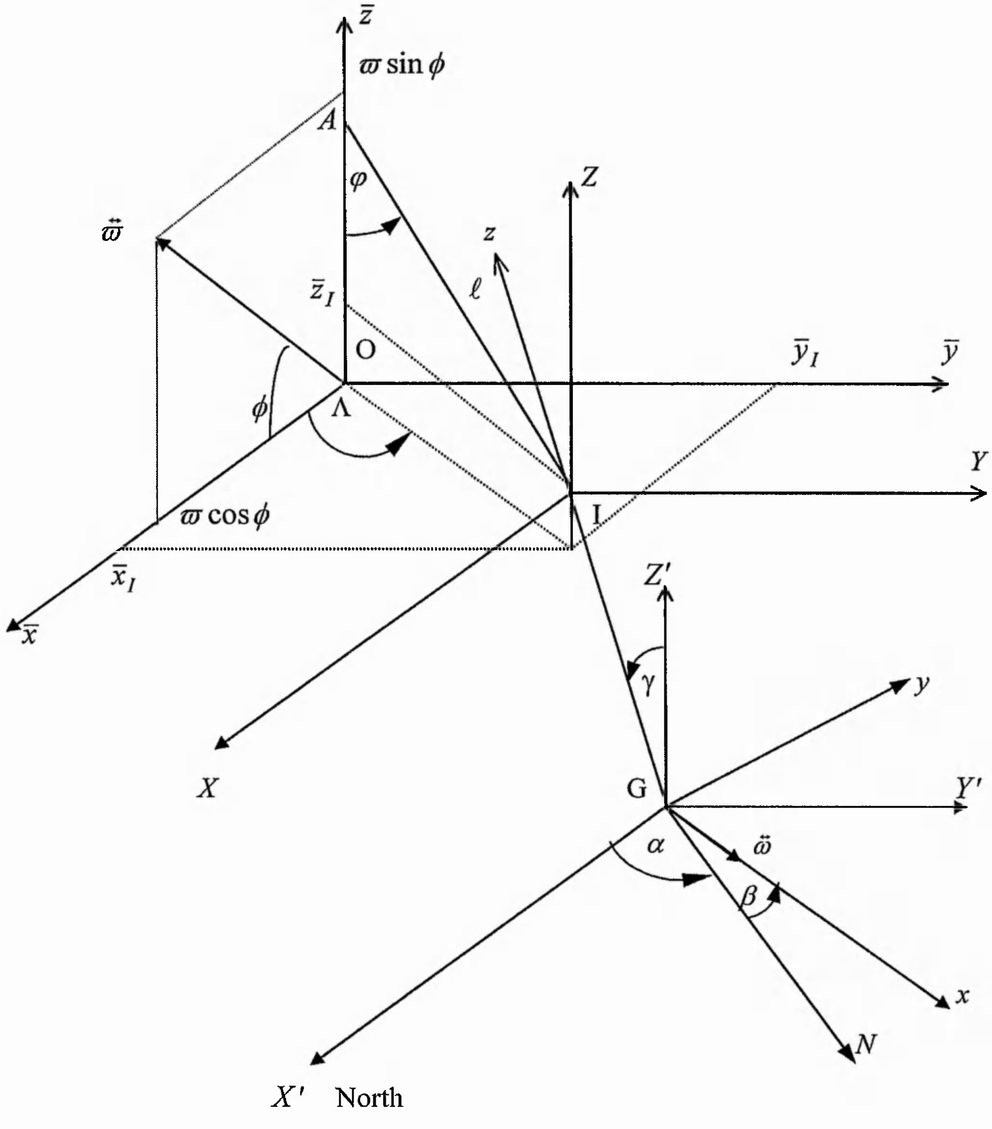
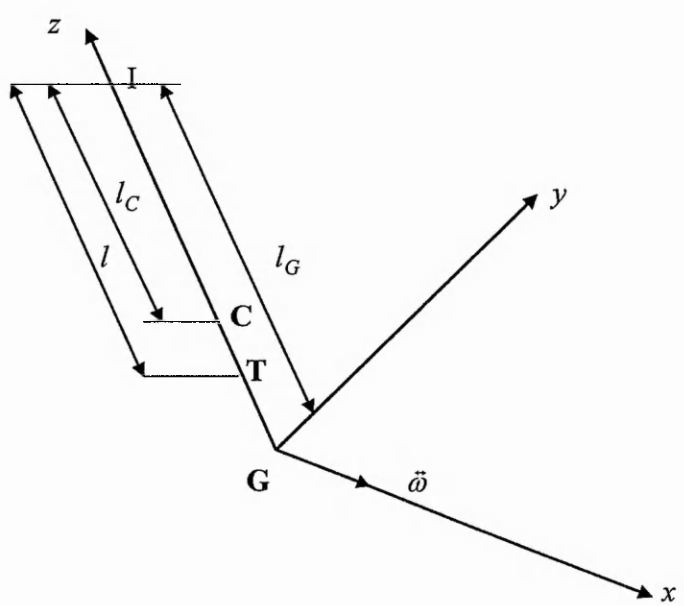


Figure 5

Details of the system G_{xyz}



Where:

C is the centre of gravity of the carriage.

T is the centre of gravity of the total system (gyroscope plus carriage).

l is the distance from I to **T** ($l = IT$).

l_C is the distance from I to **C** ($l_C = IC$).

l_G is the distance from I to **G** ($l_G = IG$).

The system $O_{\overline{xyz}}$ fixed with respect to the earth's mass centre, figure 3 rotates by a presumed constant value $\vec{\varpi}$ with respect to inertial space. All magnitudes of the vectors used in this project (vectors of position, velocity, kinetic moment etc.) derived with respect to time are related to inertial space. These derivations are absolute unless explicitly mentioned to the contrary. However, the magnitudes of components of these vectors are usually expressed in a non-inertial system, most often in the system G_{xyz} fixed with respect to the carriage, but sometimes also in the system $O_{\overline{xyz}}$. The orientation of G_{xyz} is related to inertial space by the formula:

$\overleftrightarrow{\Gamma}_C = \overleftrightarrow{\varpi} + \overleftrightarrow{\Gamma}$ (Goldstein, 1959)

Where:

$\overleftrightarrow{\Gamma}_C$ is the absolute angular velocity of the carriage.

$\overleftrightarrow{\varpi}$ is the angular velocity of the earth with respect to the fixed stars.

$\overleftrightarrow{\Gamma}$ is the instantaneous angular velocity of the system G_{xyz} with respect to $O_{\overline{xyz}}$.

The axes of these two movable systems, one with respect to the other, are usually not parallel. Using the conventional symbols of classical mechanics, the absolute derivation of the magnitude of a vector $\vec{V}$ is expressed:

$$\frac{d\vec{V}}{dt} = \left(\frac{d\vec{V}}{dt}\right)_{G_{xyz}} + \overleftrightarrow{\Gamma}_C * \vec{V}$$

Where:

$*$ is the vectorial product.

$\left(\frac{d\vec{V}}{dt}\right)_{G_{xyz}}$ denotes the vector of the derivations of the components of $\vec{V}$ in G_{xyz}.

By making, successively $\vec{V} = \vec{i}$, $\vec{j}$ and $\vec{k}$, then in the previous equation, one obtains the specific expressions:

$$\frac{d\vec{i}}{dt} = \overleftrightarrow{\Gamma}_C * \vec{i},\ \frac{d\vec{j}}{dt} = \overleftrightarrow{\Gamma}_C * \vec{j} \text{ and } \frac{d\vec{k}}{dt} = \overleftrightarrow{\Gamma}_C * \vec{k}$$ (Chester, 1979; P: 254)

The sums of the infinitely small elements, appear in calculation of the absolute kinetic moments, are denoted by the symbol "$\sum$" which allows us to specify the differential integration elements. Vectors surmounted by a double-headed arrow (↔) denote angular velocities and "moments" in order to distinguish them from ordinary vectors, which are surmounted by a single-headed arrow (→). This is because the former vectors have direction depending on the orientation of the reference system whilst the direction of ordinary vectors is independent of the orientation of the reference system. The symbol '$\in$' denotes "belonging to", for example "P$\in$(G)" means; the point P belongs to the gyroscope and "$\sum_{P\in(G)}$" denotes the sum of all the points P of the gyroscope.

1.6 Scope of the thesis:

In Chapter 2, the general equations of movement for the suspended gyroscope are derived. These equations are in the form of non-linear differential equations of second order. The equations determine the eight unknown functions of the system, $\bar{x}_I$, $\bar{y}_I$, $\bar{z}_I$, α, β, γ, ω_1 and $|\vec{S}|$. They are respectively the three co-ordinates of point, I (see figure 4), the three Euler angles, the angular velocity of the spinner and the tension of the tape. Firstly, the three Euler equations are established. These equations are modified to take account of the fact that the angular velocity of the spinner is not constant and to take account of all terms of the oscillations, which are produced by the precession torque due to a couple, applied to the spinner axis. See equations (2-65), (2-66) and (2-67). A fourth equation (2-68) is obtained explaining the fact that the point of suspension I moves on a sphere of centre at A and radius ℓ (see figure 4). The length of the tape is presumed constant. Secondly, three other equations are obtained by applying the principle of the movement of the centre of a mass (Newton's second law) to the total system (gyroscope and carriage). The equations are in vector form, equation (2-82). Finally, the eighth equation is obtained by applying the principle of kinetic moment to the gyroscope by itself, equation (2-88).

In Chapter 3, from the eight general non-linear and rigorous equations of motion for the suspended gyroscope obtained in Chapter 2, the position of the apparent equilibrium position is determined by putting all derivatives of the motion parameters to zero. Then, the equations of motion are linearised in the neighbourhood of the equilibrium position to obtain a system of five linear differential equations with constant coefficients and five unknown functions. See equations (3-58) to (3-62). The system of equations are solved by elimination of unknowns to obtain two differential equations, (3-79) and (3-82) of the sixth order. The corresponding characteristic equation has degree ten powers, equation (3-90). Having solved the differential equations and taking account of the assumption of linear change of frequency with time, the mathematical model of the motion of the moving mark is given in equation (3-95).

In Chapter 4, a new method for capturing and processing time data is described. This method makes use of the maximum data available. The method is to observe time at each instant the moving mark crosses a scale division. The method requires a video camera, video imagery, frame analysis and a computer. The practical implementation of the method is considered. The procedure for North determination by using the midpoint of swing in the equation for azimuth determination is described. Finally, the precision of the azimuth is analysed.

Chapter 5 deals with the least squares adjustments of Gyrotheodolite observations. Time observations are used in the derived mathematical model and the values of most parameters in terms of equation (5-2) and their standard deviations are found. The results are discussed and analysed. Numerical comparison with the previous models is included.

In Chapter 6, the results obtained in Chapter 5 are summerised and evaluated. Conclusions from these results are drawn. Finally, a further research and improvement section considers the significance of the results, the implications of this work and its deficiencies. This section also includes suggestions for further work to overcome the problems.
Appendices for notation, observations and least squares adjustment computations are also included at the end of thesis.

II. THE EULER EQUATIONS

2.1 Introduction:

The general equations of the motion for the suspended gyroscope are derived, following a similar procedure to that of Jeudy's derivations (Jeudy, 1981). However, in these equations, account of the fact that the angular velocity of the spinner is not constant is taken. Account of the other oscillations produced by the precession torque due to a couple, applied to the spinner axis is also taken. The motor of the gyro is suspended on a thin tape. When the spinner runs to its maximum angular velocity, it tries to rotate within the meridian plane. However, because of the angular velocity of the earth, the whole system is pulled out of its original plane. This makes the gyro accelerate about its vertical axis until its spinning axis coincides with the direction of true North. The torque in the tape causes a precession about the vertical axis. This torque is proportional to, the angular momentum of the gyro, P_a, the horizontal component, $\varpi \cos\phi$, of the vector of the earth's rotation $\vec{\varpi}$ and the sine of the horizontal angle, ε, between the spinner axis and the North (Thomas, 1976). The precession torque is given by the equation:

$\Omega_1 = P_a \, \varpi \, \cos\phi \, \sin\varepsilon$

P_a, the angular momentum of the spinner may be written as:

$P_a = P \, \omega_1$ (Vanicek, 1972)

Where P is the inertial moment of the gyroscope with respect to its axis of rotation and ω_1 is the angular velocity of the spinner.

By differentiating the equation of torque due to precession, we get:

$d\Omega_1/dt = \varpi \, \cos\phi \, \sin\varepsilon \, dP_a/dt$

Therefore, the torque due to precession changes if the angular velocity of the spinner changes. The angular momentum of the spinner may vary with time and this causes some amount of friction to the internal parts of the Gyrotheodolite. So, for this reason and because of existing damping forces, the moment of the tape torsion $c\vec{k}$ is modified to be a function of the Euler angles α, β and their derivatives. $\vec{k}$ is the unit vector of the axis G_z.

It was found by Gregerson (Gregerson, 1971) that any change in the angular velocity of the spinner causes a change in the period of swing. This problem is dealt with, in this research by assuming a linear change of the frequency of oscillation with time. This assumption is a simple representation of the solution and it is used to ensure the mathematical model is not more complex rather than is necessary.

Firstly, the three Euler equations are established, (2-65), (2-66) and (2-67). This requires calculation of the absolute kinetic moment of the carriage and the gyroscope. Then, derivation of the total kinetic moment within G_{xyz} is expressed and evaluated. The total kinetic moment is equal to the moment of external forces. These forces are, the attraction of the gravity vector of the earth and other celestial bodies, the accelerating and decelerating forces applied to the spinner, the tension of the tape, the torsion of the tape and the damping forces. Afterwards, a fourth equation (2-68) is obtained explaining the fact that the point of suspension I, moves on a sphere with centre at A and radius ℓ, see figure 4, where the length of the tape is presumed constant. Three new equations (2-82) are obtained by applying the principle of the movement of the centre of a mass (Newton's second law) to the total system (gyroscope and carriage). The equations (2-82) are in the form of three components of a vector formula. Finally, an additional equation (2-88) is obtained by applying the principle of kinetic moment to the gyroscope by itself. Thus, the general, eight equations of motion for the suspended gyroscope are derived. These equations are in the form of non-linear differential equations of second order; they determine the eight unknown parameters of the system, $\bar{x}_I$, $\bar{y}_I$, $\bar{z}_I$, α, β, γ, ω_1 and $|\vec{S}|$. These parameters are respectively the three co-ordinates of point I, figure 4, the three Euler angles, the angular velocity of the spinner and the tension of the tape.

2.2 Kinetic moment of the carriage with respect to point I:

We have by definition of the absolute kinetic moment:

$$\vec{M}_{C_I} = \sum_{P\in(C)} \vec{IP} * w\vec{V}_P \qquad \text{(Jeudy, 1981)}$$

Where:

$\vec{M}_{C_I}$ denotes the kinetic moment of the carriage with respect to the point I.

P is a current point of the carriage.

w is the mass of an infinitely small element of volume containing the point P.

$\vec{V}_P$ is the absolute velocity of the point P (with respect to the fixed stars).

$*$ is the symbol of the vectorial product.

The expression of $\vec{IP}$ may be written:

(2-1) $\vec{IP} = \vec{IG} + \vec{GP}$

Since:

$\sum w\vec{V}_P = m_C\vec{V}_C$ by definition of the centre of mass C (Goldstein, 1959)

Where:

C is the centre of mass of the carriage.

$\vec{V}_C$ denotes the absolute velocity of C.

m_C is the mass of the carriage.

Inserting the expression (2-1) into the expression for $\vec{M}_{C_I}$ above gives:

(2-1`) $\vec{M}_{C_I} = \vec{IG} * m_C\vec{V}_C + \sum_{P\in(C)} \vec{GP} * w\vec{V}_P$

Now:

(2-2) $\vec{V}_P = \vec{V}_G + \vec{\Gamma}_C * \vec{GP}$ (Jeudy, 1981)

Where:

$\vec{V}_G$ is the absolute velocity of G, the centre of mass of the gyroscope.

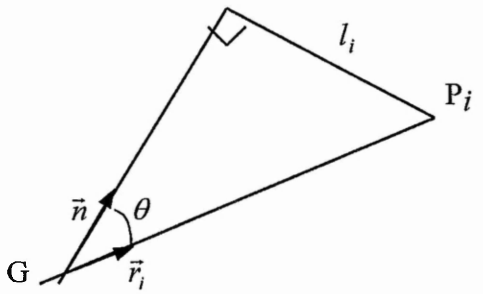
$\vec{\Gamma}_C$ is the absolute angular velocity of the carriage.

Derived below is the inertial moment in a general way:

Let $\vec{r}_i$ to be the position vector of GP$_i$, $\vec{n}$ unit vector, l_i perpendicular distance (figure 6).

<u>**Figure 6**</u>

Inertial moment



The inertial moment can be written by definition:

$$I = \sum_i w_i l_i^2 \qquad \text{(Chester, 1979)}$$

But,

$$l_i^2 = r_i^2 \sin^2\theta = (\vec{n} * \vec{r}_i)^2$$
$$= r_i^2 - r_i^2\cos^2\theta = r_i^2 - (\vec{n}.\vec{r}_i)^2$$

$$I = \sum_i w_i l_i^2 = \sum_i w_i (\vec{n} * \vec{r}_i)^2 = \sum_i w_i \{r_i^2 - (\vec{n}.\vec{r}_i)^2\}$$

If we put $\vec{r}_i$, $\vec{n}$ in their components $\vec{r}_i(x_i, y_i\ z_i)$, $\vec{n}\ (n_x, n_y, n_z)$ where $n_x^2 + n_y^2 + n_z^2 = 1$ are cosines directions.

$$I = \sum_i w_i \{(n_x^2 + n_y^2 + n_z^2)(x_i^2 + y_i^2 + z_i^2) - (n_x x_i + n_y y_i + n_z z_i)^2\}$$

$$I = I_{xx} n_x^2 + I_{yy} n_y^2 + I_{zz} n_z^2 + 2I_{yz} n_y n_z + 2I_{zx} n_z n_x + 2I_{xy} n_x n_y$$

Where:

$$I_{xx} = \sum_i w_i (y_i^2 + z_i^2), \quad I_{yy} = \sum_i w_i (z_i^2 + x_i^2), \quad I_{zz} = \sum_i w_i (x_i^2 + y_i^2)$$

$$I_{yz} = -\sum_i w_i y_i z_i, \quad I_{zx} = -\sum_i w_i z_i x_i, \quad I_{xy} = -\sum_i w_i x_i y_i$$

Since:

$$I_{yz} = I_{zx} = I_{xy} = 0$$

Then:

$$I = I_{xx} n_x^2 + I_{yy} n_y^2 + I_{zz}\, n_z^2$$

In a very similar way this expression can be proved:

$$\sum_{P\in(C)} wG\vec{P} * (\vec{\Gamma}_C * G\vec{P}) = I_{C_G}\vec{\Gamma}_C$$

$$\sum_{P\in(C)} G\vec{P} * w\vec{V}_G = m_C G\vec{C} * \vec{V}_G$$

The above two expressions result from substitution of equation (2-2) into the second term of the right-hand side of equation (2-1`). Then, we get:

(2-3) $\vec{M}_{C_I} = I\vec{G} * m_C\vec{V}_C + m_C G\vec{C} * \vec{V}_G + I_{C_G}\vec{\Gamma}_C$

For reasons of symmetry, I_{C_G} may be written as:

(2-4) $I_{C_G} = \begin{pmatrix} P' & 0 & 0 \\ 0 & Q' & 0 \\ 0 & 0 & R' \end{pmatrix}$ (Jeudy, 1981)

Where:

I_{C_G} is the inertial moment of the carriage with respect to point G in the system G_{xyz}.

P' is the inertial moment of the carriage with respect to the axis G_x.

Q' is the inertial moment of the carriage with respect to the axis G_y.

R' is the inertial moment of the carriage with respect to the axis G_z.

2.3 Kinetic moment of the gyroscope with respect to point I:

We have by definition;

$$\vec{M}_{G_I} = \sum_{P\in(G)} I\vec{P} * w\vec{V}_P \quad \text{(Jeudy, 1981)}$$

Where:

$\vec{M}_{G_I}$ denotes the kinetic moment of the gyroscope with respect to point I.

The other symbols having been explained in an earlier paragraph.

Since:

$$\vec{V}_P = \vec{V}_G + \vec{\Gamma}_G * G\vec{P}$$

Then by a derivation very similar to that in the previous section, that is, replacing C with G in (2-1') and (2-3) $\vec{M}_{G_I}$ may be put in the form:

$$(2\text{-}5)\quad \vec{M}_{G_I} = \vec{IG} * m_G\vec{V}_G + \sum_{P\in(G)} \vec{GP} * w\vec{V}_G + \sum_{P\in(G)} w\vec{GP} * (\vec{\Gamma}_G * \vec{GP})$$

Taking account of the fact that:

$\sum_{P\in(G)} w\vec{GP} = \vec{0}$, by definition of the centre of mass G.

And in a similar way for the derivation of inertial moment, (2-5) becomes:

$$(2\text{-}6)\quad \vec{M}_{G_I} = \vec{IG} * m_G\vec{V}_G + I_G\vec{\Gamma}_G$$

Where:

m_G is the mass of the gyroscope.

$\vec{\Gamma}_G$ is the absolute angular velocity of the gyroscope.

For reasons of symmetry, I_G may be written as:

$$(2\text{-}7)\quad I_G = \begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & Q \end{pmatrix} \qquad \text{(Jeudy, 1981)}$$

Where:

I_G is the inertial moment of the gyroscope with respect to point G in the system G_{xyz}.

P is the inertial moment of the gyroscope with respect to its axis of rotation G_x.

Q is the inertial moment of the gyroscope with respect to any axis perpendicular to G_x and passing through G, for example, G_y and G_z.

2.4 Movement equations:

Use the fact that the derivation with respect to time of kinetic moment of a solid body is a function of the moment of external forces affecting the body. This is proved again here, taking into account that "I" is a movable point. The absolute kinetic moment for the whole system (gyroscope and carriage) may be written as:

$$(2\text{-}8)\quad \vec{M}_I = \sum_{P\in(T)} \vec{IP} * w\vec{V}_P \qquad \text{(Jeudy, 1981)}$$

Where:

T, in equation (2-8), denotes the total system (gyroscope and carriage).

$\vec{M}_I$ is the total kinetic moment of the system, gyroscope and carriage;

$(\vec{M}_I = \vec{M}_{CI} + \vec{M}_{GI})$

By differentiating $\vec{M}_I$ in (2-8) with respect to time:

(2-9) $$\frac{d\vec{M}_I}{dt} = \sum_{P\in(T)} (\vec{V}_P - \vec{V}_I) * w\vec{V}_P + \sum_{P\in(T)} \vec{IP} * w\vec{\psi}_P$$

Where:

$\vec{\psi}_P$ denotes the differentiation of $\vec{V}_P$ with respect to time, $\vec{\psi}_P = \frac{d\vec{V}_P}{dt}$. In a general way, $\vec{\psi}_P$ describes the absolute acceleration of the point, P.

The second term of the right-hand side of (2-9) is none other than the moment of external forces affecting the total system. These forces are the gravitational forces of attraction of the earth and other celestial bodies, the accelerating and decelerating forces applied to the spinner, the tape tension, and the torsion of the tape and the damping forces. If $\vec{M}_E$ denotes the moment of external forces, then, the equation (2-9) becomes:

(2-10) $$\frac{d\vec{M}_I}{dt} = -\vec{V}_I * m\vec{V}_T + \vec{M}_E$$

Where:

$\vec{M}_E$ is the moment of external forces in the total system (gyroscope plus carriage).

m is the mass of the gyroscope and the carriage $(m = m_C + m_G)$.

2.4.1 Calculation of $\frac{d\vec{M}_I}{dt}$:

Differentiating the expression $(\vec{M}_I = \vec{M}_{CI} + \vec{M}_{GI})$ with respect to time gives:

(2-10') $$\frac{d\vec{M}_I}{dt} = \frac{d\vec{M}_{G_I}}{dt} + \frac{d\vec{M}_{C_I}}{dt},$$

Where $\vec{M}_{G_I}$ and $\vec{M}_{C_I}$ are given by (2-6) and (2-3) respectively.

By differentiation (2-3) becomes:

$$\frac{d\vec{M}_{C_I}}{dt} = (\vec{V}_G - \vec{V}_I) * m_C\vec{V}_C + I\vec{G} * m_C\vec{\psi}_C + m_C(\vec{V}_C - \vec{V}_G) * \vec{V}_G$$

$$+ m_C G\vec{C} * \vec{\psi}_G + (\frac{dI_{C_G}}{dt}\vec{\Gamma}_C)_{G_{xyz}} + \vec{\Gamma}_C * (I_{C_G}\vec{\Gamma}_C)$$

The last term is due to the absolute derivation of the vector $\vec{\Gamma}_C$.

Where $(\frac{dI_{C_G}}{dt}\vec{\Gamma}_C)_{G_{xyz}}$ denotes the vector having its components as the derivations of the components of $(I_{C_G}\vec{\Gamma}_C)$ expressed in the system G_{xyz}.

Since $\vec{V}_C * \vec{V}_G = -\vec{V}_G * \vec{V}_C$, the last equation may be written as:

$$\frac{d\vec{M}_{C_I}}{dt} = -\vec{V}_I * m_C\vec{V}_C + I\vec{G} * m_C\vec{\psi}_C + m_C G\vec{C} * \vec{\psi}_G + (\frac{dI_{C_G}}{dt}\vec{\Gamma}_C)_{G_{xyz}} + \vec{\Gamma}_C * (I_{C_G}\vec{\Gamma}_C) \quad (2\text{-}11)$$

By a similar process applied to (2-3), equation (2-6) becomes:

$$\frac{d\vec{M}_{G_I}}{dt} = (\vec{V}_G - \vec{V}_I) * m_G\vec{V}_G + I\vec{G} * m_G\vec{\psi}_G + (\frac{dI_G}{dt}\vec{\Gamma}_G)_{G_{xyz}} + \vec{\Gamma}_C * I_G\vec{\Gamma}_G \quad (2\text{-}12)$$

Since $\vec{V}_G * \vec{V}_G = 0$,

Then (2-12) becomes:

$$\frac{d\vec{M}_{G_I}}{dt} = -\vec{V}_I * m_G\vec{V}_G + I\vec{G} * m_G\vec{\psi}_G + (\frac{dI_G}{dt}\vec{\Gamma}_G)_{G_{xyz}} + \vec{\Gamma}_C * I_G\vec{\Gamma}_G \quad (2\text{-}13)$$

The right-hand side of equations (2-10'), (2-11) and (2-13) are substituted into equation (2-10) to give:

$$-\vec{V}_I * m\vec{V}_T + \vec{M}_E = -\vec{V}_I * (m_C\vec{V}_C + m_G\vec{V}_G) + I\vec{G} * (m_C\vec{\psi}_C + m_G\vec{\psi}_G) + m_C G\vec{C} * \vec{\psi}_G + (\frac{dI_{C_G}}{dt}\vec{\Gamma}_C)_{G_{xyz}} + \vec{\Gamma}_C * (I_{C_G}\vec{\Gamma}_C) + (\frac{dI_G}{dt}\vec{\Gamma}_G)_{G_{xyz}} + \vec{\Gamma}_C * (I_G\vec{\Gamma}_G) \quad (2\text{-}14)$$

Since:

$m\vec{V}_T = m_G\vec{V}_G + m_C\vec{V}_C$ and

$m\vec{\psi}_T = m_G\vec{\psi}_G + m_C\vec{\psi}_C$ (Goldstein, 1959)

So, from (2-14), the basic equation of the moment of external forces may be written as:

$$\vec{M}_E = I\vec{G} * m\vec{\psi}_T + m_C G\vec{C} * \vec{\psi}_G + (\frac{dI_{C_G}}{dt}\vec{\Gamma}_C)_{G_{xyz}} \tag{2-15}$$
$$+\vec{\Gamma}_C * (I_{C_G}\vec{\Gamma}_C) + (\frac{dI_G}{dt}\vec{\Gamma}_G)_{G_{xyz}} + \vec{\Gamma}_C * (I_G\vec{\Gamma}_G)$$

This equation explains that the derivation with respect to time of kinetic moment of a solid body (gyroscope and carriage) is a function of the moment of external forces affecting the body. Below the total kinetic moment is derived directly.

2.4.2 Calculation of the moment of the external forces $\vec{M}_E$:

By definition the moment of the external forces $\vec{M}_E$, which may be written as:

$$\vec{M}_E = \sum_{P\in(T)} I\vec{P} * w(\vec{\beta}_P + \vec{B}_P) + c\vec{k} + c_a\vec{i} - c_d\vec{i}$$

Where:

$\vec{\beta}_P$, $\vec{B}_P$ are respectively the earth's gravitational vector and the gravitational vector due to all other celestial bodies combined.

$\vec{k}$ is the unit vector of the axis G_z.

c is a function of the angles α and β and their derivatives, and thus, $c\vec{k}$ is the moment due to the tape torsion and damping forces.

$\vec{i}$ is the unit vector of the axis G_x.

c_a and c_d are the functions of accelerating and decelerating forces applied to spinner, and thus, $c_a\vec{i}$ and $c_d\vec{i}$ are the moments due to these forces.

As the last two terms in the above equation cancel each other, then $\vec{M}_E$ becomes:

$$\vec{M}_E = \sum_{P\in(T)} I\vec{P} * w(\vec{\beta}_P + \vec{B}_P) + c\vec{k} \tag{2-16}$$

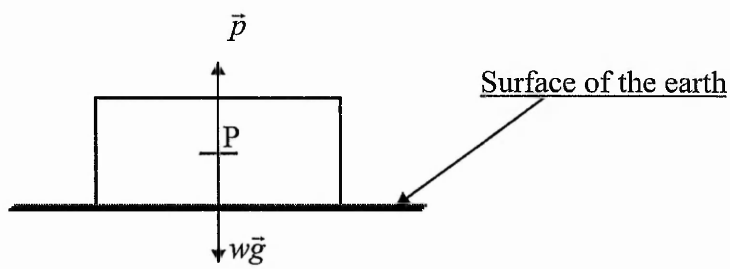
Now find the relationship between $\vec{g}$, the gravitational acceleration and $(\vec{\beta}_P + \vec{B}_P)$, the gravitational forces of attraction.

2.4.2.1 Relationship between $\vec{g}$ and $(\vec{\beta}_P + \vec{B}_P)$:

Consider a solid body of mass (w) that rests on the surface of the earth. This body is under the influence of its own weight ($w\vec{g}$) and the reacting pressure of the surface of the earth ($\vec{p}$). In a non-dynamic system these forces are in equilibrium, we have: $\vec{p} + w\vec{g} = \vec{0}$ (figure 7).

Figure 7

A body resting on the surface of the earth



P: Centre of gravity of the body being considered

If, this body is imagined to be in motion with respect to the surface of the earth, and $\vec{V}_g$ is the velocity of its centre of gravity, P, with respect to the surface of the earth. The absolute velocity of P is thus given by:

(2-17) $\vec{V}_P = \vec{V}_O + \vec{\varpi} * \vec{OP} + \vec{V}_g$ (Jeudy, 1981)

Where:

O is the centre of gravity of the earth.

Differentiating (2-17) with respect to time:

(2-18) $\vec{\psi}_P = \vec{\psi}_O + \frac{d\vec{\varpi}}{dt} * \vec{OP} + \vec{\varpi} * (\vec{V}_P - \vec{V}_O) + \vec{\psi}_g + \vec{\varpi} * \vec{V}_g$,

The last term of (2-18) is due to the fact that $\vec{\psi}_g$ is only the relative partial derivative of $\vec{V}_g$ and we want the absolute derivative of $\vec{V}_g$, which is: $\vec{\psi}_g + \vec{\varpi} * \vec{V}_g$.

Substituting $\vec{V}_P$ from (2-17) and assuming that the earth rotates with constant angular velocity and therefore $\frac{d\vec{\varpi}}{dt} = 0$. The earth rotates once every sidereal day $\varpi = \frac{2\pi}{86164} \approx 0.000073\ \frac{\text{radians}}{\text{sec.}}$, (2-18) becomes:

(2-19) $\vec{\psi}_P = \vec{\psi}_O + \vec{\varpi} * (\vec{\varpi} * \vec{OP}) + \vec{\psi}_g + 2\vec{\varpi} * \vec{V}_g$,

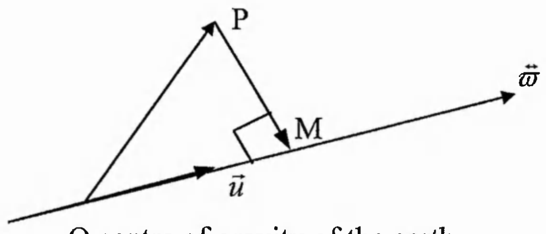
The double vectorial product is changed according to the formula:

(2-20) $\vec{\varpi} * (\vec{\varpi} * \vec{OP}) = (\vec{\varpi}.\vec{OP})\vec{\varpi} - (\vec{\varpi})^2 \vec{OP}$ (Chester, 1979)

Where $(\vec{\varpi}.\vec{OP})$ is the scalar product of $\vec{\varpi}$ and $\vec{OP}$

Figure 8

Reduction of the double vectorial product



With the unit vector $\vec{u} = \frac{\vec{\varpi}}{\varpi}$, the relation (2-20) becomes (see figure 8)

$\vec{\varpi} * (\vec{\varpi} * \vec{OP}) = \varpi^2[(\vec{u}.\vec{OP})\vec{u} - \vec{OP}] = \varpi^2[\vec{OM} - \vec{OP}]$

(2-21) $\vec{\varpi} * (\vec{\varpi} * \vec{OP}) = -\varpi^2 \vec{MP}$

This is a well-known formula, in classical mechanics, expressing the centrifugal acceleration perpendicular to the instantaneous vector of rotation. Although a particular case of a solid body in contact with the ground (figure 7) was considered, the formulae established in this section, in particular the formula (2-24), has a general character.

Having taken account of (2-21), (2-19) becomes:

(2-22) $\vec{\psi}_P = \vec{\psi}_O + \vec{\psi}_g + 2\vec{\varpi} * \vec{V}_g - \vec{MP}|\vec{\varpi}|^2$

Returning to the case when the body is in equilibrium under the forces acting upon it. In this case, $\vec{V}_g$ and $\vec{\psi}_g$ are zero. Consequently, formula (2-22) becomes:

$$\vec{\psi}_P = \vec{\psi}_O - \vec{MP}|\vec{\varpi}|^2$$

Taking account of one of the principles of mechanics that $w\vec{\psi}_P$ is equal to the sum of the forces acting on the body being considered, therefore:

$$\vec{p} + w(\vec{\beta}_P + \vec{\mathrm{B}}_P) = w\vec{\psi}_P$$

(2-23) $\vec{p} + w(\vec{\beta}_P + \vec{\mathrm{B}}_P) = w\vec{\psi}_O - w\vec{MP}|\vec{\varpi}|^2$

Since $\vec{p} + w\vec{g} = \vec{0}$, for a non-accelerating body, (2-23) becomes:

(2-24) $\vec{g} = \vec{\beta}_P + \vec{\mathrm{B}}_P - \vec{\psi}_O + \vec{MP}|\vec{\varpi}|^2$

Having taken into account of (2-24), $\vec{\vec{M}}_E$ in the previous equation (2-16) becomes:

(2-25) $\vec{\vec{M}}_E = \sum_{P \in (T)} \vec{IP} * w(\vec{g} + \vec{\psi}_O - \vec{MP}|\vec{\varpi}|^2) + c\vec{k}$

Suppose $\vec{g}$ is invariable within the volume occupied by all the possible positions of the total system, (2-25) becomes:

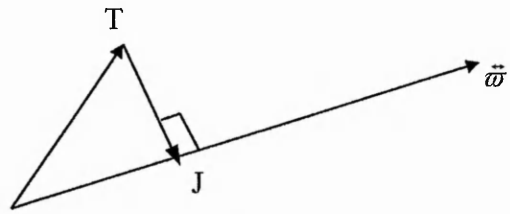
(2-26) $\vec{\vec{M}}_E = m\vec{IT} * \vec{g} + m\vec{IT} * \vec{\psi}_O - \sum_{P \in (T)} \vec{IP} * wM\vec{P}|\vec{\varpi}|^2 + c\vec{k}$

If $\vec{MP}$ is considered practically constant for all points of the total system and it is equal to $\vec{JT}$ where J is the orthogonal projection of T on $\vec{\varpi}$ passing O, see figure 9, M and J are almost co-located on the earth's rotation axis, then:

(2-27) $\sum_{P \in (T)} \vec{IP} * w\vec{MP}|\vec{\varpi}|^2 \approx m\vec{IT} * \vec{JT}|\vec{\varpi}|^2$

Figure 9

J orthogonal projection of T on $\vec{\varpi}$



O centre of gravity of the earth

Taking into account (2-27), (2-26) becomes:

(2-27′) $\vec{M}_E = m\vec{IT} * \vec{g} + m(\vec{IG} + \vec{GT}) * \vec{\psi}_O + m\vec{IT} * \vec{TJ}|\vec{\varpi}|^2 + c\vec{k}$

Because, $\vec{IT} = \vec{IG} + \vec{GT}$ and $\vec{JT} = -\vec{TJ}$

By equalling the right-hand sides of equations (2-15) and (2-27′), we get:

(2-28)

$$m\vec{IT} * \vec{g} + m\vec{IG} * (\vec{\psi}_O - \vec{\psi}_T) + (m\vec{GT} * \vec{\psi}_O - m_C\vec{GC} * \vec{\psi}_G)$$
$$+ m\vec{IT} * \vec{TJ}|\vec{\varpi}|^2 + c\vec{k} = (\frac{d(I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G)}{dt})_{G_{xyz}} + \vec{\Gamma}_C * (I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G)$$

The third term of the left-hand side of (2-28) could be simplified since:

(2-28′) $$m = m_C + m_G$$ $$\vec{GT} = \vec{GC} + \vec{CT}$$

Then the third term of the left-hand side of (2-28) may be written as:

(2-29) $$(m_C + m_G)(\vec{GC} + \vec{CT}) * \vec{\psi}_O - m_C\vec{GC} * \vec{\psi}_G =$$ $$m_C\vec{GC} * (\vec{\psi}_O - \vec{\psi}_G) + (m_C\vec{CT} + m_G\vec{GT}) * \vec{\psi}_O$$

The second term of the right-hand side of (2-29) is zero, because by definition, T is the centre of gravity of the total system:

$m_C\vec{CT} + m_G\vec{GT} = \vec{0}$

Finally, substitute (2-29) into (2-28) to get:

(2-30) $$m\vec{IT} * \vec{g} + m\vec{IG} * (\vec{\psi}_O - \vec{\psi}_T) + m_C\vec{GC} * (\vec{\psi}_O - \vec{\psi}_G) + m\vec{IT} * \vec{TJ}|\vec{\varpi}|^2 + c\vec{k}$$ $$= (\frac{d(I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G)}{dt})_{G_{xyz}} + \vec{\Gamma}_C * (I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G)$$

This equation so far, represents the three Euler equations. Next, the terms of equation (2-30) are evaluated in detail.

2.4.2.2 Evaluation of the term $m\vec{IG} * (\vec{\psi}_O - \vec{\psi}_T)$:

First calculate $(\vec{\psi}_O - \vec{\psi}_T)$:

(2-31) $\vec{V}_T = \vec{V}_I + (\vec{\varpi} + \vec{\Gamma}) * \vec{IT}$ (Jeudy, 1981)

(2-32) $\vec{V}_I = \vec{V}_A + \vec{\varpi} * \vec{AI} + \vec{V}_{I_O}$

Where:

$\vec{\Gamma}$ is the angular velocity of the system G_{xyz} with respect to the system $G_{X'Y'Z'}$ because the axes of $G_{X'Y'Z'}$ and $O_{\overline{xyz}}$ are parallel.

$\vec{V}_{I_O}$ is the velocity of point I with respect to the system $O_{\overline{xyz}}$, but:

(2-33) $\vec{V}_A = \vec{V}_O + \vec{\varpi} * \vec{OA}$

Where:

O denotes the centre of gravity of the earth.

Substitute (2-33) into (2-32) and then everything into (2-31):

(2-34) $\vec{V}_T = \vec{V}_O + \vec{V}_{I_O} + \vec{\varpi} * \vec{OT} + \vec{\Gamma} * \vec{IT}$

Because, $\vec{\varpi} * (\vec{OA} + \vec{AI}) = \vec{\varpi} * \vec{OI}$ and $\vec{\varpi} * (\vec{OI} + \vec{IT}) = \vec{\varpi} * \vec{OT}$

Differentiating (2-34) with respect to time:

(2-35)
$$\vec{\psi}_T = \vec{\psi}_O + \vec{\varpi} * (\vec{V}_T - \vec{V}_O) + \vec{\psi}_{I_O} + \vec{\varpi} * \vec{V}_{I_O} + (\frac{d\vec{\Gamma}}{dt})_{abs} * \vec{IT} + \vec{\Gamma} * (\vec{V}_T - \vec{V}_I)$$

Where the term, $(\vec{\psi}_{I_O} + \vec{\varpi} * \vec{V}_{I_O})$ is the absolute derivative of vector $\vec{V}_{I_O}$ and $(\frac{d\vec{\Gamma}}{dt})_{abs}$ denotes the absolute derivative of vector $\vec{\Gamma}$. It is assumed that: $(\frac{d\vec{\varpi}}{dt}) = 0$.

Replace $(\vec{V}_T - \vec{V}_O)$ and $(\vec{V}_T - \vec{V}_I)$ in (2-35) by their values obtained from (2-34) and (2-31) respectively, after taking into account that:

(2-36) $(\frac{d\vec{\Gamma}}{dt})_{abs} = (\frac{d\vec{\Gamma}}{dt})_{G_{xyz}} + \vec{\varpi} * \vec{\Gamma}$

Because $(\vec{\varpi}+\vec{\Gamma})*\vec{\Gamma}=\vec{\varpi}*\vec{\Gamma}$. Then, (2-35) becomes:

(2-37)

$$\vec{\psi}_T-\vec{\psi}_O=\vec{\varpi}*(\vec{\varpi}*\vec{OT})+(\frac{d\vec{\Gamma}}{dt})_{G_{xyz}}*\vec{IT}+2\vec{\varpi}*\vec{V}_{I_O}+\vec{\psi}_{I_O}+\vec{\varpi}*(\vec{\Gamma}*\vec{IT})$$
$$+(\vec{\varpi}*\vec{\Gamma})*\vec{IT}+\vec{\Gamma}*[(\vec{\varpi}+\vec{\Gamma})*\vec{IT}]$$

The first term of equation (2-37) has already been calculated for $T=P$ in the formula (2-21) (see figure 9), where:

(2-38) $\vec{\varpi}*(\vec{\varpi}*\vec{OT})=-\vec{JT}|\vec{\varpi}|^2$

The double vectorial products are changed with the formula:

(2-39) $\vec{A}*(\vec{B}*\vec{C})=(\vec{A}.\vec{C})\vec{B}-(\vec{A}.\vec{B})\vec{C}$ (Chester, 1979)

Where $\vec{A},\vec{B}$, and $\vec{C}$ are any three vectors.

Finally (2-37) becomes:

(2-40)
$$\vec{\psi}_T-\vec{\psi}_O=\vec{TJ}|\vec{\varpi}|^2+2\vec{\varpi}*\vec{V}_{I_O}+\vec{\psi}_{I_O}+(\vec{\varpi}.\vec{IT})\vec{\Gamma}-(\vec{\varpi}.\vec{\Gamma})\vec{IT}$$
$$+(\frac{d\vec{\Gamma}}{dt})_{G_{xyz}}*\vec{IT}+(\vec{IT}.\vec{\varpi})\vec{\Gamma}-(\vec{IT}.\vec{\Gamma})\vec{\varpi}$$
$$+(\vec{\Gamma}.\vec{IT})\vec{\varpi}-(\vec{\Gamma}.\vec{\varpi})\vec{IT}+(\vec{\Gamma}.\vec{IT})\vec{\Gamma}-(\vec{\Gamma})^2\vec{IT}$$

After reductions and simplifications (2-40) becomes:

(2-41)
$$\vec{\psi}_T-\vec{\psi}_O=\vec{TJ}|\vec{\varpi}|^2+2\vec{\varpi}*\vec{V}_{I_O}+\vec{\psi}_{I_O}$$
$$+[\vec{IT}.(2\vec{\varpi}+\vec{\Gamma})]\vec{\Gamma}-[\vec{\Gamma}.(2\vec{\varpi}+\vec{\Gamma})]\vec{IT}+(\frac{d\vec{\Gamma}}{dt})_{G_{xyz}}*\vec{IT}$$

To obtain a more detailed expression of (2-41) we introduce the components of $\vec{\varpi}$, $\vec{\Gamma}$ and $\vec{IT}$ into the system G_{xyz}:

(2-41`) $$\vec{\varpi}=\begin{pmatrix}D\\E\\F\end{pmatrix}_{G_{xyz}},\quad \vec{\Gamma}=\begin{pmatrix}d\\e\\f\end{pmatrix}_{G_{xyz}}\quad \text{and}\quad \vec{IT}=\begin{pmatrix}0\\0\\-l\end{pmatrix}_{G_{xyz}}\qquad (l>0)$$

Where:

D, E and F are components of $\vec{\varpi}$ in G_{xyz}.

d, e and f are components of $\vec{\Gamma}$ in G_{xyz}.

l is a distance from I to T $(l=IT)$; $l>0$

The vectorial and scalar products of (2-41) are related by these formulae (Stroud, 1995)

$$\vec{A} * \vec{B} = (A_x\vec{i} + A_y\vec{j} + A_z\vec{k}) * (B_x\vec{i} + B_y\vec{j} + B_z\vec{k})$$

$$\vec{A} * \vec{B} = (A_yB_z - A_zB_y)\vec{i} + (A_zB_x - A_xB_z)\vec{j} + (A_xB_y - A_yB_x)\vec{k}$$

and:

$$\vec{A}.\vec{B} = (A_x\vec{i} + A_y\vec{j} + A_z\vec{k}).(B_x\vec{i} + B_y\vec{j} + B_z\vec{k})$$

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Where $\vec{A}$ and $\vec{B}$ are any two vectors.

The following expressions for the last three terms of (2-41) may be written:

(2-42) $$[\vec{IT}.(2\vec{\varpi} + \vec{\Gamma})]\vec{\Gamma} = -l(2F + f)\begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

(2-43) $$-[\vec{\Gamma}.(2\vec{\varpi} + \vec{\Gamma})]\vec{IT} = \begin{pmatrix} 0 \\ 0 \\ l[d(2D+d) + e(2E+e) + f(2F+f)] \end{pmatrix}$$

(2-44) $$\left(\frac{d\vec{\Gamma}}{dt}\right)_{G_{xyz}} * \vec{IT} = \begin{pmatrix} -\dot{e}l \\ \dot{d}l \\ 0 \end{pmatrix}$$

Where:

$\dot{d}$ and $\dot{e}$ denote the derivatives of d and e with respect to time.

Then, having taken account of expressions (2-42), (2-43) and (2-44), equation (2-41) becomes:

(2-45) $$\vec{\psi}_T - \vec{\psi}_O = T\vec{J}|\vec{\varpi}|^2 + 2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O} + l\begin{pmatrix} -d(2F+f) - \dot{e} \\ -e(2F+f) + \dot{d} \\ d(2D+d) + e(2E+e) \end{pmatrix}$$

Also the expression $(\vec{\psi}_G - \vec{\psi}_O)$ is obtained from (2-45) by changing T to G, l to l_G and J to H where H denotes the orthogonal projection of G on the vector $\vec{\varpi}$ passing through O the centre of the earth (see figure 10).

$$\text{(2-46)}\quad \vec{\psi}_G - \vec{\psi}_O = \vec{GH}|\vec{\varpi}|^2 + 2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O} + l_G \begin{pmatrix} -d(2F+f) - \dot{e} \\ -e(2F+f) + \dot{d} \\ d(2D+d) + e(2E+e) \end{pmatrix}$$

Where l_G is the distance from I to G $(l_G = IG)$

Taking into account of (2-45) and (2-46), the basic equation (2-30) can thus be written:

$$m\vec{IT} * \vec{g} - m\vec{IG} * \vec{TJ}|\vec{\varpi}|^2 - (m\vec{IG} + m_C\vec{GC}) * (2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O})$$

$$\text{(2-47)}\quad -(ml\vec{IG} + m_C l_G \vec{GC}) * \begin{pmatrix} -d(2F+f) - \dot{e} \\ -e(2F+f) + \dot{d} \\ d(2D+d) + e(2E+e) \end{pmatrix} - m_C\vec{GC} * \vec{GH}|\vec{\varpi}|^2$$

$$+ m\vec{IT} * \vec{TJ}|\vec{\varpi}|^2 + c\vec{k} = \left(\frac{d(I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G)}{dt}\right)_{Gxyz} + \vec{\Gamma}_C * (I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G)$$

The three terms of (2-47) containing $|\vec{\varpi}|^2$ are equal to:

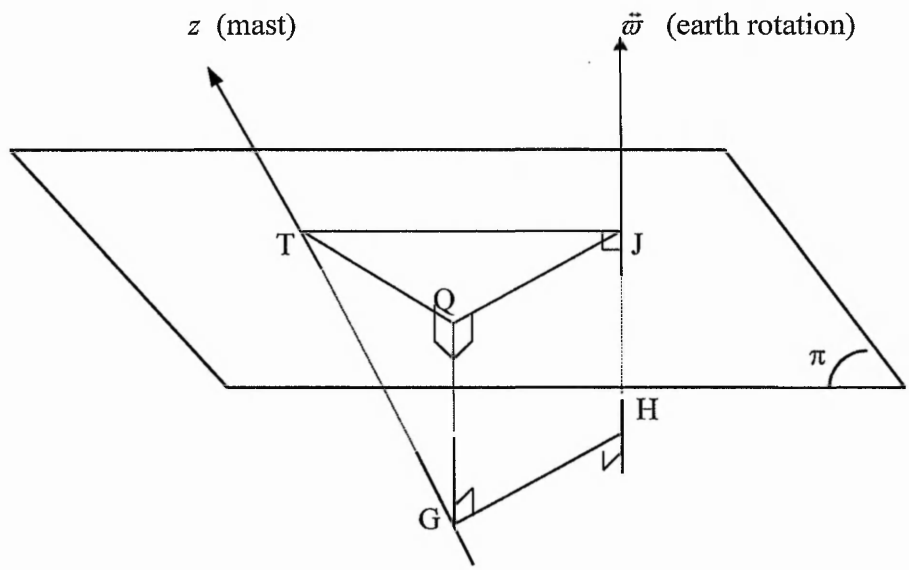
(2-48)

$$\begin{aligned}(m\vec{GT} * \vec{TJ} - m_C\vec{GC} * \vec{GH})|\vec{\varpi}|^2 &= [(m_C + m_G)(\vec{GC} + \vec{CT}) * \vec{TJ} - m_C\vec{GC} * \vec{GH}]|\vec{\varpi}|^2 \\ &= [m_C\vec{GC} * (\vec{TJ} - \vec{GH}) + (m_C\vec{CT} + m_G\vec{GT}) * \vec{TJ}]|\vec{\varpi}|^2 \\ &= m_C\vec{GC} * (\vec{TJ} - \vec{GH})|\vec{\varpi}|^2\end{aligned}$$

Because $m_C\vec{CT} + m_G\vec{GT} = \vec{0}$ by the definition of T, the centre of gravity of the whole system and $m(\vec{IT} - \vec{IG}) = m\vec{GT}$.

Figure 10

Reduction of the term containing $|\vec{\varpi}|^2$



The plane (π) is perpendicular to $\vec{\varpi}$ and passes through T, and Q is the orthogonal projection of G on (π) then, $\vec{GH} = \vec{QJ}$.

Taking account of the above explanation of figure 10:

$\vec{TJ} - \vec{GH} = \vec{TJ} - \vec{QJ} = \vec{TQ}$

The three terms of (2-47) containing $|\vec{\varpi}|^2$ are thus equal to:

(2-49) $(mI\vec{T} * \vec{TJ} - mI\vec{G} * \vec{TJ} - m_C \vec{GC} * \vec{GH})|\vec{\varpi}|^2 = m_C \vec{GC} * \vec{TQ}|\vec{\varpi}|^2$

From (2-47) the coefficient of $2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O}$ is:

$$\begin{aligned} -(mI\vec{G} + m_C\vec{GC}) &= -[(m_C + m_G)(\vec{IC} + \vec{CG}) + m_C\vec{GC}] \\ &= -[m_C\vec{IC} + m_G\vec{IG}] \\ &= -mI\vec{T} \end{aligned} \quad \text{(2-50)}$$

Because $ml = m_C l_C + m_G l_G$ by definition of T.

Finally, the expression $mlI\vec{G} + m_C l_G \vec{GC}$ in (2-47) becomes:

$$ml\vec{IG} + m_C l_G \vec{GC} = ml\begin{pmatrix}0\\0\\-l_G\end{pmatrix} + m_C l_G\begin{pmatrix}0\\0\\l_G - l_C\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\m_C l_G^2 - m_C l_G l_C - ml l_G\end{pmatrix} = -\begin{pmatrix}0\\0\\L\end{pmatrix}$$

Where l_C is a distance from I to C $(l_C = IC)$ and l_G is from I to G $(l_G = IG)$

L is the constant defined by:

(2-51) $L = -l_G(m_C l_G - m_C l_C - ml)$

But, $ml = m_C l_C + m_G l_G$, consequently the expression of L becomes:

(2-52) $L = -l_G[l_G(m_C - m_G) - 2m_C l_C]$

(2-53) $L = m_G l_G^2 + m_C l_C^2 - m_C(l_G - l_C)^2$

The term of (2-47) containing $(ml\vec{IG} + m_C l_G \vec{GC})$ will be:

(2-54) $$\begin{pmatrix}0\\0\\L\end{pmatrix} * \begin{pmatrix}-d(2F+f) - \dot{e}\\ -e(2F+f) + \dot{d}\\ d(2D+d) + e(2E+e)\end{pmatrix} = \begin{pmatrix}L(e(2F+f) - \dot{d})\\ -L(d(2F+f) + \dot{e})\\ 0\end{pmatrix}$$

Taking account of (2-47), (2-49), (2-50) and (2-54), the equation (2-30) becomes:

$$c\vec{k} + m\begin{pmatrix}0\\0\\-l\end{pmatrix} * \begin{pmatrix}g_x\\g_y\\g_z\end{pmatrix} + m_C\begin{pmatrix}0\\0\\l_G - l_C\end{pmatrix} * \begin{pmatrix}x_Q\\y_Q\\z_Q - (l_G - l)\end{pmatrix}\left|\vec{\varpi}\right|^2$$

(2-55) $$+ m\begin{pmatrix}0\\0\\l\end{pmatrix} * (2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O}) + \begin{pmatrix}L(e(2F+f) - \dot{d})\\ -L(d(2F+f) + \dot{e})\\ 0\end{pmatrix} =$$

$$\left(\frac{d(I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G)}{dt}\right)_{G_{xyz}} + \vec{\Gamma}_C * (I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G)$$

Where g_x, g_y and g_z are the co-ordinates of $\vec{g}$ in G_{xyz} and x_Q, y_Q and z_Q are the co-ordinates of Q in G_{xyz}.

2.4.2.3 Calculation of the Co-ordinates of point Q:

The point Q belongs to the plane (π) (see figure 10). Its equation is of the form:

$$xD + yE + [z - (l_G - l)]F = 0$$

This is the group of points verifying the formula: $\vec{PT}.\vec{\varpi} = 0$

Suppose r is an unknown to be determined, the co-ordinates of Q are in the form:

$x_Q = rD, \quad y_Q = rE$ and $z_Q = rF$

Substituting these co-ordinates into the equation of the plane (π):

$$rD^2 + rE^2 + [rF^2 - (l_G - l)F] = 0$$

$$r = \frac{F(l_G - l)}{(D^2 + E^2 + F^2)} = \frac{F(l_G - l)}{|\vec{\varpi}|^2}$$

Where *D*, *E* and *F* are components of the vector $\vec{\varpi}$ in the system G_{xyz}.

The co-ordinates of point Q are:

(2-56) $x_Q = \dfrac{DF(l_G - l)}{|\vec{\varpi}|^2}, \quad y_Q = \dfrac{EF(l_G - l)}{|\vec{\varpi}|^2}$ and $z_Q = \dfrac{F^2(l_G - l)}{|\vec{\varpi}|^2}$

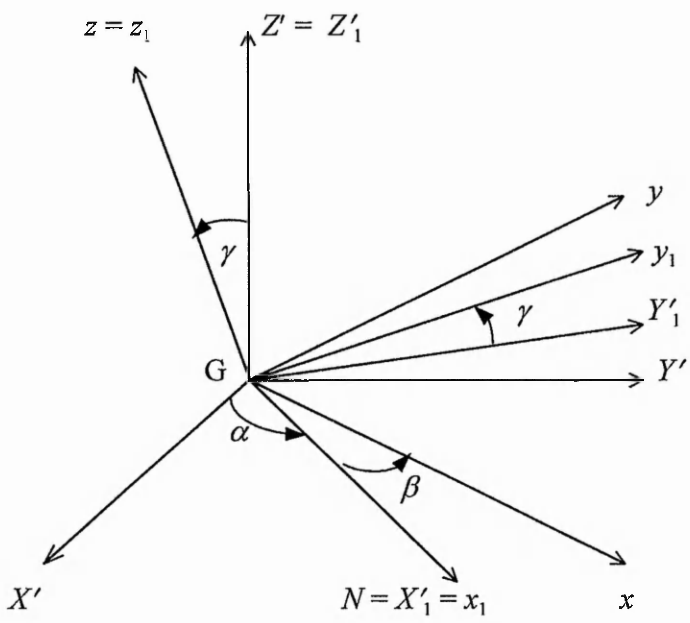
Now derive the transformation matrix between the two systems $O_{\overline{xyz}}$ and G_{xyz}.

2.4.2.4 Transformation between the two systems $O_{\overline{xyz}}$ and G_{xyz}:

The point G (figure 11), as defined in Chapter 1, is the centre of gravity of the gyroscope. It is also, the origin of the two co-ordinate systems G_{xyz} fixed in the rigid body and $G_{X'Y'Z'}$ fixed in space. Both co-ordinate systems G_{xyz} and $G_{X'Y'Z'}$ are right-handed and the planes G_{xy} and $G_{X'Y'}$ intersect along the line G_N which is perpendicular to the plane through the axes $G_{Z'}$ and G_z. Choose the orientation along G_N in such a way that the system $G_{NZ'z}$ is right-handed. The three angles, *Z'Gz*, *X'GN* and *NGx* are known as Euler angles which are defined by Leimanis (Leimanis, 1965). In this project, Euler angles are denoted by γ, α and β respectively. The angle γ lies in the range, $0 \leq \gamma < \pi$. The angle α ($0 \leq \alpha < 2\pi$) is called the angle of precession and the angle β ($0 \leq \beta < 2\pi$) the angle of proper rotation. If these angles γ, α and β are known as functions of the time t, then the position of the system G_{xyz} with respect to the system $G_{X'Y'Z'}$ is defined.

Figure 11

Euler angles α, β and γ



Among the nine possible rotations from the axes of the system G_{xyz} to the axes of the system $G_{X'Y'Z'}$ there are only three independent rotations. They are the following successive counter-clockwise rotations:

(a) R_1 ($G_{X'_1Y'_1Z'_1}$) is a rotation through the angle α about the $G_{Z'}$ – axis. $G_{X'_1}$ – axis coincides with the G_N axis.

(b) R_2 $(G_{x_1 y_1 z_1})$ is a rotation through the angle γ about the $G_{X_1'}$ – axis. G_{y_1} – axis lies in the plane $G_{Z'z_1}$ and makes the angle γ with $G_{Y_1'}$ – axis.

(c) R_3 (G_{xyz}) is a rotation through the angle β about G_{z_1} – axis until G_{x_1} – axis coincides with G_x – axis and G_{y_1} – axis with G_y – axis.

The transformation from co-ordinates fixed in the rigid body, x, y, z to space co-ordinates X', Y', Z' may be written as:

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Where $R = (R_1)(R_2)(R_3)$

The above matrix equation may be written as these formulas:

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1' \\ Y_1' \\ Z_1' \end{pmatrix}$$

$$\begin{pmatrix} X_1' \\ Y_1' \\ Z_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

And the matrix R may be written as this formula:

$$R = \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha\cos\gamma & -\sin\beta\cos\alpha - \cos\beta\sin\alpha\cos\gamma & \sin\alpha\sin\gamma \\ \cos\beta\sin\alpha + \sin\beta\cos\alpha\cos\gamma & -\sin\beta\sin\alpha + \cos\beta\cos\alpha\cos\gamma & -\cos\alpha\sin\gamma \\ \sin\gamma\sin\beta & \sin\gamma\cos\beta & \cos\gamma \end{pmatrix}$$

The inverse transformation from space co-ordinates X', Y' and Z' to the rigid body co-ordinates x, y and z may be written as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R^{-1} \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}$$

In a very similar way, ignoring the shift of origin between the two systems $O_{\overline{xyz}}$ and G_{xyz}, and supposing B is the matrix that transforms the co-ordinates $\bar{x}$, $\bar{y}$ and $\bar{z}$ of the system $O_{\overline{xyz}}$ into the co-ordinates x, y and z of the system G_{xyz}:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = B \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}$$

B has the following expression:

$B = R^{-1}$

(2-57)

$$B = \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha\cos\gamma & \cos\beta\sin\alpha + \sin\beta\cos\alpha\cos\gamma & \sin\beta\sin\gamma \\ -\sin\beta\cos\alpha - \cos\beta\sin\alpha\cos\gamma & -\sin\beta\sin\alpha + \cos\beta\cos\alpha\cos\gamma & \cos\beta\sin\gamma \\ \sin\gamma\sin\alpha & -\sin\gamma\cos\alpha & \cos\gamma \end{pmatrix}$$

Since $\vec{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$ in $O_{\overline{xyz}}$, the components of $\vec{g}$ in G_{xyz} are:

$$\text{(2-58)} \quad \begin{aligned} g_x &= -g\sin\beta\sin\gamma \\ g_y &= -g\cos\beta\sin\gamma \\ g_z &= -g\cos\gamma \end{aligned}$$

Let us calculate the vectorial products in the equation (2-55). Firstly, $(2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O})$ is evaluated in terms of components. If $\bar{x}_I$, $\bar{y}_I$ and $\bar{z}_I$ are the co-ordinates of I, in the system $O_{\overline{xyz}}$, ϕ is the latitude, the components of $\vec{\varpi}$ in $O_{\overline{xyz}}$ are:

$$\text{(2-59)} \quad \vec{\varpi} = \begin{pmatrix} \varpi\cos\phi \\ 0 \\ \varpi\sin\phi \end{pmatrix}$$

The co-ordinates of $(2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O})$ in $O_{\overline{xyz}}$ are:

$$\text{(2-60)} \quad 2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O} = 2\begin{pmatrix} \varpi\cos\phi \\ 0 \\ \varpi\sin\phi \end{pmatrix} * \begin{pmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{z}_I \end{pmatrix} + \begin{pmatrix} \ddot{x}_I \\ \ddot{y}_I \\ \ddot{z}_I \end{pmatrix}$$

Where the dot denotes the first derivative with respect to time and the double dots, the second derivative with respect to time. (2-60) becomes:

$$\text{(2-61)} \quad 2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O} = \begin{pmatrix} -2\varpi\dot{y}_I\sin\phi + \ddot{x}_I \\ 2\varpi(\dot{x}_I\sin\phi - \dot{z}_I\cos\phi) + \ddot{y}_I \\ 2\varpi\dot{y}_I\cos\phi + \ddot{z}_I \end{pmatrix}_{O_{\overline{xyz}}}$$

By pre-multiplying (2-61) by the matrix B from (2-57) we can obtain the components of ($2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O}$) in the system G_{xyz}, which are λ_x, λ_y and λ_z. Having taken account of these symbols and of equations (2-56), the left-hand side of (2-55) becomes:

$$\text{(2-62)} \quad \begin{pmatrix} m\lg_y - m_C EF(l_G - l)(l_G - l_C) - ml\lambda_y + L[e(2F+f) - \dot{d}] \\ -m\lg_x + m_C DF(l_G - l)(l_G - l_C) + ml\lambda_x - L[d(2F+f) + \dot{e}] \\ 0+0+0+0+c \end{pmatrix}$$

Where c is a function of α and β and their derivatives.

2.4.2.5 calculation of the right-hand side of the equation (2-55):

The components of the angular velocities $\vec{\Gamma}_C$, $\vec{\Gamma}_G$ and $\vec{\omega}$ are expressed in the system G_{xyz}

$$\vec{\Gamma}_C = \vec{\varpi} + \vec{\Gamma} = \begin{pmatrix} D+d \\ E+e \\ F+f \end{pmatrix}_{G_{xyz}} \quad \text{And } \vec{\Gamma}_G = \vec{\Gamma}_C + \vec{\omega}$$

Where, $\vec{\omega} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}_{G_{xyz}}$, see figure 5.

$\vec{\omega}$ is the angular velocity of the gyroscope with respect to the carriage. ω has the following values:

$\omega = \omega_0$ when $t = t_0$

and:

$\omega = \omega_1$ when $t = t_1$

Let, $t = t_1$

$$\vec{\Gamma}_G = \begin{pmatrix} D + d + \omega_1 \\ E + e \\ F + f \end{pmatrix}$$

Consequently, see equations (2-4) and (2-7):

$$I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G = \begin{pmatrix} P' & 0 & 0 \\ 0 & Q' & 0 \\ 0 & 0 & R' \end{pmatrix}\begin{pmatrix} D+d \\ E+e \\ F+f \end{pmatrix} + \begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & Q \end{pmatrix}\begin{pmatrix} D + d + \omega_1 \\ E + e \\ F + f \end{pmatrix}$$

which gives, having created the products:

$$\text{(2-63)} \quad I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G = \begin{pmatrix} (P + P')(D + d) + \omega_1 P \\ (Q + Q')(E + e) \\ (R' + Q)(F + f) \end{pmatrix}$$

The last term $\vec{\Gamma}_C * (I_{C_G}\vec{\Gamma}_C + I_G\vec{\Gamma}_G)$ of equation (2-55) has the value:

(2-64)

$$\begin{pmatrix} D+d \\ E+e \\ F+f \end{pmatrix} * \begin{pmatrix} (P + P')(D + d) + \omega_1 P \\ (Q + Q')(E + e) \\ (R' + Q)(F + f) \end{pmatrix} = \begin{pmatrix} (R' - Q')(E + e)(F + f) \\ (P + P' - R' - Q)(D + d)(F + f) + \omega_1 P(F + f) \\ (Q + Q' - P' - P)(D + d)(E + e) - \omega_1 P(E + e) \end{pmatrix}$$

Differentiate (2-63) and add it to (2-64) and then equalise the result to get (2-62) because it is equal to the right-hand side of (2-55). Three movement equations with seven unknowns, which are functions of time $\bar{x}_I, \bar{y}_I, \bar{z}_I, \alpha, \beta, \gamma$ and ω_1 are obtained. The first three equations are written below.

The first one is:

(2-65)
$$m\lg_y - m_C EF(l_G - l)(l_G - l_C) - ml\lambda_y + L[e(2F+f) - \dot{d}]$$
$$= (P+P')(\dot{D}+\dot{d}) + \dot{\omega}_1 P + (R'-Q')(E+e)(F+f)$$

The second equation is:

(2-66)
$$-m\lg_x + m_C DF(l_G - l)(l_G - l_C) + ml\lambda_x - L[d(2F+f) + \dot{e}]$$
$$= (Q+Q')(\dot{E}+\dot{e}) + (P+P'-R'-Q)(D+d)(F+f) + \omega_1 P(F+f)$$

The third equation is:

(2-67)

$$c(\alpha,\beta,\dot{\alpha},\dot{\beta}) = (R'+Q)(\dot{F}+\dot{f}) + (Q+Q'-P'-P)(D+d)(E+e) - \omega_1 P(E+e)$$

The above three equations are called also the Euler equations. A fourth equation is obtained because $\bar{x}_I$, $\bar{y}_I$ and $\bar{z}_I$ are not independent variables, the length ℓ of the tape is constant. We have the equation of a sphere with centre at A and radius ℓ:

(2-68) $\bar{x}_I^{\,2} + \bar{y}_I^{\,2} + \bar{z}_I^{\,2} - 2\bar{z}_I \ell = 0$ because $\bar{z}_A = \ell$ (Jeudy, 1981)

Three other equations may be obtained by applying the laws of mechanics of movement of the centre of gravity of the total system. Before carrying on in this direction it is useful to calculate D, E and F and d, e and f in the equations (2-65), (2-66) and (2-67).

2.4.2.6 Calculation of d, e and f the components of $\vec{\Gamma}$ in G_{xyz}:

Suppose $\vec{\dot{\alpha}}$, $\vec{\dot{\beta}}$ and $\vec{\dot{\gamma}}$ are the angular velocities of the variables α, β and γ, then:

(2-69) $\vec{\Gamma} = \vec{\dot{\alpha}} + \vec{\dot{\beta}} + \vec{\dot{\gamma}}$

From the system G_{xyz} (figures 12, 13 and 14) $\vec{\dot{\alpha}}$, $\vec{\dot{\beta}}$ and $\vec{\dot{\gamma}}$ may be written as:

$$\text{(2-70)}\quad \vec{\dot{\alpha}} = \begin{pmatrix} \dot{\alpha}\sin\gamma\sin\beta \\ \dot{\alpha}\sin\gamma\cos\beta \\ \dot{\alpha}\cos\gamma \end{pmatrix}_{G_{xyz}}, \vec{\dot{\beta}} = \begin{pmatrix} 0 \\ 0 \\ \dot{\beta} \end{pmatrix}_{G_{xyz}}, \vec{\dot{\gamma}} = \begin{pmatrix} \dot{\gamma}\cos\beta \\ -\dot{\gamma}\sin\beta \\ 0 \end{pmatrix}_{G_{xyz}} \quad \text{(Jeudy, 1981)}$$

Substitute (2-70) into (2-69) to get:

$$\text{(2-71)}\quad \vec{\Gamma} = \begin{pmatrix} \dot{\alpha}\sin\gamma\sin\beta + \dot{\gamma}\cos\beta \\ \dot{\alpha}\sin\gamma\cos\beta - \dot{\gamma}\sin\beta \\ \dot{\alpha}\cos\gamma + \dot{\beta} \end{pmatrix}_{G_{xyz}} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}_{G_{xyz}}$$

Figure 12

Angular velocities $\vec{\dot{\alpha}}, \vec{\dot{\beta}}$ and $\vec{\dot{\gamma}}$ of the variables α, β and γ

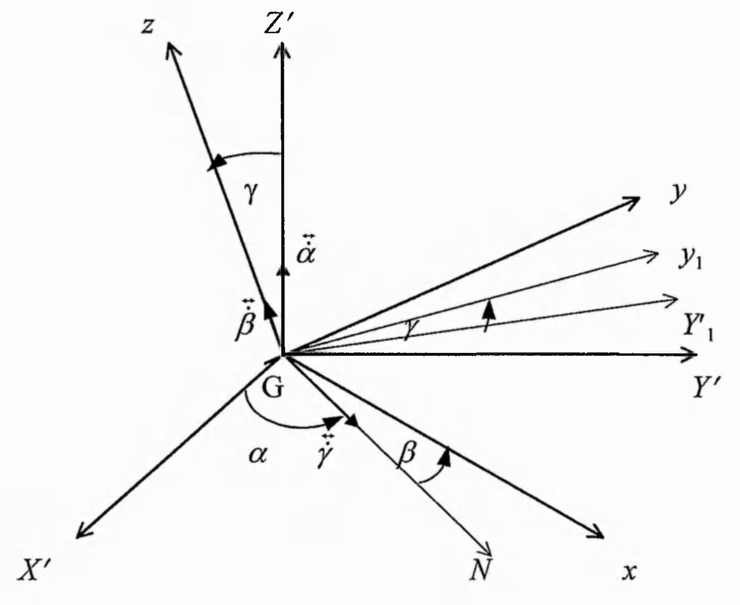


Figure 13

Components of $\dot{\alpha}$ relative to the axes G_x and G_y

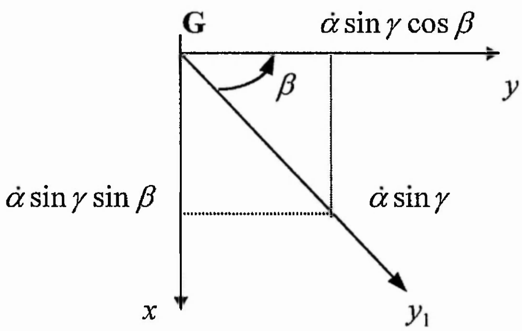
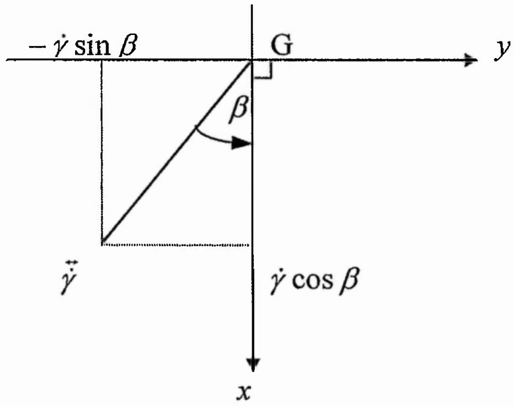


Figure 14

Components of $\dot{\gamma}$ relative to the axes G_x and G_y



Also the components of $\vec{\Gamma}$ in $G_{XYZ'}$:

(2-72) $$\vec{\dot{\alpha}} = \begin{pmatrix} 0 \\ 0 \\ \dot{\alpha} \end{pmatrix}_{G_{XYZ'}}, \vec{\dot{\beta}} = \begin{pmatrix} \dot{\beta}\sin\gamma\sin\alpha \\ -\dot{\beta}\sin\gamma\cos\alpha \\ \dot{\beta}\cos\gamma \end{pmatrix}_{G_{XYZ'}}, \vec{\dot{\gamma}} = \begin{pmatrix} \dot{\gamma}\cos\alpha \\ \dot{\gamma}\sin\alpha \\ 0 \end{pmatrix}_{G_{XYZ'}}$$

(Jeudy, 1981)

Consequently:

$$(2\text{-}73)\quad \vec{\Gamma} = \begin{pmatrix} \dot{\beta}\sin\gamma\sin\alpha + \dot{\gamma}\cos\alpha \\ -\dot{\beta}\sin\gamma\cos\alpha + \dot{\gamma}\sin\alpha \\ \dot{\beta}\cos\gamma + \dot{\alpha} \end{pmatrix}_{G_{X'Y'Z'}} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}_{G_{X'Y'Z'}}$$

Where:

p, q and r are the components of $\vec{\Gamma}$ in $G_{X'Y'Z'}$.

2.4.2.7 Calculation of D, E and F the components of $\vec{\varpi}$ in G_{xyz}:

Pre-multiplying the components of $\vec{\varpi}$ in $O_{\overline{xyz}}$, equation (2-59), by the matrix B equation (2-57) to get:

$$(2\text{-}74)\quad \begin{aligned} D &= \varpi[\cos\phi(\cos\beta\cos\alpha - \sin\beta\sin\alpha\cos\gamma) + \sin\phi\sin\beta\sin\gamma], \\ E &= \varpi[\cos\phi(-\sin\beta\cos\alpha - \cos\beta\sin\alpha\cos\gamma) + \sin\phi\cos\beta\sin\gamma], \\ F &= \varpi[\cos\phi\sin\gamma\sin\alpha + \sin\phi\cos\gamma]. \end{aligned}$$

2.4.3 Calculation of the three additional movement equations:

By applying Newton's second law to the centre of gravity, T of the whole system, this relation may be written:

$$m\vec{\psi}_T = \vec{S} + m(\vec{\beta}_T + \vec{B}_T)$$

Where:

$\vec{\beta}_T$ and $\vec{B}_T$ are respectively the terrestrial and astronomical gravitational forces.

$\vec{S}$ is the tension of the suspension tape at point I.

Since T, is the centre of gravity of the whole system, the reacting forces due to accelerating and decelerating forces applied to the spinner are zero.

We know that $\vec{\beta}_T + \vec{B}_T = \vec{g} + \vec{\psi}_O - \vec{JT}|\vec{\varpi}|^2$, see (2-24). Substitute this equation into the one above to get:

$$(2\text{-}75)\quad m(\vec{\psi}_T - \vec{\psi}_O) = \vec{S} + m\vec{g} - m\vec{JT}|\vec{\varpi}|^2$$

$(\vec{\psi}_T - \vec{\psi}_O)$ has already been evaluated by formula (2-41). But we want to express (2-75) in the system $O_{\overline{xyz}}$. To do that another expression of $(\frac{d\vec{\Gamma}}{dt})_{G_{xyz}}$ must be found:

$$(\frac{d\vec{\Gamma}}{dt})_{abs} = (\frac{d\vec{\Gamma}}{dt})_{O_{\overline{xyz}}} + \vec{\varpi} * \vec{\Gamma}$$

Having taken account of (2-36):

$$\text{(2-76)} \quad (\frac{d\vec{\Gamma}}{dt})_{O_{\overline{xyz}}} = (\frac{d\vec{\Gamma}}{dt})_{G_{xyz}}$$

Then, (2-41) becomes:

$$\text{(2-77)} \quad \begin{aligned} &\vec{\psi}_T - \vec{\psi}_O = T\vec{J}|\vec{\varpi}|^2 + 2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O} + [I\vec{T}.(2\vec{\varpi} + \vec{\Gamma})]\vec{\Gamma} \\ &-[\vec{\Gamma}.(2\vec{\varpi} + \vec{\Gamma})]I\vec{T} + (\frac{d\vec{\Gamma}}{dt})_{O_{\overline{xyz}}} * I\vec{T} \end{aligned}$$

Where the only change with respect to the equation (2-41) comes from the derivative of $\vec{\Gamma}$. For scalar products, the components of $\vec{\Gamma}$ and $I\vec{T}$ in (2-77) can still be evaluated with their components in G_{xyz} because the scalar products do not vary with a change of the orthogonal coordinate system. To obtain the components of $I\vec{T}$ in $O_{\overline{xyz}}$, the vector $(0, 0, -l)$ is pre-multiplied by the matrix B^{-1}. This gives:

$$\text{(2-78)} \quad I\vec{T} = \begin{pmatrix} -l\sin\gamma\sin\alpha \\ l\sin\gamma\cos\alpha \\ -l\cos\gamma \end{pmatrix}_{O_{\overline{xyz}}}$$

. Since B is an orthogonal matrix, $B^{-1} = B^T$.

Equation (2-61) gives the components of $(2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O})$ in $O_{\overline{xyz}}$. Substitute (2-77) into (2-75) to get:

$$\text{(2-79)} \quad \begin{aligned} &m\{2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O} + [I\vec{T}.(2\vec{\varpi} + \vec{\Gamma})]\vec{\Gamma} - [\vec{\Gamma}.(2\vec{\varpi} + \vec{\Gamma})]I\vec{T} + \\ &(\frac{d\vec{\Gamma}}{dt})_{O_{\overline{xyz}}} * I\vec{T}\} = \vec{S} + m\vec{g} \end{aligned}$$

It is necessary to express all terms of this equation in the system $O_{\overline{xyz}}$.

2.4.3.1 Calculation of the components of $\vec{S}$ in $O_{\overline{xyz}}$:

Firstly, the two vectors $\vec{S}$ and $\vec{IA}$ are collinear. $\vec{IA}$ has the following components:

$$(2\text{-}80)\quad \vec{IA} = \begin{pmatrix} -\ell\sin\varphi\cos\Lambda \\ -\ell\sin\varphi\sin\Lambda \\ \ell\cos\varphi \end{pmatrix} \quad \text{(see figure 4)}$$

The unit vector $\vec{v} = \dfrac{\vec{IA}}{|\vec{IA}|}$ has the components: $\vec{v} = \begin{pmatrix} -\sin\varphi\cos\Lambda \\ -\sin\varphi\sin\Lambda \\ \cos\varphi \end{pmatrix}$

Finally, the components of $\vec{S}$ in $O_{\overline{xyz}}$ may be written as:

$$(2\text{-}81)\quad \vec{S} = \begin{pmatrix} -|\vec{S}|\sin\varphi\cos\Lambda \\ -|\vec{S}|\sin\varphi\sin\Lambda \\ |\vec{S}|\cos\varphi \end{pmatrix}$$

Taking account of equations (2-61), (2-41`), (2-73), (2-78) and (2-81), equation (2-79) becomes:

$$m\begin{pmatrix} -2\varpi\dot{y}_I\sin\phi + \ddot{x}_I \\ 2\varpi(\dot{x}_I\sin\phi - \dot{z}_I\cos\phi) + \ddot{y}_I \\ 2\varpi\dot{y}_I\cos\phi + \ddot{z}_I \end{pmatrix} - ml(2F+f)\begin{pmatrix} p \\ q \\ r \end{pmatrix} -$$

$$(2\text{-}82)\quad m[d(2D+d)+e(2E+e)+f(2F+f)]\begin{pmatrix} -l\sin\gamma\sin\alpha \\ l\sin\gamma\cos\alpha \\ -l\cos\gamma \end{pmatrix} +$$

$$m\begin{pmatrix} -\dot{q}l\cos\gamma - \dot{r}l\sin\gamma\cos\alpha \\ \dot{p}l\cos\gamma - \dot{r}l\sin\gamma\sin\alpha \\ \dot{p}l\sin\gamma\cos\alpha + \dot{q}l\sin\gamma\sin\alpha \end{pmatrix} = \begin{pmatrix} -|\vec{S}|\sin\varphi\cos\Lambda \\ -|\vec{S}|\sin\varphi\sin\Lambda \\ |\vec{S}|\cos\varphi \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

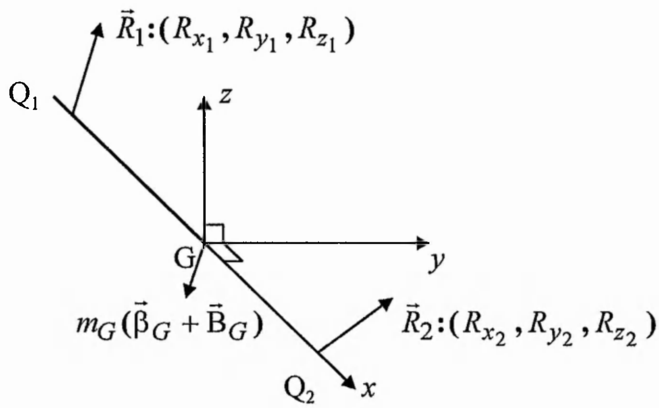
As an extra unknown is introduced, an eighth equation should be found for the function $|\vec{S}|$.

2.4.4 Calculation of the eighth equation of movement:

Taking account of the fact that the derivation of the kinetic moment of the gyroscope is equal to the moment of the forces acting on it, a very simple equation is obtained.

Figure15

Forces acting on the gyroscope



$G\overline{Q}_1 = -G\overline{Q}_2 = -s$

The gyroscope is suspended at the ends of its rotation axis G_x at Q_1 and Q_2. At these points, the reactions of the carriage on the gyroscope are $\vec{R}_1$ and $\vec{R}_2$. The gyroscope is also under the influence of the gravitational forces of attraction, but they have zero moment with respect to G, as G is the centre of gravity, figure15. The kinetic moment $\vec{M}_{G_G}$ of the gyroscope with respect to point G can be calculated as:

$$(2\text{-}83)\quad \vec{M}_{G_G} = \sum_{P \in (G)} G\vec{P} * w\vec{V}_P = \sum_{P \in (G)} G\vec{P} * w(\vec{V}_G + \vec{\Gamma}_G * G\vec{P})$$

Where $\vec{V}_P = \vec{V}_G + \vec{\Gamma}_G * G\vec{P}$

(2-84) $\vec{M}_{GG} = \sum_{P\in(G)} \vec{GP} * (w\vec{\Gamma}_G * \vec{GP}) = I_G\vec{\Gamma}_G$. See equations (2-5) and (2-6).

The differentiation of $\vec{M}_{GG}$ with respect to time can be calculated as:

$$(\frac{d\vec{M}_{GG}}{dt})_{abs} = \sum_{P\in(G)} (\vec{V}_P - \vec{V}_G) * w\vec{V}_P + \sum_{P\in(G)} \vec{GP} * w\vec{\psi}_P$$

The first term of the right-hand side of the equation above is equal to zero because G is the centre of gravity of the gyroscope. The second term represents the moment of external forces affecting the gyroscope. Therefore:

(2-85) $(\frac{d\vec{M}_{GG}}{dt})_{abs} = \vec{M}_{E_G} = G\vec{Q}_1 * \vec{R}_1 + G\vec{Q}_2 * \vec{R}_2$

Where:

$\vec{M}_{GG}$ is the kinetic moment of the gyroscope with respect to point G.

$\vec{M}_{E_G}$ is the moment of external forces acting on the gyroscope.

Substitute (2-84) into (2-85) to get:

(2-86) $(\frac{dI_G\vec{\Gamma}_G}{dt})_{abs} = G\vec{Q}_1 * \vec{R}_1 + G\vec{Q}_2 * \vec{R}_2$

Where:

$$\vec{R}_1 = \begin{pmatrix} R_{x_1} \\ R_{y_1} \\ R_{z_1} \end{pmatrix} \text{ and } \vec{R}_2 = \begin{pmatrix} R_{x_2} \\ R_{y_2} \\ R_{z_2} \end{pmatrix}, \quad G\vec{Q}_1 = -G\vec{Q}_2 = -s\vec{\imath}$$

$\vec{\imath}$ is the unit vector of the axis G_x.

s is the distance defined by; $\overline{GQ_1} = -\overline{GQ_2} = -s$ (figure15).

From the definition of the absolute derivative of a vector (Chester, 1979):

$$(\frac{dI_G\vec{\Gamma}_G}{dt})_{abc} = (\frac{dI_G\vec{\Gamma}_G}{dt})_{G_{xyz}} + \vec{\Gamma}_C * (I_G\vec{\Gamma}_G)$$

Since :

$\vec{\Gamma}_C = \vec{\Gamma} + \vec{\varpi}$

The equation (2-86) becomes:

(2-87) $$(\frac{dI_G\vec{\Gamma}_G}{dt})_{G_{xyz}} + (\vec{\varpi}+\vec{\Gamma})*(I_G\vec{\Gamma}_G) = \begin{pmatrix} -s \\ 0 \\ 0 \end{pmatrix} * \begin{pmatrix} R_{x_1} \\ R_{y_1} \\ R_{z_1} \end{pmatrix} + \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} * \begin{pmatrix} R_{x_2} \\ R_{y_2} \\ R_{z_2} \end{pmatrix}$$

$I_G\vec{\Gamma}_G$ has already been calculated in (2-63) and $\vec{\Gamma}_G = \vec{\Gamma}_C + \vec{\omega}$. Then, $I_G\vec{\Gamma}_G$ may be written as:

$$I_G\vec{\Gamma}_G = \begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & Q \end{pmatrix} \begin{pmatrix} D+d+\omega_1 \\ E+e \\ F+f \end{pmatrix}$$

Then:

$$I_G\vec{\Gamma}_G = \begin{pmatrix} P(D+d+\omega_1) \\ Q(E+e) \\ Q(F+f) \end{pmatrix}_{G_{xyz}} \quad \text{and } \vec{\varpi}+\vec{\Gamma} = \begin{pmatrix} D+d \\ E+e \\ F+f \end{pmatrix}_{G_{xyz}}$$

After substituting these expressions into (2-87) we get:

(2-88)

$$\begin{pmatrix} P(\dot{D}+\dot{d}+\dot{\omega}_1) \\ Q(\dot{E}+\dot{e}) \\ Q(\dot{F}+\dot{f}) \end{pmatrix} + \begin{pmatrix} 0 \\ (F+f)[(D+d)(P-Q)+\omega_1 P] \\ (E+e)[(D+d)(Q-P)-\omega_1 P] \end{pmatrix} = \begin{pmatrix} 0 \\ sR_{z_1} \\ -sR_{y_1} \end{pmatrix} + \begin{pmatrix} 0 \\ -sR_{z_2} \\ sR_{y_2} \end{pmatrix}$$

Finally, from (2-88) the following three equations may be written:

(2-89) $P(\dot{D}+\dot{d}+\dot{\omega}_1) = 0$

(2-90) $Q(\dot{E}+\dot{e}) + (F+f)[(D+d)(P-Q)+\omega_1 P] = s(R_{z_1} - R_{z_2})$

(2-91) $Q(\dot{F}+\dot{f}) + (E+e)[(D+d)(Q-P)-\omega_1 P] = -s(R_{y_1} - R_{y_2})$

The most useful equation is (2-89) because it does not contain the reactions R_1 and R_2. Therefore, eight equations with eight unknown terms form a system of general equations of movement for a suspended gyroscope. These equations are in a rigorous non-linearised form.

2.5 Outline of the eight movement equations:

The eight movement equations obtained in this Chapter are the three Euler equations (2-65), (2-66) and (2-67), equation (2-68), and three components of the vectorial equation (2-82) and the equation (2-89). They are written as follows:

(2-65)
$$m l g_y - m_C EF(l_G - l)(l_G - l_C) - ml\lambda_y + L[e(2F+f) - \dot{d}] = (P+P')(\dot{D}+\dot{d}) + \dot{\omega}_1 P + (R'-Q')(E+e)(F+f)$$

(2-66)
$$-m l g_x + m_C DF(l_G - l)(l_G - l_C) + ml\lambda_x - L[d(2F+f) + \dot{e}] = (Q+Q')(\dot{E}+\dot{e}) + (P+P'-R'-Q)(D+d)(F+f) + \omega_1 P(F+f)$$

(2-67)
$$c(\alpha,\beta,\dot{\alpha},\dot{\beta}) = (R'+Q)(\dot{F}+\dot{f}) + (Q+Q'-P'-P)(D+d)(E+e) - \omega_1 P(E+e)$$

(2-68) $$\bar{x}_I^{\,2} + \bar{y}_I^{\,2} + \bar{z}_I^{\,2} - 2\bar{z}_I \ell = 0$$

(2-92)
$$m(-2\varpi\dot{\bar{y}}_I \sin\phi + \ddot{\bar{x}}_I) - ml(2F+f)p - m[d(2D+d) + e(2E+e) + f(2F+f)](-l\sin\gamma\sin\alpha) + m(-\dot{q}l\cos\gamma - \dot{r}l\sin\gamma\cos\alpha) = -|\vec{S}|\sin\varphi\cos\Lambda$$

(2-93)
$$m[2\varpi(\dot{\bar{x}}_I\sin\phi - \dot{\bar{z}}_I\cos\phi) + \ddot{\bar{y}}_I] - ml(2F+f)q - m[d(2D+d) + e(2E+e) + f(2F+f)](l\sin\gamma\cos\alpha) + m(\dot{p}l\cos\gamma - \dot{r}l\sin\gamma\sin\alpha) = -|\vec{S}|\sin\varphi\sin\Lambda$$

(2-94)
$$m(2\varpi\dot{\bar{y}}_I\cos\phi + \ddot{\bar{z}}_I) - ml(2F+f)r - m[d(2D+d) + e(2E+e) + f(2F+f)](-l\cos\gamma) + m(\dot{p}l\sin\gamma\cos\alpha + \dot{q}l\sin\gamma\sin\alpha) = |\vec{S}|\cos\varphi - mg$$

(2-95) $$\dot{D} + \dot{d} + \dot{\omega}_1 = 0$$

The eight unknown parameters $\bar{x}_I$, $\bar{y}_I$, $\bar{z}_I$, α, β, γ, ω_1 and $|\vec{S}|$ are determined by these eight differential non-linear equations of second order. The unknown parameters are respectively, the three co-ordinates of the point, I in the system $O_{\overline{xyz}}$, see figure 4, the three Euler angles, which determine the position of the carriage with respect to the system $G_{XYZ'}$ parallel to $O_{\overline{xyz}}$, the angular velocity of the spinner with respect to the carriage and the tension of the suspension tape.

The general differential equations of motion for the suspended gyroscope, obtained in this Chapter, are rigorous, non-linearised and second order. These equations are fundamental for the determination of the apparent equilibrium of the suspended gyroscope and essential to obtain the equations of oscillations (Jeudy, 1982). It is difficult to solve these equations, as they are in their complex form, explicitly by analytical solutions. Therefore, these equations are linearised and solved in Chapter 3.

III. LINEARISED MOVEMENT EQUATIONS

3.1 Introduction:

From the general, non-linear, rigorous equations of movement developed, in the previous chapter the apparent position of equilibrium is determined by putting all derivatives of the motion parameters to zero. Two cases for the position of equilibrium are considered. The first (3.2.1) corresponds to the case where the moment due to tape torsion is zero while the axis of the gyroscope is on the meridian. The second (3.2.2) corresponds to the case where the moment due to tape torsion is zero while the axis of the gyroscope makes an angle ε with the meridian. Finding, the solution for the first case allows the second case to be easily solved.

Having determined the parameters of the position of equilibrium, the movement equations are linearised in the region of the equilibrium position by applying Taylor's formula and introducing the residual parameters δ, τ and μ in the same way as in Jeudy's small movements equations (Jeudy, 1982). A system of five differential linear equations with constant coefficients and five unknown functions of time are obtained. These equations are solved by elimination of three unknowns to obtain a system of two differential equations of the sixth order. The characteristic equation is a polynomial of tenth degree, which gives the ten frequencies of movement. The complete equation of motion of the moving mark of the Gyrotheodolite is of the form:

$$\sum_{i=1}^{10}[A_i e^{-\lambda_i t}\cos((\theta_i+\theta_i' t)(t-B_i))]+\boldsymbol{Q}-\Delta=0$$

The characteristic equation is of the form:

$$d_{10}\eta^{10}+d_9\eta^9+d_8\eta^8+d_7\eta^7+d_6\eta^6+d_5\eta^5+d_4\eta^4+d_3\eta^3+d_2\eta^2+d_1\eta+C=0$$

The assumed linear rate of change of the frequency with time, in the above equation, is a semi-empirical assumption, which agrees with practical observations.

The mathematical model and characteristic equation derived by Jeudy (Jeudy, 1982) "without taking into account the damping, accelerating and decelerating forces applied to the spinner" are respectively of the form:

$$\sum_{i=1}^{5}(A_i' \cos\theta_i t + A_i'' \sin\theta_i t)$$

$$d_5\eta^5 + d_4\eta^4 + d_3\eta^3 + d_2\eta^2 + d_1\eta + C = 0$$

3.2 Determination of the position of the apparent equilibrium:

Put all the derivatives of the motion parameters in the general movement equations obtained in Chapter 2, equal to zero. Namely, λ_x, λ_y and λ_z, the components of $(2\vec{\varpi} * \vec{V}_{I_O} + \dot{\psi}_{I_O})$ in the system G_{xyz}, see equation (2-61), *d*, *e* and *f* and *p*, *q* and *r*, the components of $\vec{\Gamma}$ in the systems G_{xyz} and $G_{X'Y'Z'}$ respectively, see equations (2-71) and (2-73) and the derivatives of components $\vec{\varpi}$ and $\vec{\omega}$. The following equations for the position of apparent equilibrium may be written.

From (2-65), this equation may be written:

(3-1) $m\lg_y - m_C EF(l_G - l)(l_G - l_C) = EF(R' - Q')$

From (2-66), this equation may be written:

(3-2) $-m\lg_x + m_C DF(l_G - l)(l_G - l_C) = DF(P + P' - R' - Q) + \omega_1 PF$

From (2-67), this equation may be written:

(3-3) $c(\alpha, \beta) = (Q + Q' - P' - P)DE - \omega_1 PE$

From equation (2-68), $\bar{x}_I$ and $\bar{y}_I$ represent the movement of the point I, of the suspension tape. These co-ordinates are equal to zero in the position of apparent equilibrium, (Jeudy, 1982).

From (2-92), this equation may be written:

(3-4) $S \sin\varphi \cos\Lambda = 0$

From (2-93), this equation may be written:

(3-5) $S\sin\varphi\sin\Lambda = 0$

From (2-94), this equation may be written:

(3-6) $S\cos\varphi - mg = 0$

Where φ and Λ are two angular co-ordinates for the point I, see figure 4.

All terms in equation (2-95) are equal to zero in the position of apparent equilibrium. From the last three equations, one can deduce:

(3-7) $S = mg$

Because:

(3-8) $\varphi = 0$ and Λ is indeterminate.

Let us assume from (2-95):

(3-9) $D + d + \omega_1 = J$

Where:

D, d and ω_1 are the components of $\vec{\varpi}$, $\vec{\Gamma}$ and $\vec{\omega}$ on the axis G_x respectively.

J is an arbitrary constant. J is the projection of $\vec{\Gamma}_G$, the absolute angular velocity of the gyroscope on the axis G_x. Because:

$\vec{\Gamma}_G = \vec{\Gamma}_C + \vec{\omega}$ and $\vec{\Gamma}_C = \vec{\varpi} + \vec{\Gamma}$. Then, $\vec{\Gamma}_G = \vec{\varpi} + \vec{\Gamma} + \vec{\omega}$

The vector of angular velocity of the gyroscope, $\vec{\omega}$, has only one component ω_1 on the axis G_x "the other components of $\vec{\omega}$ are from elsewhere, zero". The component d in (3-9) is equal to zero in the case of equilibrium, because it is a derivative of motion parameter, see equation (2-71). Then, (3-9) becomes:

(3-10) $D + \omega_1 = J$

Taking account of (2-74), (3-10) becomes:

(3-11) $\varpi[\cos\phi(\cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\gamma) + \sin\phi\sin\beta\sin\gamma] + \omega_1 = J$

Finally, the equations (3-1), (3-2) and (3-3) determine α, β and γ in the position of the apparent equilibrium. Two cases will be considered. The first case relates to the position where the moment due to tape torsion is zero while the axis G_x of the gyroscope is exactly in the plane of the meridian $O_{\overline{xz}}$, see figure 4 in Chapter 1. The second case relates to the position of the gyroscope where the moment due to tape torsion is zero while the axis G_x makes a certain angle, ε with the plane of the meridian $O_{\overline{xz}}$. This second case is obviously the only one, which interests us in practice because the position of zero moment about the tape has only one position with respect to meridian. Therefore, the angle, ε will be a further unknown.

3.2.1 Position of zero moment of the tape on the meridian:

This case is considered here to bring certain simplifications to the second case. The Euler angles α, β and γ are functions of time t. In equation (2-16) we defined c as a function of α, β and their derivatives. These are zero in the case of apparent equilibrium. The torsion of the suspension tape may be written as a linear equation in α and β:

(3-12) $c(\alpha, \beta) = -K(\alpha + \beta)$ (Jeudy, 1982)

Where:

K is the constant of suspension tape torsion ($K > 0$).

In this case, the position defined by the following angles is a position of apparent equilibrium:

(3-13) $\alpha = \pi/2$, $\beta = -\pi/2$, $\gamma = \gamma_0$

Where γ_0 is chosen such that the equations (3-1), (3-2) and (3-3) are satisfied. Equations (3-1) and (3-3) are already satisfied by the conditions $\beta = -\alpha = -\pi/2$. So, γ_0 may be determined by equation (3-2). Firstly, calculate D, E and F, as defined by equation (2-74) after taking into account (3-13):

$$D = \varpi(\cos\phi\cos\gamma_0 - \sin\phi\sin\gamma_0) = \varpi\cos(\phi + \gamma_0)$$

(3-14) $E = 0$

$$F = \varpi(\cos\phi\sin\gamma_0 + \sin\phi\cos\gamma_0) = \varpi\sin(\phi + \gamma_0)$$

Taking account of (2-58) the equations (3-1), (3-2) and (3-3) become:

(3-15) $-m\lg\cos\beta\sin\gamma = EF(C + R' - Q') = EFC_D$

(3-16) $m\lg\sin\beta\sin\gamma = DF(P + P' - R' - Q - C) + \omega_1 PF = F(DC_E + \omega_1 P)$

(3-17) $-K(\alpha+\beta)=(Q+Q'-P'-P)DE-\omega_1 PE=E(DC_F-\omega_1 P)$

Where the constants C_D, C_E and C_F are defined by relations:

(3-18) $C_D=C+R'-Q'$

(3-19) $C_E=P+P'-R'-Q-C$

(3-20) $C_F=Q+Q'-P'-P$

And C is the constant equal to $m_C(l_G-l)(l_G-l_C)$, which is also equal to $m(l_G-l)^2$. This can be proved as it is shown below, by the definition of the centre of mass (Goldstein, 1959):

(3-21) $ml=m_C l_C+m_G l_G$

$$m_C(l_G-l)(l_G-l_C)=m_C l_G{}^2-m_C l_G l_C - m_C l l_G+m_C l l_C$$

Substitute $m_C l_C=ml-m_G l_G$ into the above equation:

$$m_C(l_G-l)(l_G-l_C)=m_C l_G{}^2-ml_G l+m_G l_G{}^2-m_C l l_G+ml^2-ml_G l$$

$$m_C(l_G-l)(l_G-l_C)=(m_C+m_G)l_G{}^2-l_G l(m_C+m_G)+ml^2-ml_G l$$

$$C=m_C(l_G-l)(l_G-l_C)=m(l_G-l)^2$$

Since $E=0$ from equations (3-14), equations (3-15) and (3-17) are effectively satisfied by $\beta=-\pi/2$ and $\alpha=\pi/2$. Put D and F from (3-14) into (3-16) to get the equation defines γ_0:

(3-22) $-ml g\sin\gamma_0=\varpi\sin(\phi+\gamma_0)[\varpi C_E\cos(\phi+\gamma_0)+\omega_1 P]$

Ignore the term $\varpi C_E\cos(\phi+\gamma_0)$ in front of $\omega_1 P$ because ω_1 is a large number. The angular velocity of the spinner, ω_1 is approximately 22,000 r.p.m, which is about 2300 radians/sec. The angular velocity of the earth ϖ is about 0.000073 radians/sec. So, the term $\varpi^2 C_E\cos(\phi+\gamma_0)\sin(\phi+\gamma_0)$ in (3-22) is negligible. Equation (3-22) becomes:

$$-ml g\sin\gamma_0=\varpi\omega_1 P[\sin\phi\cos\gamma_0+\cos\phi\sin\gamma_0]$$

$$-(ml g+\varpi\omega_1 P\cos\phi)\sin\gamma_0=\varpi\omega_1 P\sin\phi\cos\gamma_0$$

γ_0 is a small angle therefore; the right-hand side of the last equation becomes: $\varpi\omega_1 P sin\phi$. Then:

(3-23) $\tan\gamma_0 \approx \sin\gamma_0 \approx \dfrac{-PJ\varpi\sin\phi}{m\,\mathrm{lg} + PJ\varpi\cos\phi}$

Because γ_0 is a small angle and $\omega_1 \approx J$, D is very small relative to ω_1, The exact relationship is given by formula (3-10):

$\omega_1 + D = J$

Put D from (3-14) into (3-10) to get:

(3-24) $\omega_1 + \varpi\cos(\phi + \gamma_0) = J$

The angle γ_0 in formula (3-23) corresponds to a very small swing of the axis of the gyroscope away from North.

3.2.2 Position of zero moment of the tape outside of the meridian:

We define in this case that the gyroscope makes an angle, ε with the meridian while the moment of the tape torsion is zero. The function c may be written as:

(3-25) $c(\alpha,\beta) = -K(\alpha + \beta - \varepsilon)$ (Jeudy, 1982)

Where:

ε is the azimuth of the axis G_x while the gyroscope is not spinning.

Further unknowns δ, τ and μ are used and given in these formulae.

(3-26) $\alpha = \pi/2 + \delta$

(3-27) $\beta = -\pi/2 + \tau$

(3-28) $\gamma = \gamma_0 + \mu$

Having determined the parameters of the position of equilibrium, the movement equations are linearised in the region of the equilibrium position. Substitute these further unknowns δ, τ and μ into the equations (3-15), (3-16) and also into this equation from (3-17):

(3-29) $-K(\alpha + \beta - \varepsilon) = E(DC_F - \omega_1 P)$

D, E and F may be expressed as functions of δ, τ and μ. From (2-74), D, E and F can be written with first order approximation taking account of the fact that δ, τ and μ are small:

$$D = \varpi \cos(\phi + \gamma_0) - \mu\varpi \sin(\phi + \gamma_0)$$

(3-30) $$E = -\delta\varpi \cos\phi - \tau\varpi \cos(\phi + \gamma_0)$$

$$F = \varpi \sin(\phi + \gamma_0) + \mu\varpi \cos(\phi + \gamma_0)$$

Substitute D, E and F from equations (3-30) into equations (3-15), (3-16) and (3-29) to get, in a first order approximation, these equations:

(3-31) $$-ml g \tau \sin\gamma_0 = -\delta\varpi^2 C_D \cos\phi \sin(\phi + \gamma_0) - \frac{\tau\varpi^2 C_D \sin 2(\phi + \gamma_0)}{2}$$

(3-32) $$-ml g(\sin\gamma_0 + \mu\cos\gamma_0) = [\varpi \sin(\phi + \gamma_0) + \mu\varpi \cos(\phi + \gamma_0)] \{C_E[\varpi \cos(\phi + \gamma_0) - \mu\varpi \sin(\phi + \gamma_0)] + \omega_1 P\}$$

(3-33) $$-K(\delta + \tau - \varepsilon) = [-\delta\varpi \cos\phi - \tau\varpi \cos(\phi + \gamma_0)]\{C_F[\varpi \cos(\phi + \gamma_0)] - \omega_1 P\}$$

Where α, β and γ are replaced from equations (3-26), (3-27) and (3-28) and where negligibly small second order terms in the right-hand side of (3-33) have been ignored. If we develop the products of the factors of the equation (3-32), ignoring the term in μ^2, we will obtain a constant plus the term of degree 1 in μ.

$$\mu[ml g \cos\gamma_0 + \varpi^2 C_E \cos 2(\phi + \gamma_0) + \varpi\omega_1 P \cos(\phi + \gamma_0)] = -ml g \sin\gamma_0 - \varpi^2 C_E \frac{\sin 2(\phi + \gamma_0)}{2} - \varpi\omega_1 P \sin(\phi + \gamma_0)$$

Substitute equation (3-22) into the above equation to get:

$$\mu[ml g \cos\gamma_0 + \varpi^2 C_E \cos 2(\phi + \gamma_0) + \varpi\omega_1 P \cos(\phi + \gamma_0)] = 0$$

Since the coefficient of μ is not zero the solution is:

(3-34) $$\mu = 0$$

The equations (3-31) and (3-33) allow us to calculate 2 values δ_0 and τ_0 corresponding to the apparent equilibrium. After developing (3-31) and (3-33):

(3-35) $$[-ml g \sin\gamma_0 + \varpi^2 C_D \frac{\sin 2(\phi + \gamma_0)}{2}]\tau_0 + [\varpi^2 C_D \cos\phi \sin(\phi + \gamma_0)]\delta_0 = 0$$

(3-36) $$[P\omega_1\varpi \cos(\phi + \gamma_0) - \varpi^2 C_F \cos^2(\phi + \gamma_0) + K]\tau_0 + [P\omega_1\varpi \cos\phi - \varpi^2 C_F \cos\phi \cos(\phi + \gamma_0) + K]\delta_0 = K\varepsilon$$

With an insignificant error, the equations (3-35) and (3-36) can be replaced by the following equations:

(3-37) $(m\lg\sin\gamma_0)\tau_0 - [\varpi^2 C_D \cos\phi \sin(\phi+\gamma_0)]\delta_0 = 0$

(3-38) $(PJ\varpi\cos\phi + K)(\tau_0 + \delta_0) = K\varepsilon$

The effect of neglected terms is less than 10^{-10} because ϖ and γ_0, in radians, are very small.

From (3-37):

(3-37′) $$\tau_0 = \frac{\varpi^2 C_D \cos\phi \sin(\phi+\gamma_0)}{m\lg\sin\gamma_0}\delta_0$$

And from (3-38):

(3-38′) $$(\delta_0 + \tau_0) = \frac{K\varepsilon}{PJ\varpi\cos\phi + K}$$ (figure 16)

From these formulae (3-37′) and (3-38′) the angle of precession α is the most affected while, the angle of proper rotation β is only affected by a very small variation (τ_0), (Jeudy, 1981).

This can be seen from the numerator $[\varpi^2 C_D \cos\phi \sin(\phi+\gamma_0)]\delta_0$ in (3-37′) which is very small. If we substitute τ_0 from (3-37′) into (3-38′) we get:

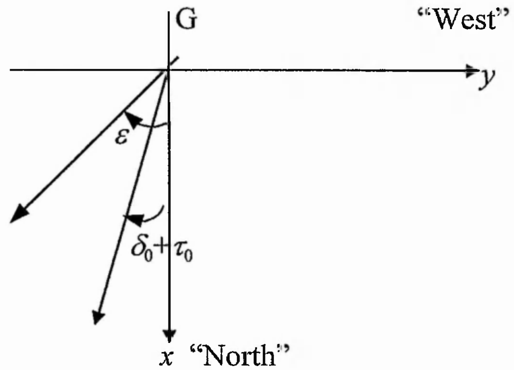
$$\delta_0 = \frac{K\varepsilon m\lg\sin\gamma_0}{[m\lg\sin\gamma_0 + \varpi^2 C_D \cos\phi \sin(\phi+\gamma_0)][PJ\varpi\cos\phi + K]}$$

From this equation we can see that the denominator is very small.

The angle (δ_0 + τ_0) as shown in figure 16, represents the angle between the gyroscope axis of rotation and meridian (Jeudy, 1982).

Figure 16

The influence of the position of zero moment of the tape on the position of the apparent equilibrium



3.3 Equations of small movements (oscillations):

These equations are obtained from the general movement equations obtained in Chapter 2, by making the assumptions that the unknown functions δ, τ, μ, $\bar{x}_I$, $\bar{y}_I$ and their derivatives are very small in the first order. Therefore, ignoring all the terms of an order above one in the movement equations, one gets the equations of "small movements" or oscillations. Firstly, useful relationships are calculated.

3.3.1 Calculation of the functions in the neighbourhood of equilibrium:

All functions obtained for the determination of the movement equations are expressed as functions of the unknowns δ, τ and μ. Then, the movement equations are linearised in the region of equilibrium.

3.3.1.1 Calculation of g_x, g_y and g_z:

The exact formulae for calculating g_x, g_y and g_z are the formulae (2-58). Substitute further variables δ, τ and μ defined by equations (3-26), (3-27) and (3-28) into (2-58). By keeping only terms of the first order, we obtain the following equations:

(3-39) $g_x = g\sin\gamma_0 + \mu g\cos\gamma_0$

(3-40) $g_y = -\tau g\sin\gamma_0$

(3-41) $g_z = -g\cos\gamma_0 + \mu g\sin\gamma_0$

3.3.1.2 Calculation of the matrix B:

The exact formula is the formula (2-57). After linearisation (that is to say after elimination of terms of order above one):

(3-42) $$B = \begin{pmatrix} \cos\gamma_0 - \mu\sin\gamma_0 & \tau + \delta\cos\gamma_0 & -(\sin\gamma_0 + \mu\cos\gamma_0) \\ -\delta - \tau\cos\gamma_0 & 1 & \tau\sin\gamma_0 \\ \sin\gamma_0 + \mu\cos\gamma_0 & \delta\sin\gamma_0 & \cos\gamma_0 - \mu\sin\gamma_0 \end{pmatrix}$$

3.3.1.3 Calculation of λ_x, λ_y and λ_z components of $2\vec{\varpi} * \vec{V}_{I_O} + \vec{\psi}_{I_O}$ in the system G_{xyz} from equation (2-61):

The point, I moves on a sphere with centre at A (figure 4) and radius ℓ such that:

$$\bar{x}_I^{\,2} + \bar{y}_I^{\,2} + \bar{z}_I^{\,2} - 2\bar{z}_I\ell = 0$$

If $\bar{x}_I$ and $\bar{y}_I$ are of the first order then, $\bar{z}_I$ is of second order because:

$$\bar{z}_I = \frac{\bar{x}_I^{\,2} + \bar{y}_I^{\,2} + \bar{z}_I^{\,2}}{2\ell}$$

This will be the same for successive derivatives of $\bar{z}_I$. Then, $\dot{\bar{z}}_I$ and $\ddot{\bar{z}}_I$ in the right-hand side of (2-61) may be ignored. Pre-multiplying (2-61) by matrix B as given in (3-42) and ignoring all small terms above first order, we obtain:

$$\lambda_x = -2\varpi\dot{y}_I \sin(\phi + \gamma_0) + \ddot{x}_I \cos\gamma_0$$

$$(3\text{-}43) \quad \lambda_y = 2\varpi\dot{x}_I \sin\phi + \ddot{y}_I$$

$$\lambda_z = 2\varpi\dot{y}_I \cos(\phi + \gamma_0) + \ddot{x}_I \sin\gamma_0$$

3.3.1.4 Calculation of D, E and F :

These components are obtained by pre-multiplying the vector $\vec{\varpi} = (\varpi\cos\phi,\ 0,\ \varpi\sin\phi)$ (2-59) by matrix B as given in (3-42). The result is already given in formulae (3-30)

3.3.1.5 Calculation of $\dot{D}$, $\dot{E}$ and $\dot{F}$:

These components are obtained by differentiating formulae (3-30) with respect to time:

$$\dot{D} = -\dot{\mu}\varpi \sin(\phi + \gamma_0)$$

$$(3\text{-}44) \quad \dot{E} = -\dot{\delta}\varpi\cos\phi - \dot{\tau}\varpi\cos(\phi + \gamma_0)$$

$$\dot{F} = \dot{\mu}\varpi\cos(\phi + \gamma_0)$$

3.3.1.6 Calculation of d, e, f and their derivatives:

From (2-71) one can deduce:

$$d = -\dot{\delta}\sin\gamma_0$$

$$(3\text{-}45) \quad e = \dot{\mu}$$

$$f = \dot{\delta}\cos\gamma_0 + \dot{\tau}$$

and:

$$\dot{d} = -\ddot{\delta}\sin\gamma_0$$

$$(3\text{-}46) \quad \dot{e} = \ddot{\mu}$$

$$\dot{f} = \ddot{\delta}\cos\gamma_0 + \ddot{\tau}$$

Substitute D, E and F from (3-30) and d, e and f from (3-45) into the term $m[d(2D+d)+e(2E+e)+f(2F+f)]$ of the left-hand side of equations (2-92), (2-93) and (2-94). Ignoring negligibly small second order terms, we get:

$$m[d(2D+d)+e(2E+e)+f(2F+f)] = 2m\varpi[\dot{\delta}\sin\phi + \dot{\tau}\sin(\phi+\gamma_0)]$$

3.3.1.7 Calculation of p, q, r and their derivatives:

From (2-73) one can deduce:

(3-45`) $$p = \dot{\tau}\sin\gamma_0 \quad q = \dot{\mu} \quad r = \dot{\tau}\cos\gamma_0 + \dot{\delta}$$

and: (3-46`) $$\dot{p} = \ddot{\tau}\sin\gamma_0 \quad \dot{q} = \ddot{\mu} \quad \dot{r} = \ddot{\tau}\cos\gamma_0 + \ddot{\delta}$$

3.3.1.8 Outline of the previous results:

The results obtained in section (3.3.1) above are reassembled below. Five equations of oscillations need to be calculated to determine five unknown parameters, δ, τ, μ, $\bar{x}_I$ and $\bar{y}_I$. The first three equations may be obtained from the three Euler equations (2-65), (2-66) and (2-67). The last two equations may be obtained from (2-92) and (2-93). The functions $|\vec{S}|$ and ω_1, the tension of the tape and the angular velocity of the gyroscope have been eliminated with the aid of equations (2-94) and (2-95) respectively. The function $\bar{z}_I$ may be ignored because of the reasoning in (3.3.1.3).

These results are outlined from (3.3.1):

(3-39) $g_x = g\sin\gamma_0 + \mu g\cos\gamma_0$

(3-40) $g_y = -\tau g\sin\gamma_0$

(3-41) $g_z = -g\cos\gamma_0 + \mu g\sin\gamma_0$

(3-26) $\alpha = \pi/2 + \delta$

(3-27) $\beta = -\pi/2 + \tau$

(3-28) $\gamma = \gamma_0 + \mu$

$$D = \varpi\cos(\phi+\gamma_0) - \mu\varpi\sin(\phi+\gamma_0)$$

(3-30) $$E = -\delta\varpi\cos\phi - \tau\varpi\cos(\phi+\gamma_0)$$

$$F = \varpi\sin(\phi+\gamma_0) + \mu\varpi\cos(\phi+\gamma_0)$$

$$\dot{D} = -\dot{\mu}\varpi\sin(\phi+\gamma_0)$$

(3-44) $$\dot{E} = -\dot{\delta}\varpi\cos\phi - \dot{\tau}\varpi\cos(\phi+\gamma_0)$$

$$\dot{F} = \dot{\mu}\varpi\cos(\phi+\gamma_0)$$

(3-45) $$d = -\dot{\delta}\sin\gamma_0,\quad e = \dot{\mu},\quad f = \dot{\delta}\cos\gamma_0 + \dot{\tau}$$

and (3-46) $$\dot{d} = -\ddot{\delta}\sin\gamma_0,\quad \dot{e} = \ddot{\mu},\quad \dot{f} = \ddot{\delta}\cos\gamma_0 + \ddot{\tau}$$

(3-45′) $$p = \dot{\tau}\sin\gamma_0,\quad q = \dot{\mu},\quad r = \dot{\tau}\cos\gamma_0 + \dot{\delta}$$

and (3-46′) $$\dot{p} = \ddot{\tau}\sin\gamma_0,\quad \dot{q} = \ddot{\mu},\quad \dot{r} = \ddot{\tau}\cos\gamma_0 + \ddot{\delta}$$

$$\lambda_x = -2\varpi\dot{\bar{y}}_I\sin(\phi+\gamma_0) + \ddot{\bar{x}}_I\cos\gamma_0$$

(3-43) $$\lambda_y = 2\varpi\dot{\bar{x}}_I\sin\phi + \ddot{\bar{y}}_I$$

$$\lambda_z = 2\varpi\dot{\bar{y}}_I\cos(\phi+\gamma_0) + \ddot{\bar{x}}_I\sin\gamma_0$$

We also need the relation:

$$m[d(2D+d) + e(2E+e) + f(2F+f)] = 2m\varpi[\dot{\delta}\sin\phi + \dot{\tau}\sin(\phi+\gamma_0)]$$

3.3.2 Calculation of the first equation of oscillations:

The first general equation of movement (2-65) is linearised using the residual parameters δ, τ and μ. Using the results calculated in section (3.3.1) and taking account of the fact that $\dot{D}+\dot{d}+\dot{\omega}_1=0$ from equation (2-95), the equation (2-65) becomes:

$$
\begin{aligned}
&-ml g\tau\sin\gamma_0+C\varpi\sin(\phi+\gamma_0)[\delta\varpi\cos\phi+\tau\varpi\cos(\phi+\gamma_0)]\\
&-ml(2\varpi\dot{\bar{x}}_I\sin\phi+\ddot{\bar{y}}_I)+L[2\dot{\mu}\varpi\sin(\phi+\gamma_0)+\ddot{\delta}\sin\gamma_0]=\\
&P'(-\dot{\mu}\varpi\sin(\phi+\gamma_0)-\ddot{\delta}\sin\gamma_0)\\
&+(R'-Q')\varpi\sin(\phi+\gamma_0)[-\delta\varpi\cos\phi-\tau\varpi\cos(\phi+\gamma_0)+\dot{\mu}]
\end{aligned}
\tag{3-47}
$$

Where all negligibly small second order terms have been ignored.

And where $C=m_C(l_G-l)(l_G-l_C)$

For reasons of simplification, The coefficients of δ, τ, μ, $\bar{x}_I$, $\bar{y}_I$ and their derivatives are indicated by a^i_j. Where superscript i denotes the order in which the equations appear below (1, 2, 3, 4 and 5) and subscript j the order of the parameter being considered, according to the following table:

δ	τ	μ	x_I	y_I	$\dot{\delta}$	$\dot{\tau}$	$\dot{\mu}$	$\dot{x}_I$	$\dot{y}_I$	$\ddot{\delta}$	$\ddot{\tau}$	$\ddot{\mu}$	$\ddot{x}_I$	$\ddot{y}_I$	function
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	j

$\dddot{\delta}$	$\dddot{\tau}$	$\dddot{\mu}$	$\dddot{x}_I$	$\dddot{y}_I$	$\tilde{\delta}$	$\tilde{\tau}$	$\tilde{\mu}$	$\tilde{x}_I$	$\tilde{y}_I$	function
16	17	18	19	20	21	22	23	24	25	j

Where ~ indicates to the fourth derivative.

Put equation (3-47) in order of the coefficients a^i_j to get:

$$\begin{aligned}
&\delta[\varpi^2(C+R'-Q')\cos\phi\sin(\phi+\gamma_0)] \\
&+\tau[-m\lg\sin\gamma_0+\frac{\varpi^2}{2}(C+R'-Q')\sin2(\phi+\gamma_0)] \\
(3\text{-}48)\quad &+\dot{\mu}[\varpi(P'+Q'-R'+2L)\sin(\phi+\gamma_0)] \\
&+\dot{\ddot{x}}_I[-2ml\varpi\sin\phi] \\
&+\ddot{\delta}[(L+P')\sin\gamma_0] \\
&+\ddot{\ddot{y}}_I[-ml]=0
\end{aligned}$$

If:

$a_1^1=\varpi^2(C+R'-Q')\cos\phi\sin(\phi+\gamma_0)$

$a_2^1=-m\lg\sin\gamma_0+\frac{\varpi^2}{2}(C+R'-Q')\sin2(\phi+\gamma_0)$

$a_2^1\approx-m\lg\sin\gamma_0+\varpi^2(C+R'-Q')\sin\phi\cos\phi$

$a_8^1=\varpi(P'+Q'-R'+2L)\sin(\phi+\gamma_0)$

$a_9^1=-2ml\varpi\sin\phi$

$a_{11}^1=(L+P')\sin\gamma_0$

$a_{15}^1=-ml$

Then, equation (3-48) becomes:

(3-48`) $\delta(a_1^1)+\tau(a_2^1)+\dot{\mu}(a_8^1)+\dot{\ddot{x}}_I(a_9^1)+\ddot{\delta}(a_{11}^1)+\ddot{\ddot{y}}_I(a_{15}^1)=0$

This is the first equation of oscillations.

3.3.3 Calculation of the second equation of oscillations:

In the same way as in equation (3-47), the second general equation of movement (2-66) is linearised using the residual parameters δ, τ and μ. Substitute g_x, λ_x, D, F, d, f, $\dot{E}$ and $\dot{e}$ from their equations in paragraph (3.3.1.8) into equation (2-66), to get after simplification:

$$- ml(g\sin\gamma_0 + \mu g\cos\gamma_0) + C\varpi^2[\frac{\sin 2(\phi+\gamma_0)}{2} + \mu\cos 2(\phi+\gamma_0)] +$$

$$ml[-2\varpi\dot{y}_I \sin(\phi+\gamma_0) + \ddot{x}_I \cos\gamma_0] - L[-2\dot{\delta}\varpi\sin\gamma_0 \sin(\phi+\gamma_0) + \dot{\mu}] =$$

(3-49)

$$(Q+Q')[-\dot{\delta}\varpi\cos\phi - \dot{\tau}\varpi\cos(\phi+\gamma_0) + \dot{\mu}] +$$

$$(P'-R'-Q)[\dot{\tau}\varpi\cos(\phi+\gamma_0) +$$

$$\dot{\delta}\varpi\cos(\phi+2\gamma_0) + \mu\varpi^2\cos 2(\phi+\gamma_0) + \frac{\varpi^2\sin 2(\phi+\gamma_0)}{2}] +$$

$$PJ[\varpi\sin(\phi+\gamma_0) + \mu\varpi\cos(\phi+\gamma_0) + \dot{\delta}\cos\gamma_0 + \dot{\tau}]$$

Where all negligibly small second order terms have been ignored.

And where $d + D + \omega_1 = J$ from (3-9)

Put (3-49) in the order of increasing value of the subscript *j* to get:

(3-50)

$$\mu[-m\lg\cos\gamma_0 - PJ\varpi\cos(\phi+\gamma_0) + \varpi^2\cos 2(\phi+\gamma_0)(C-P'+R'+Q)]$$

$$+\dot{\delta}[-PJ\cos\gamma_0 + 2L\varpi\sin\gamma_0\sin(\phi+\gamma_0) + \varpi(Q+Q')\cos\phi + \varpi(Q+R'-P')\cos(\phi+2\gamma_0)]$$

$$+\dot{\tau}[-PJ + \varpi(2Q+Q'+R'-P')\cos(\phi+\gamma_0)]$$

$$+\dot{y}_I[-2ml\varpi\sin(\phi+\gamma_0)]$$

$$+\dot{\mu}[-(L+Q+Q')]$$

$$+\ddot{x}_I(ml\cos\gamma_0)$$

$$+[-m\lg\sin\gamma_0 + \varpi^2(C-P'+R'+Q)\frac{\sin 2(\phi+\gamma_0)}{2} - PJ\varpi\sin(\phi+\gamma_0)] = 0$$

If:

$$a_3^2 = -ml g\cos\gamma_0 - PJ\varpi\cos(\phi+\gamma_0) + \varpi^2\cos 2(\phi+\gamma_0)(C - P' + R' + Q)$$
$$a_3^2 \approx -ml g - PJ\varpi\cos\phi$$

$$a_6^2 = -PJ\cos\gamma_0 + 2L\varpi\sin\gamma_0\sin(\phi+\gamma_0) + \varpi(Q+Q')\cos\phi + \varpi(Q + R' - P')\cos(\phi+2\gamma_0)$$
$$a_7^2 = -PJ + \varpi(2Q + Q' + R' - P')\cos(\phi+\gamma_0)$$
$$a_6^2 \approx a_7^2 \approx -PJ + \varpi(2Q + Q' + R' - P')\cos\phi$$

$$a_{10}^2 = -2ml\varpi\sin(\phi+\gamma_0)$$

$$a_{13}^2 = -(L + Q + Q')$$

$$a_{14}^2 = ml\cos\gamma_0 \approx ml$$

Taking into account of (3-24) $J = \omega_1 + \varpi\cos(\phi+\gamma_0)$ and having taken account of (3-22), the last term of (3-50) is zero:

$$\begin{aligned}
&\varpi^2 C_E\sin(\phi+\gamma_0)\cos(\phi+\gamma_0) + \varpi\omega_1 P\sin(\phi+\gamma_0) \\
&+\varpi^2(C - P' + R' + Q)\sin(\phi+\gamma_0)\cos(\phi+\gamma_0) \\
&- P\varpi\omega_1\sin(\phi+\gamma_0) - P\varpi^2\sin(\phi+\gamma_0)\cos(\phi+\gamma_0) = 0
\end{aligned}$$

Where $C_E = P + P' - R' - Q - C$ from (3-19)

Then, (3-50) becomes:

(3-50`) $\mu(a_3^2) + \dot{\delta}(a_6^2) + \dot{\tau}(a_7^2) + \dot{y}_I(a_{10}^2) + \ddot{\mu}(a_{13}^2) + \ddot{x}_I(a_{14}^2) = 0$

This is the second equation of oscillations.

3.3.4 Calculation of the third equation of oscillations:

In the same way as in equation (3-49), taking into account of equations (3-30), (3-44), (3-45) and (3-46) and also taking into account of the fact that: $d + D + \omega_1 = J$ (3-9), equation (2-67) becomes:

$$\begin{aligned}
&- K(\delta + \tau - \varepsilon) - \lambda\dot{\delta} - \lambda\dot{\tau} = (R' + Q)[\dot{\mu}\varpi\cos(\phi+\gamma_0) + \ddot{\delta}\cos\gamma_0 + \ddot{\tau}] \\
\text{(3-51)}\quad &+ (Q + Q' - P')[-\delta\varpi\cos\phi - \tau\varpi\cos(\phi+\gamma_0) + \dot{\mu}]\varpi\cos(\phi+\gamma_0) \\
&- JP[-\delta\varpi\cos\phi - \tau\varpi\cos(\phi+\gamma_0) + \dot{\mu}]
\end{aligned}$$

Where all negligibly small second order terms have been ignored and:

$$c(\alpha,\beta,\dot{\alpha},\dot{\beta}) = -K(\alpha+\beta-\varepsilon) - \lambda(\dot{\alpha}+\dot{\beta})$$
$$c(\alpha,\beta,\dot{\alpha},\dot{\beta}) = -K(\delta+\tau-\varepsilon) - \lambda(\dot{\delta}+\dot{\tau})$$

Where λ is the damping factor.

Put (3-51) in the order of increasing value of *j* to get:

$$\begin{aligned}
&\delta[-PJ\varpi\cos\phi - K + \varpi^2(Q+Q'-P')\cos\phi\cos(\phi+\gamma_0)] \\
&+\tau[-PJ\varpi\cos(\phi+\gamma_0) - K + \varpi^2(Q+Q'-P')\cos^2(\phi+\gamma_0)] \\
&+\dot{\delta}[-\lambda] \\
&+\dot{\tau}[-\lambda] \\
&+\dot{\mu}[PJ - \varpi(2Q+Q'+R'-P')\cos(\phi+\gamma_0)] \\
&+\ddot{\delta}[-(R'+Q)\cos\gamma_0] \\
&+\ddot{\tau}[-(R'+Q)] \\
&+K\varepsilon = 0
\end{aligned} \qquad (3\text{-}52)$$

If:

$$a_1^3 = -PJ\varpi\cos\phi - K + \varpi^2(Q+Q'-P')\cos\phi\cos(\phi+\gamma_0)$$

$$a_2^3 = -PJ\varpi\cos(\phi+\gamma_0) - K + \varpi^2(Q+Q'-P')\cos^2(\phi+\gamma_0)$$

$$a_6^3 = a_7^3 = -\lambda$$

$$a_8^3 = PJ - \varpi(2Q+Q'+R'-P')\cos(\phi+\gamma_0)$$
$$a_8^3 \approx PJ - \varpi(2Q+Q'+R'-P')\cos\phi$$

$$a_{11}^3 = -(R'+Q)\cos\gamma_0$$

$$a_{12}^3 = -(R'+Q) \approx a_{11}^3$$

Such that $-a_8^3$ is also equal to a_6^2 and a_7^2.

Then, equation (3-52) becomes:

(3-52`) $\delta(a_1^3) + \tau(a_2^3) + \dot{\delta}(a_6^3) + \dot{\tau}(a_7^3) + \dot{\mu}(a_8^3) + \ddot{\delta}(a_{11}^3) + \ddot{\tau}(a_{12}^3) = -K\varepsilon$

This is the third equation of oscillations.

3.3.5 Calculation of the fourth equation of oscillations:

We see from (2-94) that $|\vec{S}| = mg$ if d, e, f, r, $\dot{p}$, $\dot{q}$, $\dot{\bar{y}}_I$ and $\ddot{\bar{z}}_I$ are negligibly small. All of these are put equal to zero in the position of equilibrium. As $|\vec{S}|$ is multiplied in (2-92) and (2-93) by a negligibly small $(\sin\varphi)$, we only have to consider that $|\vec{S}| = mg$ for the substitution in these equations. By considering figure 4, the functions $\bar{x}_I$ and $\bar{y}_I$ will be introduced by these relations:

(3-53) $\sin\varphi\cos\Lambda = \dfrac{\bar{x}_I}{\ell}$ and $\sin\varphi\sin\Lambda = \dfrac{\bar{y}_I}{\ell}$ (Jeudy, 1982)

Taking into account of equations (3-26), (3-28), (3-30), (3-45), (3-45′), (3-46′) and (3-53), equation (2-92) becomes:

(3-54) $m(-2\varpi\dot{\bar{y}}_I\sin\phi + \ddot{\bar{x}}_I) + 2ml\varpi(\dot{\delta}\sin\phi\sin\gamma_0) + m(-\ddot{\mu}l\cos\gamma_0) = -mg\dfrac{\bar{x}_I}{\ell}$

Where all negligibly small second order terms have been ignored. Put (3-54) in the order of increasing value of *j* to get:

(3-55) $$\bar{x}_I(\frac{mg}{\ell}) + \dot{\delta}(2ml\varpi\sin\phi\sin\gamma_0) + \dot{\bar{y}}_I(-2m\varpi\sin\phi) + \ddot{\mu}(-ml\cos\gamma_0) + \ddot{\bar{x}}_I(m) = 0$$

If:

$a_4^4 = \dfrac{mg}{\ell}$

$a_6^4 = 2ml\varpi\sin\phi\sin\gamma_0$

$a_{10}^4 = -2m\varpi\sin\phi$

$a_{13}^4 = -ml\cos\gamma_0 \approx -ml$

$a_{14}^4 = m$

Then, equation (3-55) becomes:

(3-55`) $\bar{x}_I(a_4^4) + \dot{\delta}(a_6^4) + \dot{\bar{y}}_I(a_{10}^4) + \ddot{\mu}(a_{13}^4) + \ddot{\bar{x}}_I(a_{14}^4) = 0$

This is the fourth equation of oscillations.

3.3.6 Calculation of the fifth equation of oscillations:

In the same way as in equation (3-54), taking into account of equations (3-26), (3-28), (3-30), (3-45), (3-45′), (3-46′) and (3-53), equation (2-93) becomes:

(3-56)

$$m(2\varpi\dot{\bar{x}}_I \sin\phi + \ddot{\bar{y}}_I) - ml\dot{\mu}[2\varpi \sin(\phi+\gamma_0)]$$
$$+ 2ml\varpi(\delta \sin\gamma_0 + \delta\mu\cos\gamma_0)[\dot{\delta}\sin\phi + \dot{\tau}\sin(\phi+\gamma_0)]$$
$$+ ml[\ddot{\tau}\sin\gamma_0(\cos\gamma_0 - \mu\sin\gamma_0) - \ddot{\tau}\cos\gamma_0(\sin\gamma_0 + \mu\cos\gamma_0)] = \frac{-mg}{\ell}\bar{y}_I.$$

Finally (3-56) becomes, after removal of negligibly small second order terms:

$$\text{(3-57)} \quad \begin{aligned} &\bar{y}_I(\frac{mg}{\ell}) + \dot{\mu}[-2ml\varpi\sin(\phi+\gamma_0)] \\ &+ \dot{\bar{x}}_I(2m\varpi\sin\phi) + \ddot{\bar{y}}_I(m) = 0 \end{aligned}$$

If:

$$a_5^5 = \frac{mg}{\ell}$$

$$a_8^5 = -2ml\varpi\sin(\phi+\gamma_0)$$

$$a_9^5 = 2m\varpi\sin\phi$$

$$a_{15}^5 = m$$

Then, equation (3-57) becomes:

$$\text{(3-57')} \quad \bar{y}_I(a_5^5) + \dot{\mu}(a_8^5) + \dot{\bar{x}}_I(a_9^5) + \ddot{\bar{y}}_I(a_{15}^5) = 0$$

This is the fifth equation of oscillations.

So, the five equations obtained form the following system:

(3-58) $a_1^1\delta + a_2^1\tau + a_8^1\dot{\mu} + a_9^1\dot{\bar{x}}_I + a_{11}^1\ddot{\delta} + a\ddot{\bar{y}}_I = 0$ from (3-48′)

(3-59) $a_3^2\mu + b\dot{\delta} + b\dot{\tau} + c\dot{\bar{y}}_I + a_{13}^2\ddot{\mu} - a\ddot{\bar{x}}_I = 0$ from (3-50′)

(3-60) $t\delta + t'\tau - \lambda\dot{\delta} - \lambda\dot{\tau} - b\dot{\mu} + s\ddot{\delta} + s\ddot{\tau} = -K\varepsilon$ from (3-52′)

(3-61) $r\bar{x}_I + a_6^4\dot{\delta} + i\dot{\bar{y}}_I + a\ddot{\mu} + a'\ddot{\bar{x}}_I = 0$ from (3-55′)

(3-62) $r\bar{y}_I + c\dot{\mu} - i\dot{\bar{x}}_I + a'\ddot{\bar{y}}_I = 0$ from (3-57′)

Where the coefficients are introduced for simplification and they are defined as follows:

(3-63)

$$a_1^1 = \varpi^2 (C + R' - Q')\cos\phi \sin(\phi + \gamma_0)$$

$$a_2^1 \approx -m\lg\sin\gamma_0 + \varpi^2 (C + R' - Q')\sin\phi\cos\phi$$

$$a_8^1 = \varpi(P' + Q' - R' + 2L)\sin(\phi + \gamma_0)$$

$$a_9^1 = -2ml\varpi\sin\phi = li$$

$$a_{11}^1 = (L + P')\sin\gamma_0$$

$$a_3^2 \approx -m\lg - PJ\varpi\cos\phi$$

$$a_{13}^2 = -(L + Q + Q')$$

$$a_6^4 = 2ml\varpi\sin\phi\sin\gamma_0$$

$$a = -ml = a_{15}^1 \approx -a_{14}^2 \approx a_{13}^4$$

$$b = -PJ + \varpi(2Q + Q' + R' - P')\cos\phi \approx a_6^2 \approx a_7^2 \approx -a_8^3$$

$$c = -2ml\varpi\sin(\phi + \gamma_0) = a_{10}^2 = a_8^5$$

$$t = -PJ\varpi\cos\phi - K + \varpi^2(Q + Q' - P')\cos^2\phi \approx a_1^3$$

$$t' = -PJ\varpi\cos\phi - K + \varpi^2(Q + Q' - P')\cos^2(\phi + \gamma_0) \approx a_2^3$$

$$s = -(R' + Q) = a_{12}^3 \approx a_{11}^3$$

$$r = \frac{mg}{\ell} = a_4^4 = a_5^5$$

$$i = -2m\varpi\sin\phi = a_{10}^4 = -a_9^5$$

$$a' = m = a_{14}^4 = a_{15}^5$$

$$a_6^3 = a_7^3 = -\lambda$$

The calculations made so far in this paragraph (3.3) are summarised above. Out of the eight movement equations obtained in Chapter 2, there are only five linear differential equations with constant coefficients and five unknown functions of time remaining. The first three linear equations correspond to the three Euler equations (2-65), (2-66) and (2-67), where the unknown ω_1 has been eliminated with the aid of equation (2-95). The fourth equation of oscillations is obtained from equation (2-92), where the tape tension was eliminated with the aid of equation (2-94) and consideration of the negligibly small terms such as those explained in (3.3.5). The fifth equation of oscillations is obtained from equation (2-93). Equation (2-68) does not give rise to a linear equation because it contains terms, which are all of second order. Th function $\bar{z}_I$ may be ignored according to the reasoning in (3.3.1.3).

The system of equations (3-58) to (3-62) could be resolved in different ways. The complete solution is the sum of a particular solution and a general solution for the whole system. The particular solution as it is given by Jeudy (Jeudy, 1982) is:

$\bar{x}_I = \bar{y}_I = \mu = 0$

and:

$\delta = \delta_0$, $\tau = \tau_0$

The general solution is found in an exponential form $e^{\eta t}$ where η is a coefficient to be determined. Direct substitution into the system leads to the calculation of five equations in five unknowns. However, it is easier if we eliminate three unknowns μ, $\bar{x}_I$ and $\bar{y}_I$, as this reduces the system of equations to two equations with two unknowns. This second approach is adopted because it is simple. The characteristic equation is a polynomial in η to powers of ten.

3.3.7 Elimination of the function μ:

From (3-60):

$$(3\text{-}64)\quad \dot{\mu} = \frac{1}{b}(t\delta + t'\tau - \lambda\dot{\delta} - \lambda\dot{\tau} + s\ddot{\delta} + s\ddot{\tau} + K\varepsilon)$$

This equation will be substituted into each of the remaining equations, for example, (3-58), (3-59), (3-61) and (3-62). Substitute (3-64) into (3-58) to give:

$$(3\text{-}65)\quad \begin{aligned} &\delta(a_1^1 + a_8^1\frac{t}{b}) + \tau(a_2^1 + a_8^1\frac{t'}{b}) + \dot{\delta}(-\frac{\lambda}{b}a_8^1) + \dot{\tau}(-\frac{\lambda}{b}a_8^1) \\ &+ \dot{\bar{x}}_I(a_9^1) + \ddot{\delta}(a_{11}^1 + a_8^1\frac{s}{b}) + \ddot{\tau}(a_8^1\frac{s}{b}) + \ddot{\bar{y}}_I(a) + (a_8^1\frac{K\varepsilon}{b}) = 0 \end{aligned}$$

Differentiate (3-59) and substitute from (3-64) to give:

$$(3\text{-}66)\quad \begin{aligned} &\frac{a_3^2}{b}(t\delta + t'\tau - \lambda\dot{\delta} - \lambda\dot{\tau} + s\ddot{\delta} + s\ddot{\tau} + K\varepsilon) + b\ddot{\delta} + b\ddot{\tau} + c\ddot{\bar{y}}_I \\ &+ \frac{a_{13}^2}{b}(t\ddot{\delta} + t'\ddot{\tau} - \lambda\dddot{\delta} - \lambda\dddot{\tau} + s\widetilde{\delta} + s\widetilde{\tau}) - a\dddot{\bar{x}}_I = 0 \end{aligned}$$

Differentiate (3-64) and substitute it into (3-61) to give:

$$(3\text{-}67)\quad r\bar{x}_I + a_6^4\dot{\delta} + i\dot{\bar{y}}_I + \frac{a}{b}(t\dot{\delta} + t'\dot{\tau} - \lambda\ddot{\delta} - \lambda\ddot{\tau} + s\dddot{\delta} + s\dddot{\tau}) + a'\ddot{\bar{x}}_I = 0$$

Substitute (3-64) into (3-62) to give:

$$(3\text{-}68)\quad r\bar{y}_I + \frac{c}{b}(t\delta + t'\tau - \lambda\dot{\delta} - \lambda\dot{\tau} + s\ddot{\delta} + s\ddot{\tau} + K\varepsilon) - i\dot{\bar{x}}_I + a'\ddot{\bar{y}}_I = 0$$

3.3.8 Elimination of the function $\bar{x}_I$:

From (3-68):

$$(3\text{-}69)\quad \dot{\bar{x}}_I = \frac{1}{i}[\frac{c}{b}(t\delta + t'\tau - \lambda\dot{\delta} - \lambda\dot{\tau} + s\ddot{\delta} + s\ddot{\tau} + K\varepsilon) + r\bar{y}_I + a'\ddot{\bar{y}}_I]$$

This equation will be substituted into each of the remaining equations, for example, (3-65), (3-66) and (3-67). Substitute (3-69) into (3-65), after having taken account of the fact that $\frac{a_9^1}{i} = l$, the result is:

$$(3\text{-}70)\quad \begin{aligned}&\delta(a_1^1 + a_8^1\frac{t}{b} + lc\frac{t}{b}) + \tau(a_2^1 + a_8^1\frac{t'}{b} + lc\frac{t'}{b}) \\ &+\dot{\delta}(\frac{-\lambda}{b}a_8^1 - \frac{\lambda}{b}lc) + \dot{\tau}(\frac{-\lambda}{b}a_8^1 - \frac{\lambda}{b}lc) \\ &+\ddot{\delta}(a_{11}^1 + a_8^1\frac{s}{b} + lc\frac{s}{b}) + \ddot{\tau}(a_8^1\frac{s}{b} + lc\frac{s}{b}) \\ &+\bar{y}_I(lr) + \ddot{\bar{y}}_I(a + a'l) + [(\frac{a_8^1}{b} + \frac{lc}{b})K\varepsilon] = 0\end{aligned}$$

The coefficient of $\ddot{\bar{y}}_I$ is zero as one can see by referring to the value of coefficient, a, given in (3-63). For reasons of simplification, the coefficients of (3-70) are indicated by b_j^i where superscript i denotes the order in which the equations appear below (1, 2 and 3) and subscript j denotes the order of the function as previously defined for coefficients a_j^i:

$$b_1^1 = a_1^1 + (a_8^1 + lc)\frac{t}{b}$$

$$b_2^1 = a_2^1 + (a_8^1 + lc)\frac{t'}{b}$$

$$b_6^1 = -\frac{\lambda}{b}(a_8^1 + lc) = b_7^1$$

$$b_{11}^1 = a_{11}^1 + (a_8^1 + lc)\frac{s}{b}$$

$$b_{12}^1 = (a_8^1 + lc)\frac{s}{b}$$

$$b_5^1 = lr = \frac{m\lg}{\ell}$$

$$(\frac{a_8^1 + lc}{b})K\varepsilon = C_1$$

One can see from (3-63) that:

$$a_8^1 + lc = \varpi(P' + Q' - R' + 2L - 2ml^2)\sin(\phi + \gamma_0)$$

Then, equation (3-70) may be written with coefficients b_j^i as:

$$(3\text{-}71)\quad b_1^1\delta + b_2^1\tau + b_6^1\dot{\delta} + b_7^1\dot{\tau} + b_{11}^1\ddot{\delta} + b_{12}^1\ddot{\tau} + b_5^1\bar{y}_I + C_1 = 0$$

Differentiate (3-69) twice and substitute it into (3-66) to get:

$$\delta(\frac{a_3^2 t}{b}) + \tau(\frac{a_3^2 t'}{b}) + \dot{\delta}(-\frac{\lambda}{b}a_3^2) + \dot{\tau}(-\frac{\lambda}{b}a_3^2)$$

$$+\ddot{\delta}(\frac{a_3^2 s}{b} + b + \frac{a_{13}^2 t}{b} - \frac{act}{ib}) + \ddot{\tau}(\frac{a_3^2 s}{b} + b + \frac{a_{13}^2 t'}{b} - \frac{act'}{ib})$$

$$+\dddot{\delta}(-\frac{\lambda}{b}a_{13}^2 + \frac{\lambda}{ib}ac) + \dddot{\tau}(-\frac{\lambda}{b}a_{13}^2 + \frac{\lambda}{ib}ac) \tag{3-72}$$

$$+\widetilde{\delta}(\frac{a_{13}^2 s}{b} - \frac{asc}{ib}) + \widetilde{\tau}(\frac{a_{13}^2 s}{b} - \frac{asc}{ib})$$

$$+\ddot{\bar{y}}_I(c - \frac{ar}{i}) + \widetilde{\bar{y}}_I(-\frac{aa'}{i}) + \frac{a_3^2}{b}K\varepsilon = 0$$

If:

(3-73)

$$b_1^2 = a_3^2 \frac{t}{b}$$

$$b_2^2 = a_3^2 \frac{t'}{b}$$

$$b_6^2 = b_7^2 = -\frac{\lambda}{b}a_3^2$$

$$b_{11}^2 = [a_3^2 \frac{s}{b} + b + (a_{13}^2 - \frac{a}{i}c)\frac{t}{b}]$$

$$b_{12}^2 = [a_3^2 \frac{s}{b} + b + (a_{13}^2 - \frac{a}{i}c)\frac{t'}{b}]$$

$$b_{16}^2 = b_{17}^2 = \frac{\lambda}{b}(\frac{ac}{i} - a_{13}^2)$$

$$b_{21}^2 = b_{22}^2 = [(a_{13}^2 - \frac{a}{i}c)\frac{s}{b}]$$

$$b_{15}^2 = c - \frac{a}{i}r \approx \frac{-m\lg}{2\varpi\ell\sin\phi}$$

$$b_{25}^2 = -\frac{a}{i}a' = \frac{-ml}{2\varpi\sin\phi}$$

Then, equation (3-72) may be written as:

$$b_1^2\delta + b_2^2\tau + b_6^2\dot{\delta} + b_7^2\dot{\tau} + b_{11}^2\ddot{\delta} + b_{12}^2\ddot{\tau} + b_{16}^2\dddot{\delta} + b_{17}^2\dddot{\tau}$$
$$+ b_{21}^2\widetilde{\delta} + b_{22}^2\widetilde{\tau} + b_{15}^2\ddot{\bar{y}}_I + b_{25}^2\widetilde{\bar{y}}_I + \frac{a_3^2}{b}K\varepsilon = 0 \tag{3-74}$$

Differentiate (3-67) and substitute from (3-69) and also from (3-69) differentiated twice to get:

$$
\begin{aligned}
&\delta(\frac{r}{i}\frac{c}{b}t)+\tau(\frac{r}{i}\frac{c}{b}t')+\dot{\delta}(-\frac{\lambda}{ib}rc)+\dot{\tau}(-\frac{\lambda}{ib}rc)\\
&+\ddot{\delta}(a_6^4+t\frac{a}{b}+\frac{r}{i}\frac{c}{b}s+\frac{a'}{i}\frac{c}{b}t]+\ddot{\tau}(\frac{r}{i}\frac{c}{b}s+\frac{a}{b}t'+\frac{a'}{i}\frac{c}{b}t')\\
\text{(3-75)}\quad &+\dddot{\delta}(-\frac{\lambda}{b}a-\frac{\lambda}{ib}a'c)+\dddot{\tau}(-\frac{\lambda}{b}a-\frac{\lambda}{ib}a'c)\\
&+\widetilde{\delta}(\frac{a}{b}s+\frac{c}{b}s\frac{a'}{i})+\widetilde{\tau}(\frac{a}{b}s+\frac{a'}{i}\frac{c}{b}s)+\bar{y}_I(\frac{r^2}{i})\\
&+\ddot{\bar{y}}_I(i+\frac{2ra'}{i})+\widetilde{\bar{y}}_I(\frac{a'^2}{i})+(\frac{r}{i}\frac{c}{b}K\varepsilon)=0
\end{aligned}
$$

If:

$$b_1^3=\frac{r}{i}\frac{c}{b}t$$

$$b_2^3=r\frac{c}{ib}t'$$

$$b_6^3=b_7^3=-\frac{\lambda}{ib}rc$$

$$b_{11}^3=[a_6^4+(a+a'\frac{c}{i})\frac{t}{b}+\frac{r}{i}\frac{c}{b}s]$$

$$b_{12}^3=[(a+a'\frac{c}{i})\frac{t'}{b}+\frac{r}{i}\frac{c}{b}s]$$

$$b_{16}^3=b_{17}^3=-\frac{\lambda}{b}(a+\frac{a'c}{i})$$

$$b_{21}^3=b_{22}^3=[(a+a'\frac{c}{i})\frac{s}{b}]$$

$$b_5^3=\frac{r^2}{i}$$

$$b_{15}^3=(i+\frac{2a'r}{i})$$

$$b_{25}^3=\frac{a'^2}{i}$$

Equation (3-75) now becomes:

$$
\begin{aligned}
&b_1^3\delta+b_2^3\tau+b_6^3\dot{\delta}+b_7^3\dot{\tau}+b_{11}^3\ddot{\delta}+b_{12}^3\ddot{\tau}+b_{16}^3\dddot{\delta}+b_{17}^3\dddot{\tau}\\
\text{(3-76)}\quad &+b_{21}^3\widetilde{\delta}+b_{22}^3\widetilde{\tau}+b_5^3\bar{y}_I+b_{15}^3\ddot{\bar{y}}_I+b_{25}^3\widetilde{\bar{y}}_I+\frac{r}{i}\frac{c}{b}K\varepsilon=0
\end{aligned}
$$

3.3.9 Elimination of the function $\bar{y}_I$:

From (3-71):

$$\text{(3-77)} \quad \bar{y}_I = \frac{-1}{b_5^1}(b_1^1\delta + b_2^1\tau + b_6^1\dot{\delta} + b_7^1\dot{\tau} + b_{11}^1\ddot{\delta} + b_{12}^1\ddot{\tau} + C_1)$$

Where: $C_1 = (\frac{a_8^1 + lc}{b})K\varepsilon$

In the same way as in the previous paragraph, this equation (3-77) will be substituted into each of the remaining equations, for example, (3-74) and (3-76). Substitute the second and fourth derivatives of (3-77) into (3-74) to get:

$$\text{(3-78)} \quad \begin{aligned} & b_1^2\delta + b_6^2\dot{\delta} + \ddot{\delta}(b_{11}^2 - \frac{b_{15}^2}{b_5^1}b_1^1) + \dddot{\delta}(b_{16}^2 - \frac{b_{15}^2}{b_5^1}b_6^1) \\ & + \tilde{\delta}(b_{21}^2 - \frac{b_{15}^2}{b_5^1}b_{11}^1 - \frac{b_{25}^2}{b_5^1}b_1^1) + \breve{\delta}(-\frac{b_{25}^2}{b_5^1}b_6^1) + \hat{\delta}(-\frac{b_{25}^2}{b_5^1}b_{11}^1) \\ & + b_2^2\tau + b_7^2\dot{\tau} + \ddot{\tau}(b_{12}^2 - \frac{b_{15}^2}{b_5^1}b_2^1) + \dddot{\tau}(b_{17}^2 - \frac{b_{15}^2}{b_5^1}b_7^1) + \\ & \tilde{\tau}(b_{22}^2 - \frac{b_{15}^2}{b_5^1}b_{12}^1 - \frac{b_{25}^2}{b_5^1}b_2^1) + \breve{\tau}(-\frac{b_{25}^2}{b_5^1}b_7^1)\hat{\tau}(-\frac{b_{25}^2}{b_5^1}b_{12}^1) + C_2 = 0 \end{aligned}$$

Where symbols ~, ∨ and ∧ denote the fourth, fifth and sixth derivatives respectively and:

$$C_2 = \frac{a_3^2}{b}K\varepsilon$$

Again for reasons of simplification, suppose symbols c_j^i denote the coefficients of (3-78) where superscript i stands for the order in which the equations appear below (and takes the value 1 or 2) and subscript j the order of the function in accordance with the following table:

δ	$\dot{\delta}$	$\ddot{\delta}$	$\dddot{\delta}$	$\tilde{\delta}$	$\breve{\delta}$	$\hat{\delta}$	τ	$\dot{\tau}$	$\ddot{\tau}$	$\dddot{\tau}$	$\tilde{\tau}$	$\breve{\tau}$	$\hat{\tau}$	function
1	2	3	4	5	6	7	8	9	10	11	12	13	14	j

Then, equation (3-78) becomes:

(3-79)
$$\delta c_1^1 + \dot{\delta} c_2^1 + \ddot{\delta} c_3^1 + \dddot{\delta} c_4^1 + \tilde{\delta} c_5^1 + \breve{\delta} c_6^1 + \hat{\delta} c_7^1$$
$$+ \tau c_8^1 + \dot{\tau} c_9^1 + \ddot{\tau} c_{10}^1 + \dddot{\tau} c_{11}^1 + \tilde{\tau} c_{12}^1 + \breve{\tau} c_{13}^1 + \hat{\tau} c_{14}^1 + C_2 = 0$$

Where:

(3-80)

$$c_1^1 = b_1^2$$

$$c_2^1 = b_6^2$$

$$c_3^1 = b_{11}^2 - \frac{b_{15}^2}{b_5^1} b_1^1$$

$$c_4^1 = b_{16}^2 - \frac{b_{15}^2}{b_5^1} b_6^1$$

$$c_5^1 = b_{21}^2 - \frac{b_{15}^2}{b_5^1} b_{11}^1 - \frac{b_{25}^2}{b_5^1} b_1^1$$

$$c_6^1 = -\frac{b_{25}^2}{b_5^1} b_6^1$$

$$c_7^1 = -\frac{b_{25}^2}{b_5^1} b_{11}^1$$

$$c_8^1 = b_2^2$$

$$c_9^1 = b_7^2$$

$$c_{10}^1 = b_{12}^2 - \frac{b_{15}^2}{b_5^1} b_2^1$$

$$c_{11}^1 = b_{17}^2 - \frac{b_{15}^2}{b_5^1} b_7^1$$

$$c_{12}^1 = b_{22}^2 - \frac{b_{15}^2}{b_5^1} b_{12}^1 - \frac{b_{25}^2}{b_5^1} b_2^1$$

$$c_{13}^1 = -\frac{b_{25}^2}{b_5^1} b_7^1$$

$$c_{14}^1 = -\frac{b_{25}^2}{b_5^1} b_{12}^1$$

Substitute equation (3-77) and its second and fourth derivatives into equation (3-76) to get:

$$
\begin{aligned}
(3\text{-}81)\quad & \delta(b_1^3 - \frac{b_5^3}{b_5^1} b_1^1) + \tau(b_2^3 - \frac{b_5^3}{b_5^1} b_2^1) + \dot{\delta}(b_6^3 - \frac{b_5^3}{b_5^1} b_6^1) + \dot{\tau}(b_7^3 - \frac{b_5^3}{b_5^1} b_7^1) \\
& + \ddot{\delta}(b_{11}^3 - \frac{b_5^3}{b_5^1} b_{11}^1 - \frac{b_{15}^3}{b_5^1} b_1^1) + \ddot{\tau}(b_{12}^3 - \frac{b_5^3}{b_5^1} b_{12}^1 - \frac{b_{15}^3}{b_5^1} b_2^1) \\
& + \dddot{\delta}(b_{16}^3 - \frac{b_{15}^3}{b_5^1} b_6^1) + \dddot{\tau}(b_{17}^3 - \frac{b_{15}^3}{b_5^1} b_7^1) \\
& + \widetilde{\delta}(b_{21}^3 - \frac{b_{15}^3}{b_5^1} b_{11}^1 - \frac{b_{25}^3}{b_5^1} b_1^1) + \widetilde{\tau}(b_{22}^3 - \frac{b_{15}^3}{b_5^1} b_{12}^1 - \frac{b_{25}^3}{b_5^1} b_2^1) \\
& + \breve{\delta}(-\frac{b_{25}^3}{b_5^1} b_6^1) + \breve{\tau}(-\frac{b_{25}^3}{b_5^1} b_7^1) \\
& + \hat{\delta}(-\frac{b_{25}^3}{b_5^1} b_{11}^1) + \hat{\tau}(-\frac{b_{25}^3}{b_5^1} b_{12}^1) + (-\frac{b_5^3}{b_5^1} C_1 + \frac{r}{i}\frac{c}{b} K\varepsilon) = 0
\end{aligned}
$$

Where:

$$C_1 = \frac{a_8^1 + lc}{b} K\varepsilon$$

Writing (3-81) with coefficients c_j^i:

$$
\begin{aligned}
(3\text{-}82)\quad & \delta c_1^2 + \dot{\delta} c_2^2 + \ddot{\delta} c_3^2 + \dddot{\delta} c_4^2 + \widetilde{\delta} c_5^2 + \breve{\delta} c_6^2 + \hat{\delta} c_7^2 \\
& + \tau c_8^2 + \dot{\tau} c_9^2 + \ddot{\tau} c_{10}^2 + \dddot{\tau} c_{11}^2 + \widetilde{\tau} c_{12}^2 + \breve{\tau} c_{13}^2 + \hat{\tau} c_{14}^2 + C_3 = 0
\end{aligned}
$$

Where:

$$C_3 = -\frac{b_5^3}{b_5^1} C_1 + \frac{r}{i}\frac{c}{b} K\varepsilon$$

The coefficients of (3-82) have the following expressions:

(3-83)

$$c_1^2 = b_1^3 - \frac{b_5^3}{b_5^1} b_1^1$$

$$c_2^2 = b_6^3 - \frac{b_5^3}{b_5^1} b_6^1$$

$$c_3^2 = b_{11}^3 - \frac{b_5^3}{b_5^1} b_{11}^1 - \frac{b_{15}^3}{b_5^1} b_1^1$$

$$c_4^2 = b_{16}^3 - \frac{b_{15}^3}{b_5^1} b_6^1$$

$$c_5^2 = b_{21}^3 - \frac{b_{15}^3}{b_5^1} b_{11}^1 - \frac{b_{25}^3}{b_5^1} b_1^1$$

$$c_6^2 = -\frac{b_{25}^3}{b_5^1} b_6^1$$

$$c_7^2 = -\frac{b_{25}^3}{b_5^1} b_{11}^1$$

$$c_8^2 = b_2^3 - \frac{b_5^3}{b_5^1} b_2^1$$

$$c_9^2 = b_7^3 - \frac{b_5^3}{b_5^1} b_7^1$$

$$c_{10}^2 = b_{12}^3 - \frac{b_5^3}{b_5^1} b_{12}^1 - \frac{b_{15}^3}{b_5^1} b_2^1$$

$$c_{11}^2 = b_{17}^3 - \frac{b_{15}^3}{b_5^1} b_7^1$$

$$c_{12}^2 = b_{22}^3 - \frac{b_{15}^3}{b_5^1} b_{12}^1 - \frac{b_{25}^3}{b_5^1} b_2^1$$

$$c_{13}^2 = \frac{-b_{25}^3 b_7^1}{b_5^1}$$

$$c_{14}^2 = \frac{-b_{25}^3 b_{12}^1}{b_5^1}$$

3.3.10 Resolving the system of differential equations:

The system defined by equations (3-82) and (3-79) is a system of two differential linear equations, of the sixth order, with constant coefficients:

(3-84)

$$\delta c_1^1 + \dot{\delta} c_2^1 + \ddot{\delta} c_3^1 + \dddot{\delta} c_4^1 + \tilde{\delta} c_5^1 + \breve{\delta} c_6^1 + \hat{\delta} c_7^1 + \tau c_8^1 + \dot{\tau} c_9^1 + \ddot{\tau} c_{10}^1 + \dddot{\tau} c_{11}^1 + \tilde{\tau} c_{12}^1 + \breve{\tau} c_{13}^1 + \hat{\tau} c_{14}^1 + C_2 = 0$$

$$\delta c_1^2 + \dot{\delta} c_2^2 + \ddot{\delta} c_3^2 + \dddot{\delta} c_4^2 + \tilde{\delta} c_5^2 + \breve{\delta} c_6^2 + \hat{\delta} c_7^2 + \tau c_8^2 + \dot{\tau} c_9^2 + \ddot{\tau} c_{10}^2 + \dddot{\tau} c_{11}^2 + \tilde{\tau} c_{12}^2 + \breve{\tau} c_{13}^2 + \hat{\tau} c_{14}^2 + C_3 = 0$$

The solution of these two equations is of the type:

(3-85) $\delta = \delta' e^{\eta t}$ and $\tau = \tau' e^{\eta t}$ (Tierney, 1985)

Where, t is the time, δ' and τ' are constants, their values are not zero and η is a parameter, which needs to be determined.

Substitute δ and τ from (3-85) into the equations (3-84) to get:

(3-86)

$$\delta'(c_1^1 + \eta c_2^1 + \eta^2 c_3^1 + \eta^3 c_4^1 + \eta^4 c_5^1 + \eta^5 c_6^1 + \eta^6 c_7^1) + \tau'(c_8^1 + \eta c_9^1 + \eta^2 c_{10}^1 + \eta^3 c_{11}^1 + \eta^4 c_{12}^1 + \eta^5 c_{13}^1 + \eta^6 c_{14}^1) = 0$$

$$\delta'(c_1^2 + \eta c_2^2 + \eta^2 c_3^2 + \eta^3 c_4^2 + \eta^4 c_5^2 + \eta^5 c_6^2 + \eta^6 c_7^2) + \tau'(c_8^2 + \eta c_9^2 + \eta^2 c_{10}^2 + \eta^3 c_{11}^2 + \eta^4 c_{12}^2 + \eta^5 c_{13}^2 + \eta^6 c_{14}^2) = 0$$

The constants C_2 and C_3 may be obtained from the initial conditions, for example, $\delta, \tau, \mu, \bar{x}_I, \bar{y}_I, \dot{\delta}, \dot{\tau}, \dot{\mu}, \dot{\bar{x}}_I, \dot{\bar{y}}_I$ and $t = 0$. Thus, from (3-84), these two equations may be written:

$\delta_0 c_1^1 + \tau_0 c_8^1 + C_2 = 0$ and $\delta_0 c_1^2 + \tau_0 c_8^2 + C_3 = 0$

The equations (3-86) could perhaps be considered as forming a system of two equations with two unknowns δ' and τ'. Writing (3-86) into matrix form:

$$\text{(3-86')} \quad \begin{bmatrix} (c_1^1 + \eta c_2^1 + \ldots\ldots + \eta^6 c_7^1) & (c_8^1 + \eta c_9^1 + \ldots\ldots\ldots\ldots + \eta^6 c_{14}^1) \\ (c_1^2 + \eta c_2^2 + \ldots\ldots + \eta^6 c_7^2) & (c_8^2 + \eta c_9^2 + \ldots\ldots\ldots\ldots + \eta^6 c_{14}^2) \end{bmatrix} \begin{bmatrix} \delta' \\ \tau' \end{bmatrix} = 0$$

To find solutions for δ' and τ' where neither of them are zero, the determinant of the matrix system in (3-86') must be zero. The determinant is:

(3-87)

$$\eta^{12}(c_7^1c_{14}^2 - c_7^2c_{14}^1) +$$

$$\eta^{11}(c_6^1c_{14}^2 + c_7^1c_{13}^2 - c_7^2c_{13}^1 - c_6^2c_{14}^1) +$$

$$\eta^{10}(c_6^1c_{13}^2 + c_5^1c_{14}^2 + c_7^1c_{12}^2 - c_6^2c_{13}^1 - c_5^2c_{14}^1 - c_7^2c_{12}^1) +$$

$$\eta^9(c_6^1c_{12}^2 + c_5^1c_{13}^2 + c_7^1c_{11}^2 + c_4^1c_{14}^2 - c_{12}^1c_6^2 - c_{13}^1c_5^2 - c_{11}^1c_7^2 - c_{14}^1c_4^2) +$$

$$\eta^8(c_5^1c_{12}^2 + c_7^1c_{10}^2 + c_3^1c_{14}^2 + c_6^1c_{11}^2 + c_4^1c_{13}^2 - c_5^2c_{12}^1 - c_7^2c_{10}^1 - c_3^2c_{14}^1 - c_6^2c_{11}^1 - c_4^2c_{13}^1) +$$

$$\eta^7(c_7^1c_9^2 + c_2^1c_{14}^2 + c_6^1c_{10}^2 + c_3^1c_{13}^2 + c_5^1c_{11}^2 + c_4^1c_{12}^2 - c_7^2c_9^1 - c_2^2c_{14}^1 - c_6^2c_{10}^1 - c_3^2c_{13}^1 - c_5^2c_{11}^1 - c_4^2c_{12}^1) +$$

$$\eta^6(c_7^1c_8^2 + c_1^1c_{14}^2 + c_6^1c_9^2 + c_2^1c_{13}^2 + c_5^1c_{10}^2 + c_3^1c_{12}^2 + c_4^1c_{11}^2$$
$$- c_8^1c_7^2 - c_{14}^1c_1^2 - c_9^1c_6^2 - c_{13}^1c_2^2 - c_5^2c_{10}^1 - c_3^2c_{12}^1 - c_4^2c_{11}^1) +$$

$$\eta^5(c_6^1c_8^2 + c_1^1c_{13}^2 + c_5^1c_9^2 + c_2^1c_{12}^2 + c_4^1c_{10}^2 + c_3^1c_{11}^2 - c_8^1c_6^2 - c_{13}^1c_1^2 - c_9^1c_5^2 - c_{12}^1c_2^2 - c_4^2c_{10}^1 - c_3^2c_{11}^1) +$$

$$\eta^4(c_5^1c_8^2 + c_1^1c_{12}^2 + c_4^1c_9^2 + c_2^1c_{11}^2 + c_3^1c_{10}^2 - c_8^1c_5^2 - c_{12}^1c_1^2 - c_9^1c_4^2 - c_2^2c_{11}^1 - c_3^2c_{10}^1) +$$

$$\eta^3(c_4^1c_8^2 + c_1^1c_{11}^2 + c_3^1c_9^2 + c_2^1c_{10}^2 - c_8^1c_4^2 - c_{11}^1c_1^2 - c_9^1c_3^2 - c_{10}^1c_2^2) +$$

$$\eta^2(c_3^1c_8^2 + c_2^1c_9^2 + c_1^1c_{10}^2 - c_8^1c_3^2 - c_9^1c_2^2 - c_{10}^1c_1^2) +$$

$$\eta(c_1^1c_9^2 + c_2^1c_8^2 - c_8^1c_2^2 - c_9^1c_1^2) +$$

$$(c_1^1c_8^2 - c_8^1c_1^2) = 0$$

Taking into account of (3-80) and (3-83), it can be shown that the coefficients of η^{12} and η^{11} are zero. The coefficient of η^{12} is:

(3-88) $$c_7^1c_{14}^2 - c_7^2c_{14}^1 = \left(\frac{-b_{25}^2}{b_5^1}b_{11}^1\right)\left(\frac{-b_{25}^3}{b_5^1}b_{12}^1\right) - \left(\frac{-b_{25}^3}{b_5^1}b_{11}^1\right)\left(\frac{-b_{25}^2}{b_5^1}b_{12}^1\right) = 0$$

The coefficient of η^{11} is:

(3-89)

$$c_6^1c_{14}^2 + c_7^1c_{13}^2 - c_7^2c_{13}^1 - c_6^2c_{14}^1 = [(\frac{-b_{25}^2}{b_5^1}b_6^1)(\frac{-b_{25}^3}{b_5^1}b_{12}^1) + (\frac{-b_{25}^2}{b_5^1}b_{11}^1)(\frac{-b_{25}^3}{b_5^1}b_7^1)]$$

$$-[(\frac{-b_{25}^3}{b_5^1}b_{11}^1)(\frac{-b_{25}^2}{b_5^1}b_7^1) - (\frac{-b_{25}^3}{b_5^1}b_6^1)(\frac{-b_{25}^2}{b_5^1}b_{12}^1)] = 0$$

To aid simplification, replace the coefficients of the parameter η in equation (3-87) by symbols d_j, where j is the power of the parameter η:

$$\begin{aligned}&\eta^{10}(d_{10}) + \eta^9(d_9) + \eta^8(d_8) + \eta^7(d_7) + \eta^6(d_6) + \eta^5(d_5) \\ &+ \eta^4(d_4) + \eta^3(d_3) + \eta^2(d_2) + \eta(d_1) + C_4 = 0\end{aligned} \quad (3\text{-}90)$$

Where $C_4 = c_1^1c_8^2 - c_8^1c_1^2 = b_1^2(b_2^3 - \frac{b_5^3}{b_5^1}b_2^1) - b_2^2(b_1^3 - \frac{b_5^3}{b_5^1}b_1^1)$

3.3.11 General solution of the system of equations (3-84):

The solution of the system of equations (3-84) is of the type:

$\delta = \delta' e^{\eta t}$ and $\tau = \tau' e^{\eta t}$. The characteristic equation (3-90), obtained in the previous paragraph (3.3.10) is a polynomial in η to powers of ten:

$$d_{10}\eta^{10} + d_9\eta^9 + d_8\eta^8 + d_7\eta^7 + d_6\eta^6 + d_5\eta^5 + d_4\eta^4 + d_3\eta^3 + d_2\eta^2 + d_1\eta + C_4 = 0$$

Where d_1, d_2,....., d_{10} and C_4 are constants. The roots of the above equation, in general, are η_i where, i = 1, 2, 3, 4, 5,....10.

In the same way as explained in (Jeudy, 1982), suppose the frequencies of oscillations are equal to squares roots of η_i, $\theta_i = \sqrt{\eta_i}$, then, we have ten frequencies. The general solution of the system of equations (3-84) is of the form of sine and cosine waves of differing periods:

$$\begin{aligned}\delta &= \delta_0 + \sum_{i=1}^{10}(A_i' \cos\theta_i t + A_i'' \sin\theta_i t) \\ \tau &= \tau_0 + \sum_{i=1}^{10}\sigma_i(A_i' \cos\theta_i t + A_i'' \sin\theta_i t)\end{aligned} \quad (3\text{-}91)$$

Where, A'_i, A''_i, δ_0, τ_0 and σ_0 are constant. A'_i and A''_i are arbitrary constants and depend on the initial conditions. δ_0 and τ_0 are determined by equations:

$$\delta_0 c_1^1 + \tau_0 c_8^1 + C_2 = 0$$

$$\delta_0 c_1^2 + \tau_0 c_8^2 + C_3 = 0$$

These equations are equivalent to equations (3-35) and (3-36). The justification may be obtained by substituting the constants c_1^1, c_8^1, c_1^2, c_8^2, C_2 and C_3.

Finally, σ_i may be found from (3-86) by the formula:

$$\sigma_i = \frac{-(c_1^1 + \eta_i c_2^1 + \eta_i^2 c_3^1 + \eta_i^3 c_4^1 + \eta_i^4 c_5^1 + \eta_i^5 c_6^1 + \eta_i^6 c_7^1)}{(c_8^1 + \eta_i c_9^1 + \eta_i^2 c_{10}^1 + \eta_i^3 c_{11}^1 + \eta_i^4 c_{12}^1 + \eta_i^5 c_{13}^1 + \eta_i^6 c_{14}^1)}$$

The system of equations (3-91) may be replaced by:

$$\delta = \delta_0 + \sum_{i=1}^{10} [A_i e^{-\lambda_i t} \cos(\theta_i (t - B_i))]$$

(3-92)

$$\tau = \tau_0 + \sum_{i=1}^{10} \sigma_0 [A_i e^{-\lambda_i t} \cos(\theta_i (t - B_i))]$$

Since:

$$\sin\theta_i t = \cos(\frac{\pi}{2} - \theta_i t) \quad \text{and} \quad \cos\theta_i t + \cos(\frac{\pi}{2} - \theta_i t) = \sqrt{2}\cos(\theta_i t - \frac{\pi}{4})$$

And taking account of a damping factor $e^{-\lambda t}$ (King, 1987), which is introduced because of friction and air resistance damping the oscillation lightly in practice.

The frequencies depend, among other factors, on ω_1 the angular velocity of the gyroscope. It was shown by Gregerson (Gregerson, 1971a) that any change in the power driving the motor of gyro spinner produces a change in the period of oscillation, which is also affected by temperature. In the field, for example, in a tunnel or mine the power source is a battery, which with time runs down. Suppose when the battery runs down the period of oscillation is small and of the form of a linear function with time, then the term θ_i in equations (3-92) may be replaced by (θ_i + $\theta'_i t$). The system of equations (3-92) becomes:

$$\delta = \delta_0 + \sum_{i=1}^{10} [A_i e^{-\lambda_i t} \cos((\theta_i + \theta_i' t)(t - B_i))]$$

(3-93)

$$\tau = \tau_0 + \sum_{i=1}^{10} \sigma_0 [A_i e^{-\lambda_i t} \cos((\theta_i + \theta_i' t)(t - B_i))]$$

Where:

A_i are the magnitudes of oscillations.

θ'_i are the rate of change of frequencies of oscillations due to the change in the rate of angular velocity of the gyro spinner.

λ_i are the damping coefficients.

B_i are the times at positive turning points.

And the corresponding periods may be obtained by the formula:

$$T_i = \frac{2\pi}{\theta_i + \theta_i' t}$$

3.3.12 Calculation of $(\delta + \tau)$, $\bar{y}_I$, $\bar{x}_I$ and μ:

Obtaining the value of $(\delta + \tau)$ is the most interesting function because it is equal to the angle between the axis of rotation G_x of the gyroscope and the meridian plane. From equations (3-93), we get:

(3-94) $$\delta + \tau = \delta_0 + \tau_0 + \sum_{i=1}^{10} (1 + \sigma_i)[A_i e^{-\lambda_i t} \cos((\theta_i + \theta_i' t)(t - B_i))]$$

The function $\bar{y}_I$, denotes the East-West movement of the point of suspension tape, I. The function may be obtained after substituting δ, τ and their derivatives into equation (3-77).

The other two unknown functions $\bar{x}_I$ and μ may be obtained from integrating the equations (3-69) and (3-64) respectively.

3.3.13 Equation of motion of the moving mark of the Gyrotheodolite:

The small movements, oscillations of the gyro, can be seen through an eyepiece in the form of a light mark moving against scale divisions. The centre is at zero with + 15 divisions on the left and – 15 divisions on the right. When the Gyrotheodolite is directed towards the true North the moving mark should point to zero. However, in practice the actual centre of oscillation is offset by an amount ***Q*** from zero. Taking into account of ***Q***, the mid-point of swing, the equation of motion of the moving mark from (3-93) may be written as:

$$(3\text{-}95)\quad \sum_{i=1}^{10}[A_i e^{-\lambda_i t}\cos((\theta_i+\theta_i' t)(t-B_i))]+\boldsymbol{Q}-\Delta=0$$

Where:

Q is the mid-point of oscillations.

Δ is the scale reading.

The mathematical model and characteristic equation derived by Jeudy (Jeudy, 1982) "without taking account of the damping, accelerating and decelerating forces applied to the spinner" are respectively of the form:

$$\sum_{i=1}^{5}(A_i'\cos\theta_i t + A_i''\sin\theta_i t)$$

$$d_5\eta^5+d_4\eta^4+d_3\eta^3+d_2\eta^2+d_1\eta+C=0$$

Jeudy (Jeudy et al. 1981) determined the parameters of the above equation and the mid-point of swing by using an electronic registration device. They used a few observations of time. The individual values obtained in that research cannot be assessed from the observations. In this research, the values of parameters in terms of equation (3-95) and their standard deviations are found by using least squares adjustment techniques. The time observations are taken with the aid of a video camera over a period of about three hours. The method is described in Chapter 4. It makes use of the maximum data available from the Gyrotheodolite. A precise determination of values of ***Q***, the mid-point of swing leads to a high precision of azimuth determination.

IV. AZIMUTH DETERMINATION

4.1 Introduction:

Different methods, for example, turning point, transit and amplitude methods are used for azimuth determination using a suspended gyroscope. These methods are described adequately in general literature. Usually, the measured quantities are the time, scale divisions and turning points. The observer, in conventional observations, takes a few observations of time during the period of one or two complete oscillations of the moving mark. Using a stopwatch for timing in these methods is inaccurate and highly personal. Gregerson (Gregerson, 1971 and 1972) showed by experiments that timings on sharp visual signals have a mean error of about 0.12 seconds of time. Taking account of the fact that the moving mark of the Gyrotheodolite GAK1 is not very sharp, the quality of timing is estimated to be not more than 0.2 seconds at best. The observations are used in an observation equation to determine uniquely the centre of oscillations without any degrees of freedom. In such a method, the centre of oscillations is determined by using the minimum data available so the standard error of individual results cannot be assessed from the observations.

In this research, the reasons of deriving a new mathematical model and using a new method for observations are:

- To account for the great volume of observations that can be obtained.

- To take advantage of an increase in the quality of timing by using a video camera and video frame analysis.

- To consider all mathematical terms which relate to the oscillations. These reflect the physical changes to the internal parts of the Gyrotheodolite, especially those due to the effect of changes in the angular velocity of the spinner and those due to friction and the damping forces.

4.2 Description of the observations and experiments:

Two sets of time observations, see appendix B, were used in this research. The first set took about one and half-hours. The battery was not full charged and after two and half-hours was flat. This is similar to the practical case, which may occur in the field. The second set took about three hours and the battery was fully charged at the start. The method used in this work required a video camera to observe the oscillations of the moving mark. The camera was set to look through the gyro eyepiece onto the gyro scale and also, to record each time the moving mark crossed a scale division. The Gyrotheodolite was set up on a pillar, figure 17 to avoid vibrations that may have occurred by using a tripod. The gyro spinner was run up to its full speed. The only power source, which drove the motor was a battery. The mast was carefully dropped to make sure that the moving mark did not go beyond the limit of the scale divisions. The moving mark was allowed to settle down for a few minutes before observations were taken.

Figure 17

Gyrotheodolite with a video camera

The observations are summarised in the table below:

Set of observations	Range of oscillations	Number of readings	Observations per oscillation	complete oscillations
1	-11 +10	552	44	> 12
2	-11 +11	1165	46	> 25

This table shows that more than twelve oscillations of the moving mark were completed in the first set of observations and more than twenty-five oscillations were completed in the second set of observations. Only one or two oscillations of the moving mark are involved in conventional methods of observation. Therefore, this method leads to a substantial increase in the quantity of observations. In this method, many observations of time may be taken in a single oscillation of the moving mark. The readings of each scale division are repeated many times, therefore, the degrees of freedom are greatly increased.

4.2.1 Automated data capture of the maximum available data:

Observations of time at scale divisions are captured on videotape. The time was recorded when the moving mark crossed a scale division. In the laboratory, a time code was put on the videotape to extract each time observation. With the aid of video frames, the quality of timing improved from approximately 0.2 seconds, at best by manual methods to the time equivalent of one frame. One frame = 1/25 seconds = 0.04 seconds. The accuracy of timing increased five times. Therefore, automated data capture leads to a great increase in the quantity and quality of observations. The good handling of data in terms of quality and quantity lets us not only increase the accuracy of the timing method but also to consider a new mathematical model for the theory of the suspended gyroscope, for the Wild GAK1. This model takes into account of all potentially significant terms that affect the changes in the physical environment of the suspended gyroscope, that is, to account for accelerations.

4.2.2 Semi-automated data processing:

Video imagery using frame analysis is selected and the time data is extracted with the aid of a microcomputer, "laptop", see figure 18. After processing the time data, it is used in a rigorous mathematical model and processed by least squares techniques, which lead to high quality solutions and statistical assessments.

Figure 18

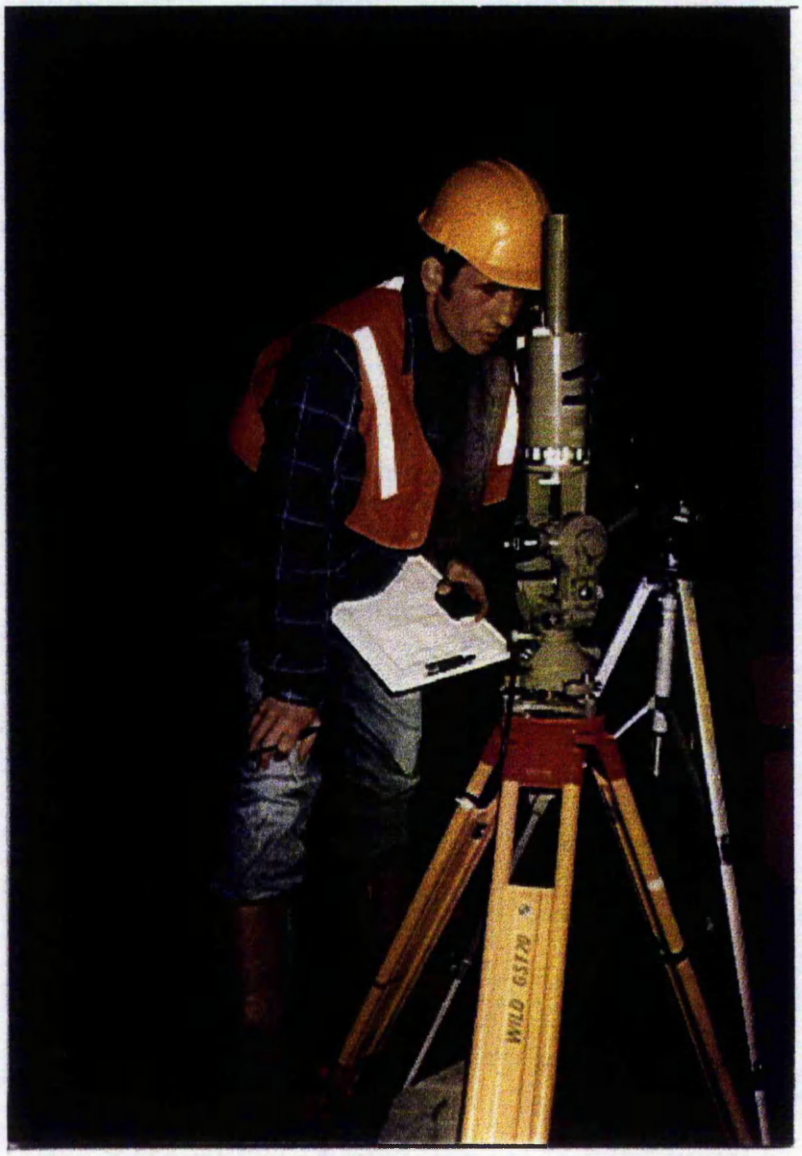
Data processing

4.3 Procedure of determining the North:

The theodolite is directed approximately to North to within a few minutes of arc. The gyro spinner is run up to full speed. The motor is driven by a battery, which runs down with time. The mast is dropped carefully, and the gyro is released to seek true North and oscillate about the meridian.

<u>Figure 19</u>

Underground observations in railway tunnel under the River Severn



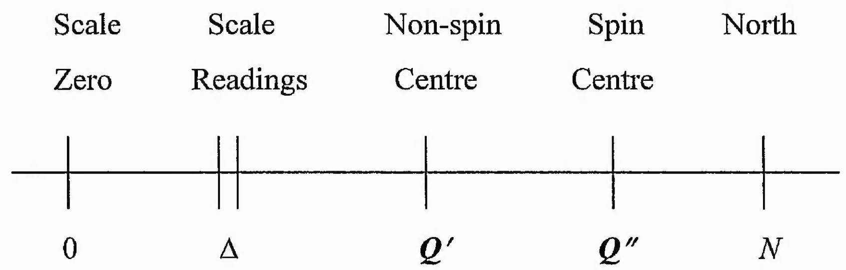
In fact, the gyro swings under the influence of many forces. Some of those forces are the precession torque due to a couple applied to the spinner axis, the torque due to the twisting tape, decelerating forces applied to the spinner due to the change in the rate of its angular velocity and finally the gravity forces. Therefore, the swinging gyro defines the direction in which the forces are in equilibrium rather than the direction of the North. The problem of determining azimuth is broken down into two parts (Breach, 1983). Firstly, determine the mid-point of swing in the spin and non-spin mode, equation (3-95) in Chapter 3, which is written once again here:

$$\sum_{i=1}^{10}[A_i e^{-\lambda_i t}\cos((\theta_i + \theta_i' t)(t - B_i))] + \boldsymbol{Q} - \Delta = 0$$

Secondly, relate the mid-point of swing, $\boldsymbol{Q}$ determined from the last equation to equation of Z, the reading on the horizontal circle of the theodolite, which is equivalent to North. See figure 20.

Figure 20

Diagram of the scale of the Wild GAK1 gyroscope



In figure 20:

Δ is the scale reading, the moving mark appears on the gyro scale as a double line.

$\boldsymbol{Q'}$ is the position that the moving mark would take when the gyro is in the non-spin mode. This position would not be at the scale zero because the system cannot be adjusted that precisely.

$\boldsymbol{Q''}$ is the position that the moving mark would take when the gyro is in the spin mode.

N is the direction of astronomical North.

The position of scale zero depends on determining of two torques affecting on the gyroscope oscillations. The first torque is due to precession, which may be written as:

(4-1) $\Omega_1 = P_a \varpi \cos\phi \sin\varepsilon$ (Thomas, 1976)

Where, ϖ and ϕ are constants defined in previous chapters and appendix A.

P_a, is the angular momentum of the spinner.

ε is an angle between the spinner axis and the North.

Since ε, in radians is a very small angle the above equation becomes:

(4-2) $\Omega_1 = K_1\varepsilon$

Where:

Ω_1 is a torque due to precession.

K_1 is the precession torque per unit angle e.g. arc second. K_1 is constant at a given latitude.

From figure 20, equation (4-2) may be written:

(4-3) $\Omega_1 = K_1(N - \Delta)s$

Where, s is the value of one scale unit in angular measure, for example, arc seconds.

The second torque is due to tape twisting, which may be written as:

(4-4) $\Omega_2 = K_2\varepsilon'$ (Gregerson, 1972)

Where:

Ω_2 is the torque due to tape twisting.

K_2 is the tape twisting torque per unit angle e.g. arc second.

ε' is the rotation from the tape zero axis.

From figure 20, equation (4-4) may be written as:

(4-5) $\Omega_2 = K_2(\boldsymbol{Q}' - \Delta)s$

The total torque is the sum of the two torques Ω_1 and Ω_2:

$$\Omega = \Omega_1 + \Omega_2 = K_1(N - \Delta)s + K_2(\boldsymbol{Q}' - \Delta)s$$

$$\Omega = s[K_1N + K_2\boldsymbol{Q}' - \Delta(K_1 + K_2)]$$

(4-6) $$\Omega = s(K_1 + K_2)[\frac{K_1N + K_2\boldsymbol{Q}'}{K_1 + K_2} - \Delta]$$

The ratio between the two torques Ω_1 and Ω_2 may be written as:

(4-7) $$K = \frac{K_2}{K_1}$$

At the mid-point of swing, the centre of oscillation, the total torque is zero, that is, when $\Delta = \boldsymbol{Q}''$:

Substitute Δ in equation (4-6) by $\boldsymbol{Q}''$ and put Ω equal to zero, we get:

(4-8) $$\frac{K_1 N + K_2 \boldsymbol{Q}'}{K_1 + K_2} - \boldsymbol{Q}'' = 0$$

Then:

(4-9) $$\boldsymbol{Q}'' = \frac{K_1 N + K_2 \boldsymbol{Q}'}{K_1 + K_2} = \frac{N + \frac{K_2}{K_1}\boldsymbol{Q}'}{1 + \frac{K_2}{K_1}}$$

Taking account of (4-7), the last equation becomes:

(4-10) $$\boldsymbol{Q}'' = \frac{N + K\boldsymbol{Q}'}{1 + K}$$

Where, K is the constant of the torque ratio. Equation (4-10) may be re-arranged to get:

(4-11) $$N = \boldsymbol{Q}''(1 + K) - K\boldsymbol{Q}'$$

Suppose Z, is the reading on the horizontal circle of the theodolite, which is equivalent to North. Z may be written as:

(4-12) $$Z = \Theta + sN + E$$ (Thomas, 1976)

Where, Θ is the reading on the horizontal circle when the theodolite is clamped up ready for observations of the gyro.

E is an instrument constant.

Substitute (4-11) into (4-12) to get:

(4-13) $$Z = \Theta + s\boldsymbol{Q}''(1 + K) - sK\boldsymbol{Q}' + E$$

Finally, the azimuth of the line of a reference point may be written as:

(4-14) $$A_Z = H_R - \Theta - s[\boldsymbol{Q}''(1 + K) - K\boldsymbol{Q}'] - E$$

Where, H_R is the observed horizontal circle reading to a reference point.

4.4 Practical application and limitations:

The observational process consists of a pre-orientation towards the North; this can be done from the use of the gyro in the unclamped method. The mast is dropped in the spin mode and the observer tries to rotate the theodolite slowly to keep following the moving mark on the zero division. At the extremity of the oscillation the theodolite is clamped up and the horizontal circle is read. A provisional estimate of North may be achieved by taking the mean of two successive readings. The pre-orientation to North must be determined as accurately as possible to ensure accurate azimuth determination and to reduce errors due to the centrifugal force of earth rotation, (Halmos, 1977). This process may take about 15 minutes and achieve a precision of a few minutes of arc. The Gyrotheodolite may be set up on a tripod or pillar. A pillar is preferable, because a wooden tripod may twist in damp or sunny conditions and may be knocked easily. The duration of 15-20 minutes of pre-orientation of the North is essential for the instrument to reach its equilibrium temperature before making any observations. Then, the observer is ready to make the required observations to determine the azimuth to a reference point. These observations include:

- Two observations of the horizontal circle readings H_R to a reference point.
- Two observations of the horizontal circle readings Θ_1 and Θ_2 with the theodolite pointing about half a degree to the West and East of North.
- Observations of time versus scale divisions for determining the mid-point of swing twice in the spin mode and once in the non-spin mode. To reduce timing errors, a video camera is used to observe and record time on videotape at each instant the moving mark crosses a scale division. The data is extracted with the aid of a "laptop" computer and by using the video imagery with frame analysis.

The video camera is set up carefully to focus on the eyepiece of the gyro scale. The observer makes sufficient observations, about 1-2 hours, to determine the mid-point of swing for each spin and non-spin mode of the spinner. The method makes use of the maximum data available from the Gyrotheodolite.

A complete oscillation of the moving mark takes about eight minutes in the spin mode and about one minute in non-spin mode. In conventional methods, using a stopwatch for timing it is not possible to make good observations in the non-spin mode. The period between two successive observations of the moving mark can be less than one second, which is too rapid to make any sensible observations. However, using video imagery with frame analysis and with the facilities of "pause", "backward" and "forward", it is possible to obtain each time to the nearest frame as stated in (4.2.1).

The weight of Wild GAK1 including the video camera is approximately about 6 kg, this is not considered an excessive in comparison, for example, with the weight of Gyromat-2000. The latter is a bulky piece of equipment with a large mass of about 16 kg, which is difficult for handling in confined spaces, especially in the underground environment (Eyre et al. 1995). The second hand cost of the Wild GAK1 including the equipment used in this method is in the order of £5,000 to £10,000 while the cost of Gyromat-2000 is understood to be in the order of £50,000 to £70,000. Upgrading the observing and computing procedures applied to the Wild GAK1, using the methods described in this work, a precision of azimuth determination to $\pm 3''$ may be obtained. The result is comparable with the precision obtained by Gyromat-2000.

The method is very simple and safe to be used for practical application in surface and subsurface baseline orientation, for example, for azimuth determination in underground tunnels. Precautions need to be considered for the stability of the equipment and protection from sunshine and wind when used on the surface. Observations using a Wild GAK1 with video camera were carried out by Mr. Breach, the supervisor of this project in the railway tunnel under the River Severn, figures 19 and 21. However, additional precautions need to be considered if this method used in a mine, the system must be acceptable for mine safety for electrical equipment.

The observations are processed in an appropriate rigorous mathematical model, equation (3-95) to determine the mid-point of swing and other parameters. The adjustment by using least squares techniques leads to high quality solutions and statistical assessments. The programme runs on Excel spreadsheets. The dislevelment of the theodolite especially, when it is in the Prime Vertical will affect each reading on the gyro scale by a constant error for each set up of the instrument. However, precise levelling of the Gyrotheodolite may be achieved when it is set on a pillar and this may reduce the dislevelment error according to Breach (Breach, 1983). Also the deflection of the vertical is neglected in the mathematical model (3-95). However, this error is negligible in surveying flat areas but may need to be taken into account in geodetic applications (Halmos, 1977).

Figure 21

Making observations by Wild GAK1 with a video camera in the railway tunnel under the River Severn

Gyrotheodolites can be used in densification of geodetic networks for the orientation of surface and subsurface traverses and for azimuth determination in many industrial applications, for example, railway tunnels. They can be used at all times and under circumstances in environments where ground control is limited and the view of the sky is restricted. The azimuths determined by Gyrotheodolites are equivalent to astronomical azimuths because in both cases they are defined by the vertical and by true North. However, gyroscopic azimuths are determined without depending on astronomical observations, for which different observational conditions apply.

4.5 The precision of azimuth determination:

Equation (4-14) relates the values Q, which are the midpoint of swing in the spin and non-spin mode, and determines the azimuth of a reference point. If two sets of observations are made in the spin mode, each with the theodolite pointing about a half degree to West and East of North respectively. Then equation (4-13) may be written for both sets of observations as follows:

(4-15) $Z = \Theta_1 + sQ_1''(1+K) - sKQ' + E$ (for one side of North).

(4-16) $Z = \Theta_2 + sQ_2''(1+K) - sKQ' + E$ (for the other side of North).

Subtract (4-15) from (4-16) to get:

(4-17) $1+K = \dfrac{\Theta_2 - \Theta_1}{s(Q_1'' - Q_2'')}$ (Thomas, 1982)

Where:

Θ_1 and Θ_2 are the horizontal circle readings with the theodolite pointing about a half degree to the West and East of North respectively.

Q''_1 and Q''_2 are the midpoints of swing for the two sets of observations in the spin mode.

Substitute (4-17) into equation (4-14) to get:

(4-18) $A_Z = H_R - \Theta - [\dfrac{\Theta_2 - \Theta_1}{(Q_1'' - Q_2'')}(Q'' - Q') + sQ'] - E$

Apply the law of error propagation from (Nassar, 1985) and use the variances of parameters of equation (4-18) to get:

$$\begin{aligned}(4-19)\quad \sigma_E^2 &= \sigma_{A_Z}^2 + \sigma_{H_R}^2 + \sigma_{\Theta}^2 + \sigma_s^2 Q'^2 + (\sigma_{\Theta_1}^2 + \sigma_{\Theta_2}^2)(\frac{Q'' - Q'}{Q_1'' - Q_2''})^2 \\ &+ \sigma_{Q'}^2[s - (\frac{\Theta_2 - \Theta_1}{Q_1'' - Q_2''})]^2 + \sigma_{Q''}^2(\frac{\Theta_2 - \Theta_1}{Q_1'' - Q_2''})^2 + (\sigma_{Q_1''}^2 + \sigma_{Q_2''}^2)[\frac{(\Theta_2 - \Theta_1)(Q'' - Q')}{(Q_1'' - Q_2'')^2}]^2\end{aligned}$$

If $\sigma_\Theta = \sigma_{\Theta_1} = \sigma_{\Theta_2}$ and $\sigma_{Q''} = \sigma_{Q''_1} = \sigma_{Q''_2} = \sigma_{Q'}$. Equation (4-19) becomes:

$$\sigma_E^2 = \sigma_{Az}^2 + \sigma_{H_R}^2 + \sigma_s{}^2 Q'^2 + \sigma_\Theta^2[1 + 2(\frac{Q''-Q'}{Q''_1 - Q''_2})^2]$$

(4-20)

$$+\sigma_{Q'}^2[s^2 - 2s(\frac{\Theta_2 - \Theta_1}{Q''_1 - Q''_2}) + 2(\frac{\Theta_2 - \Theta_1}{Q''_1 - Q''_2})^2(1 + (\frac{Q''-Q'}{Q''_1 - Q''_2})^2)]$$

Assume these reasonable values for the parameters of equation (4-20):

$\sigma_{H_R} = 1''$

$\sigma_{Q''} = 0.0012$ scale divisions (from computations).

$\sigma_\Theta = 1''$

$\boldsymbol{Q''} = 0.2$ scale divisions.

$\boldsymbol{Q'} = 0.1$

Q''_1 - $Q''_2 = 2$ scale divisions.

Θ_2 - $\Theta_1 = 1° = 3600$ seconds.

The value of s, one unit of scale division is given in Wild GAK1 manual as 10′ or 0.19^gr^. Therefore, there is a difference of 15.6″ between the two values of s mentioned in Wild GAK1 manual.

Suppose $\sigma_S = 15.6''$ therefore,

$\sigma_{Az}^2 = \sigma_E^2 + 11.2$ (arc second)^2

The precision of the azimuth to a reference point depends among other factors on the determination of the instrument constant, E. Practically, it is impossible to determine E perfectly. The precision of E, the instrument constant, may be considered in the range from ± 1″ to ± 5″, then σ_{Az}, the precision of the azimuth will be in the range from ± 3.5″ to ± 6.2″, which is usually acceptable from the practical point of view.

In Chapter 5, two sets of time observation were used to test the validity of the mathematical model. Results by least squares adjustment for most parameters, in terms of equation (3-95), and their standard deviations were found.

V. GYROTHEODOLITE ADJUSTMENT COMPUTATIONS

5.1 Mathematical model:

Measured quantities are the "observations" obtained through the data collecting process. They are used in observation equation. The mathematical model represents the mathematical relationship between x the unknown parameters and l the observables. The mathematical model may be written as:

(5-1) $F(x, l) = 0$

In our case, we find in Chapter 3 the equation of motion of the moving mark may be written in the term of equation (5-1) as:

(5-2) $F(x,l) = \sum_{i=1}^{10} A_i e^{-\lambda_i t} \cos((\theta_i + \theta_i' t)(t - B_i)) + \boldsymbol{Q} - \Delta = 0$

The vector of unknown parameters is:

(5-3) $x = (A,\ \theta,\ \theta',\ B,\ \lambda,\ \boldsymbol{Q})$

Suppose the scale divisions are equally spaced. This will depend on the manufacturer's precision. Therefore, they are not considered as observable quantities. The only observable quantity is the time. The vector of time observations may be written as:

(5-4) $l = (t_1, t_2, \ldots\ldots\ldots t_n)$

Where:

n is the number of observations.

5.2 Least squares adjustments:

The mathematical model (5-2) is a function of the observations and the parameters. The least squares adjustment follows a "general form". The mathematical model is linearised as a Taylor series. The solution of which is given in most standard textbooks as:

(5-5) $\boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{v} + \boldsymbol{b} = 0$ (Vanicek, 1982)

Where:

$\boldsymbol{A}$ and $\boldsymbol{B}$ are design matrices.

$\boldsymbol{b}$ is a misclosure vector.

The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and the vector b may be written as:

$$\boldsymbol{A} = \frac{\partial F}{\partial \boldsymbol{x}}, \quad \boldsymbol{B} = \frac{\partial F}{\partial \boldsymbol{l}}, \quad \text{and} \quad \boldsymbol{b} = F(\boldsymbol{l}_0, \boldsymbol{x})$$

Where:

$\boldsymbol{x}$ and $\boldsymbol{l}$ are the adjusted parameters and observations respectively.

$\boldsymbol{l}_0$ represents the original observations.

$\boldsymbol{v}$ is the vector of residuals for time observations.

$\hat{\boldsymbol{x}}$ is the vector of corrections to the provisional values of parameters.

F represents the complete non-linear model given by (5-2).

For n observation equations, the elements of (5-5) are of dimensions:

$$\underset{n\times u}{\boldsymbol{A}}\,\underset{u\times 1}{\hat{\boldsymbol{x}}} + \underset{n\times n}{\boldsymbol{B}}\,\underset{n\times 1}{\boldsymbol{v}} + \underset{n\times 1}{\boldsymbol{b}} = 0 \qquad (5\text{-}6)$$

Where;

n is the number of observation equations; each observable has one observation equation.

u is the number of unknown parameters.

The elements of the matrix $\boldsymbol{A}$ are formed by taking the partial derivatives of (5-2) with respect to the parameters (A, θ, θ', B, λ, $\boldsymbol{Q}$). Similarly, take the partial derivatives of (5-2) with respect to time observations to form the elements of the matrix $\boldsymbol{B}$. Since the time is the only observable quantity, $\boldsymbol{B}$ is a non-unit diagonal matrix.

If $i = 1$ in terms of equation (5-2), u the number of parameters will be six. Then, the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for one observation equation may be written as:

$$\underset{1\times 6}{\boldsymbol{A}} = \begin{bmatrix} \frac{\partial F}{\partial A} & \frac{\partial F}{\partial \theta} & \frac{\partial F}{\partial \theta'} & \frac{\partial F}{\partial B} & \frac{\partial F}{\partial \lambda} & \frac{\partial F}{\partial \boldsymbol{Q}} \end{bmatrix} \text{ and } \underset{1\times 1}{\boldsymbol{B}} = \frac{\partial F}{\partial t_1} \qquad (5\text{-}7)$$

(5-8)

$$\underset{1\times 1}{\boldsymbol{B}} = -Ae^{-\lambda t_1}[(2\theta' t_1 - \theta' B + \theta)\sin((\theta + \theta' t_1)(t_1 - B)) + \lambda\cos((\theta + \theta' t_1)(t_1 - B))]$$

Where:

$$\frac{\partial F}{\partial A} = e^{-\lambda t_1}\cos((\theta + \theta' t_1)(t_1 - B))$$

$$\frac{\partial F}{\partial \theta} = -(t_1 - B)Ae^{-\lambda t_1}\sin((\theta + \theta' t_1)(t_1 - B))$$

$$\frac{\partial F}{\partial \theta'} = -t_1(t_1 - B)Ae^{-\lambda t_1}\sin((\theta + \theta' t_1)(t_1 - B))$$

$$\frac{\partial F}{\partial B} = (\theta + \theta' t_1)Ae^{-\lambda t_1}\sin((\theta + \theta' t_1)(t_1 - B))$$

$$\frac{\partial F}{\partial \lambda} = -t_1 Ae^{-\lambda t_1}\cos((\theta + \theta' t_1)(t_1 - B))$$

$$\frac{\partial F}{\partial Q} = 1$$

The vector $\boldsymbol{b}$ is a column vector, $\boldsymbol{b}$ in terms of equation (5-2) for one observation equation may be written as:

(5-9) $\underset{1\times1}{\boldsymbol{b}} = \Delta_1 - Ae^{-\lambda t_1}\cos((\theta + \theta' t_1)(t_1 - B)) - \boldsymbol{Q}$

This equation expresses the difference between the observed quantity Δ_{obs} and the computed quantity Δ_{com}. Equation (5-9) may be written as:

(5-10) $\underset{1\times1}{\boldsymbol{b}} = \Delta_{1obs} - \Delta_{1com}$

Therefore, in terms of equation (5-2) if i = 1, equation (5-6) for one linearised observation equation may be written as:

(5-11)

$$\underset{1\times6}{\begin{bmatrix} \frac{\partial F}{\partial A} & \frac{\partial F}{\partial \theta} & \frac{\partial F}{\partial \theta'} & \frac{\partial F}{\partial B} & \frac{\partial F}{\partial \lambda} & \frac{\partial F}{\partial \boldsymbol{Q}} \end{bmatrix}} \underset{6\times1}{\begin{bmatrix} dA \\ d\theta \\ d\theta' \\ dB \\ d\lambda \\ d\boldsymbol{Q} \end{bmatrix}} + \underset{1\times1}{\begin{bmatrix} \frac{\partial F}{\partial t_1} \end{bmatrix}} \underset{1\times1}{\begin{bmatrix} \boldsymbol{v}_{t_1} \end{bmatrix}} + \underset{1\times1}{\begin{bmatrix} \Delta_{1obs} - \Delta_{1com} \end{bmatrix}} = 0$$

Where, dA, $d\theta$, $d\theta'$, dB, $d\lambda$ and $d\boldsymbol{Q}$ are corrections to the provisional values of parameters A, θ, θ', B, λ and $\boldsymbol{Q}$.

Equation (5-11) for n observation equations may be written as:

(5-12)

$$\begin{bmatrix} \frac{\partial F}{\partial A} & \frac{\partial F}{\partial \theta} & \frac{\partial F}{\partial \theta'} & \frac{\partial F}{\partial B} & \frac{\partial F}{\partial \lambda} & \frac{\partial F}{\partial Q} \\ .... & .... & .... & .... & .... & .... \\ .... & .... & .... & .... & .... & .... \\ (\frac{\partial F}{\partial A})_n & (\frac{\partial F}{\partial \theta})_n & (\frac{\partial F}{\partial \theta'})_n & (\frac{\partial F}{\partial B})_n & (\frac{\partial F}{\partial \lambda})_n & (\frac{\partial F}{\partial Q})_n \end{bmatrix}_{n\times 6} \begin{bmatrix} dA & .... & .... & (dA)_n \\ d\theta & .... & .... & (d\theta)_n \\ d\theta' & .... & .... & (d\theta')_n \\ dB & .... & .... & (dB)_n \\ d\lambda & .... & .... & (d\lambda)_n \\ dQ & .... & .... & (dQ)_n \end{bmatrix}_{6\times n} +$$

$$\begin{bmatrix} \frac{\partial F}{\partial t_1} & 0 & 0 & .... & 0 \\ 0 & \frac{\partial F}{\partial t_2} & 0 & .... & 0 \\ 0 & 0 & .... & .... & 0 \\ 0 & 0 & .... & .... & 0 \\ 0 & 0 & .... & .... & \frac{\partial F}{\partial t_n} \end{bmatrix}_{n\times n} \begin{bmatrix} v_{t_1} \\ v_{t_2} \\ .... \\ .... \\ v_{t_n} \end{bmatrix}_{n\times 1} + \begin{bmatrix} \Delta_{1obs} - \Delta_{1com} \\ \Delta_{2obs} - \Delta_{2com} \\ .... \\ .... \\ \Delta_{nobs} - \Delta_{ncom} \end{bmatrix}_{n\times 1} = 0$$

The adjusted vector of estimated parameters might be written as:

(5-13) $\boldsymbol{x} = \boldsymbol{x}_0 + \hat{\boldsymbol{x}}$

Where, $\boldsymbol{x}_0$ is an initial approximation to the parameters and

(5-14) $\hat{\boldsymbol{x}} = (\boldsymbol{A}^T M^{-1} \boldsymbol{A})^{-1} (\boldsymbol{A}^T M^{-1} \boldsymbol{b})$ (Mikhail, 1981) and (Vanicek, 1982)

Where;

$M = \boldsymbol{B} W^{-1} \boldsymbol{B}^T$

W^{-1} is the inverse of the weight matrix for the observations.

The variance-covariance matrix for the estimated parameters may be written as:

(5-15) $C_x = \hat{\sigma}_0{}^2 (\boldsymbol{A}^T M^{-1} \boldsymbol{A})^{-1}$

Where $\hat{\sigma}_0{}^2$ is computed by formula:

$$\hat{\sigma}_0{}^2 = \frac{\boldsymbol{v}^T W \boldsymbol{v}}{n-u}$$

$\boldsymbol{v}$, the vector of residuals for time observations may be written as:

(5-16) $\boldsymbol{v} = -W^{-1} \boldsymbol{B}^T M^{-1} (A\hat{\boldsymbol{x}} + \boldsymbol{b})$

The adjusted observations may be written as:

(5-17) $\boldsymbol{l} = \boldsymbol{l}_0 + \boldsymbol{v}$

5.3 The weight of observations:

The observations are time at each moment the moving mark crosses a scale division. Since the scale divisions are considered equally spaced, the weight of each observed time should be considered. The correct weighting of each observed time may be given as a function of either the distance of its corresponding scale division from the midpoint of swing or the velocity of the moving mark. However, Breach (Breach, 1983) found that taking different weights for each observable time had no effect on the determination of the centre of oscillation $\boldsymbol{Q}$ or on its standard deviation σ_Q. Therefore, all time observations should have equal weight. The weight matrix appears in equations (5-14), (5-15) and (5-16) is a unit diagonal matrix.

5.4 Simplification of computed matrices:

Since the only observed quantity is time, the second design matrix $\boldsymbol{B}$ is a non-unit diagonal matrix. Therefore, there is no need to construct $\boldsymbol{B}$ in the adjustment in its full dimensions $n \times n$ where n will be equal to the number of observation equations. Assuming the weight matrix is a unit matrix in equations (5-14), (5-15) and (5-16), we get the following equations:

(5-18) $\hat{\boldsymbol{x}} = (\boldsymbol{A}^T \boldsymbol{A})^{-1}(\boldsymbol{A}^T \boldsymbol{b})$

(5-19) $C_x = \hat{\sigma}_0{}^2 (\boldsymbol{A}^T \boldsymbol{A})^{-1}$

(5-20) $\boldsymbol{v} = -\boldsymbol{B}^T(\boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{b})$

Where, $\hat{\boldsymbol{x}}$ is the vector of corrections to the provisional parameters, C_x is the variance-covariance matrix for the estimated parameters and $\boldsymbol{v}$ is the vector of residuals for time observations. $\hat{\sigma}_0{}^2$ in this case will be:

$$\hat{\sigma}_0{}^2 = \frac{\boldsymbol{v}^T \boldsymbol{v}}{n-u}$$

Suppose $\boldsymbol{r}$ is the corresponding residuals vector for scale divisions, $\boldsymbol{r}$ may be written as:

$$\boldsymbol{r} = \boldsymbol{B}\boldsymbol{v}$$

(5-21) $\boldsymbol{r} = -(\boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{b})$

5.5 Some computing considerations:

The observations used in this project are two sets of 552 observations of time for the first set and 1165 observations of time for the second set. The moving mark oscillates in the spin mode. The extent of the swing in both the positive and negative directions is from –11 to +10 for the first set of observations. The oscillations involve 44 times at scale divisions per oscillations. However, the extent of the swing for the second set of observations is from –11 to +11. The oscillations involve 46 times at scale divisions per oscillations. Thus, there is more than twelve complete oscillations for the first set of observations and more than twenty-five complete oscillations for the second set of observations. The observations are used in the mathematical model to determine the midpoint of swing and other parameters in terms of the equation:

(5-2) $$\sum_{i=1}^{10} A_i e^{-\lambda_i t} \cos((\theta_i + \theta_i' t)(t - B_i)) + \boldsymbol{Q} - \Delta = 0$$

Least squares adjustment techniques are used to compute the different parameters for equation (5-2). See appendix C. Since the initial approximate values of the parameters are not known, good provisional values of the unknown parameters are used to get a convergent solution by an iterative least squares process. Since some parameters are numerically very much larger than others are, setting all parameters to zero, as initial values did not work. The problem of finding the provisional values of the parameters consists of the following steps:

- Put $\theta' = 0$ in equation (5-2), compute the provisional values of the parameters by non-rigorous means.
- Put $\theta' = 0$ and $\lambda = 0$ in equation (5-2), compute the values of the parameters by iterative least squares using the provisional values derived from the previous step.
- Put $\theta' = 0$, use the values obtained in the previous step to compute all the values of the parameters in equation (5-2) up to the next value of i by iterative least squares.
- Repeat these three steps to compute the values of the parameters in equation (5-2) up to the next value of i by iterative least squares.

5.6 Results from using the first set of time observations:

The first set of observations took about one and half-hours, the battery was not full charged and after two and half-hours was flat. The values of these observations are shown in table B1, appendix B.

In all tables appear below, the precision, standard deviations of the parameters A, θ, θ', B, λ and $\boldsymbol{Q}$ are computed from the diagonal elements, the variances, of the matrix $(A^T A)^{-1}$, see equation (5-19) and appendix C, by the formula: $\sigma_x = \sqrt{\sigma_x{}^2}\,\hat{\sigma}_0$ where, x is any parameter.

The parameters A, θ, θ', B, λ and $\boldsymbol{Q}$ have the following units:

A, the magnitude of the oscillations is in scale division units, one scale unit is about 600″ arc seconds.

θ and θ', the frequency of the oscillations and the rate of change in the frequency are in radians per frame and radians per $(\text{frame})^2$ respectively.

B, the phase is in frames.

λ, the coefficient of damping is unitless.

$\boldsymbol{Q}$, the centre of oscillations is in scale division units.

A least squares adjustment was performed in a step by step manner for the mathematical model (5-2) $\sum_{i=1}^{10} A_i e^{-\lambda_i t}\cos((\theta_i + \theta_i' t)(t - B_i)) + \boldsymbol{Q} - \Delta = 0$. This represents the equation of motion of the moving mark of the Gyrotheodolite. The model is in the form of a damped harmonic motion with ten periods for the oscillations. Put i = 1 in equation (5-2) to obtain the first model of the observation equation (5-22) for the first period of oscillations:

(5-22) $A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + \boldsymbol{Q} - \Delta = 0$

The provisional values of the parameters in terms of this equation (5-22) are found following the steps explained in paragraph (5.5). Where t, is time and its units are frames. See "table B1, appendix B". Table 1 and figure 22 summarise the results for values of these parameters and their standard deviations computed by a least squares adjustment.

5.6.1 Results from using the observation equation (5-22):

Table 1

Parameters A, θ, θ', B, λ, Q and their standard deviations using the first set of data

Parameters	Values	σ
A	10.98296	0.004172
θ	0.000592	1.23E-08
θ'	9.29E-13	8.31E-14
B	9273.193	0.55987
λ	8.53E-08	4.38E-09
Q	-0.1028	0.001126
$\hat{\sigma}_0$	0.0272	

Figure 22

Residuals 1 of the first set of time observations

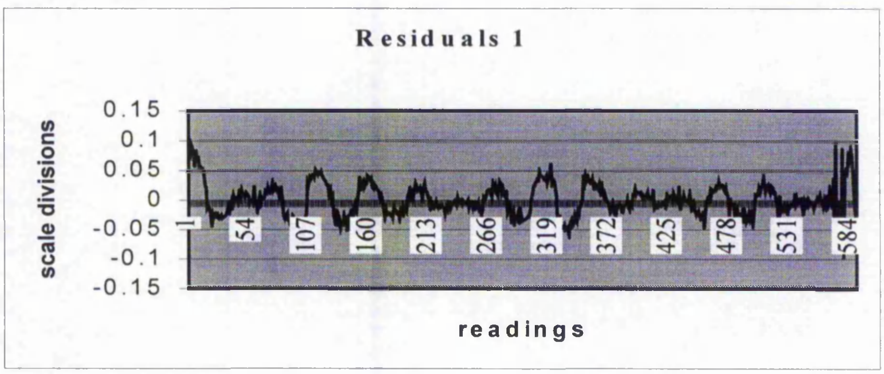


Table 1 shows that the main period of oscillations ($2\pi/\theta$), is about seven minutes, 424.5 seconds, which is well known from the experiments. Most of the conventional observation methods use the mathematical model of a damped harmonic motion with just one period of oscillation (EMR, 1975). This model is inadequate for precise azimuth determination. The standard deviation of θ', the change in the frequency of oscillations is much less than its value, $\frac{\theta'}{\sigma_{\theta'}} = 11.2$.

This suggests practical evidence for including the term $\theta' t$ in the mathematical model. The midpoint of swing, $\boldsymbol{Q}$ and its standard deviation are shown in scale divisions, which in arc seconds are respectively, 61″.68 and 0″.67. Figure 22 shows the observation residuals, values of which are shown in appendix C. From value of $\hat{\sigma}_0$ and the residuals, we can deduce whether the model may be improved or not. The value of $\hat{\sigma}_0$ in the experimental case was 0.0272.

Next, put $i = 2$ in equation (5-2) to obtain the second model of the observation equation (5-23) for the second set of oscillations:

$$\text{(5-23)} \quad A_1 e^{-\lambda_1 t} \cos[(\theta_1 + \theta_1' t)(t - B_1)] + A_2 e^{-\lambda_2 t} \cos[(\theta_2 + \theta_2' t)(t - B_2)] + \boldsymbol{Q} - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 1. The values of t are in frames, $t = t_1, t_2 .....t_n$, see table B1, appendix B, and n is the number of observations

Table 2 and figure 23 summarise the results of values for the parameters of equation (5-23) and their standard deviations computed by a least squares adjustment.

5.6.2 Results from using the observation equation (5-23):

Table 2

Parameters A_1, θ_1, θ'_1, B_1, λ_1, $\boldsymbol{Q}$, A_2, θ_2, θ'_2, B_2, λ_2

and their standard deviations using the first set of data

Parameters	Values	σ
A_1	11.00503	0.003392
θ_1	0.000592	1.61E-08
θ'_1	4.48E-13	1.18E-13
B_1	9277.126	0.637706
λ_1	1.1E-07	3.72E-09
$\boldsymbol{Q}$	-0.10486	0.000618
A_2	0.044418	0.00472
θ_2	0.000677	6.42E-06
θ'_2	-1.7E-11	5.47E-11
B_2	6547.588	189.8212
λ_2	8.87E-06	1.78E-06
$\hat{\sigma}_0$	0.0145	

Figure 23

Residuals 2 of the first set of time observations

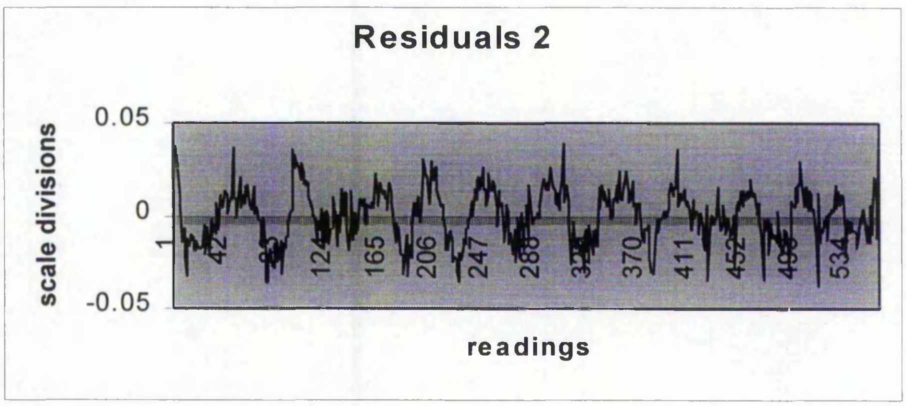


Table 2 shows that the value of $\hat{\sigma}_0$ improved from 0.0272 to 0.0145. The standard deviation of the midpoint of swing, $\boldsymbol{Q}$ improved from 0.001126 scale divisions (0″.67) to 0.000618 scale divisions (0″.37). The second oscillation period, ($2\pi/\theta_2$), which is 371.2 seconds is about 87% of the main period of oscillation, the corresponding amplitude is 0.0444 scale divisions (26″.64). Using a set with very few of time observations and another type of the Gyrotheodolite, Jeudy (Jeudy et al. 1981) found the second period of oscillation to be about half of the main period and the corresponding amplitude about 70″. In our case, the main period is approximately the same, the second period of oscillation is much longer.

Now put $i = 3$ in equation (5-2) to obtain the third model of the observation equation (5-24) for the third set of oscillations:

(5-24) $$\sum_{i=1}^{3} A_i e^{-\lambda_i t} \cos[(\theta_i + \theta_i' t)(t - B_i)] + \boldsymbol{Q} - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 2. The values of t are in frames, $t = t_1, t_2 \ldots . t_n$, see table B1, appendix B, and n is the number of observations

Table 3 and figure 24 summarise the results of values for the parameters of equation (5-24) and their standard deviations computed by a least squares adjustment.

5.6.3 Results from using the observation equation (5-24):

Table 3

Parameters A_1, θ_1, θ'_1, B_1, λ_1, $\boldsymbol{Q}$, A_2, θ_2, θ'_2, B_2, λ_2, A_3, θ_3, θ'_3, B_3, λ_3 and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.97568	0.059684
θ_1	0.000592	1.2E-06
θ'_1	3.26E-12	5.35E-12
B_1	9242.709	96.612
λ_1	8.03E-08	4.54E-08
$\boldsymbol{Q}$	-0.10441	0.000378
A_2	0.101236	0.388769
θ_2	0.00063	5.8E-05
θ'_2	1.48E-10	3.05E-10
B_2	3436.839	1682.654
λ_2	1.41E-05	3.87E-05
A_3	0.174148	0.707454
θ_3	0.000573	4.84E-05
θ'_3	-3.4E-10	3.25E-10
B_3	21812.25	528.0803
λ_3	2.43E-05	4.44E-05
$\hat{\sigma}_0$	0.0087	

Figure 24

Residuals 3 of the first set of time observations

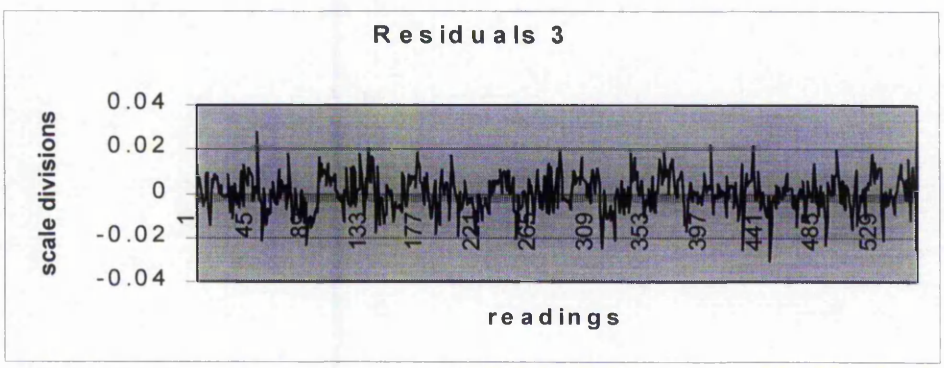


Table 3 shows that the value of $\hat{\sigma}_0$ improved from 0.0145 to 0.0087. However, the ratios between some parameters and their standard deviations are much smaller than one. This suggests that the values of some parameters are insignificant. Therefore, we reduced one of these parameters at a time to see its effect on the values of the other parameters and their respective standard deviations. Table 3 shows that $\dfrac{\lambda_2}{\sigma_{\lambda_2}} = 0.36$, then let us, put $\lambda_2 = 0$ to see the effect of neglecting λ_2 upon the other parameters, equation (5-24) becomes:

$$\begin{aligned} &A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + A_2 \cos[(\theta_2 + \theta_2' t)(t - B_2)] \\ &+ A_3 e^{-\lambda_3 t}\cos[(\theta_3 + \theta_3' t)(t - B_3)] + Q - \Delta = 0 \end{aligned} \tag{5-25}$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 3. The values of t are in frames, $t = t_1, t_2 .... t_n$, see table B1, appendix B, and n is the number of observations

Table 3a and figure 25 summarise the results of values for the parameters of equation (5-25) and their standard deviations computed by a least squares adjustment.

5.6.3.1 Results from using the observation equation (5-25):

Table 3a

Parameters A_1, θ_1, θ'_1, B_1, λ_1, $\boldsymbol{Q}$, A_2, θ_2, θ'_2, B_2, A_3, θ_3, θ'_3, B_3, λ_3 and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.98186	0.007327
θ_1	0.000592	1.73E-07
θ'_1	3.17E-12	9.42E-13
B_1	9254.717	10.43212
λ_1	7.76E-08	7.96E-09
$\boldsymbol{Q}$	-0.10438	0.000378
A_2	0.035627	0.007107
θ_2	0.000654	5.43E-06
θ'_2	-2.9E-11	2.82E-11
B_2	3872.234	464.4054
A_3	0.082806	0.062044
θ_3	0.000564	1.56E-05
θ'_3	-2.5E-10	9.93E-11
B_3	22202.39	434.1262
λ_3	1.4E-05	7.11E-06
$\hat{\sigma}_0$	0.0088	

Figure 25

Residuals 3a of the first set of time observations

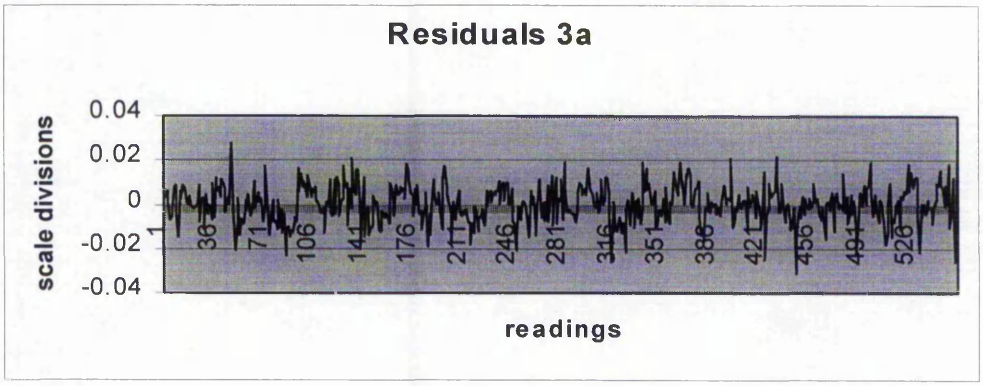


Table 3a shows that the ratios between the values of most parameters and their standard deviations have been improved. However, the ratio between θ'_2 and its standard deviation is just about one. Let us, put $\theta'_2 = 0$ in equation (5-25) to see the effect of neglecting θ'_2 upon the other parameters, equation (5-25) becomes:

$$(5\text{-}26)\quad \begin{aligned} &A_1 e^{-\lambda_1 t} \cos[(\theta_1 + \theta'_1 t)(t - B_1)] + A_2 \cos(\theta_2 (t - B_2) \\ &+ A_3 e^{-\lambda_3 t} \cos[(\theta_3 + \theta'_3 t)(t - B_3)] + \boldsymbol{Q} - \Delta = 0 \end{aligned}$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 3a. The values of t are in frames, $t = t_1, t_2.....t_n$, see table B1, appendix B, and n is the number of observations

Table 3b and figure 26 summarise the results of values of the parameters of equation (5-26) and their standard deviations computed by a least squares adjustment.

5.6.3.2 Results from using the observation equation (5-26):

Table 3b

Parameters A_1, θ_1, θ'_1, B_1, λ_1, $\boldsymbol{Q}$, A_2, θ_2, B_2, A_3, θ_3, θ'_3, B_3, λ_3 and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.98135	0.006363
θ_1	0.000592	1.15E-07
θ'_1	3.09E-12	6.55E-13
B_1	9256.621	6.298556
λ_1	7.94E-08	7.35E-09
$\boldsymbol{Q}$	-0.10436	0.000377
A_2	0.035093	0.00535
θ_2	0.00065	3.5E-06
B_2	3693.42	400.9385
A_3	0.06544	0.02946
θ_3	0.000557	1.07E-05
θ'_3	-1.9E-10	6.59E-11
B_3	22143.32	388.8328
λ_3	1.11E-05	4.03E-06
$\hat{\sigma}_0$	0.0088	

Figure 26

Residuals 3b of the first set of time observations

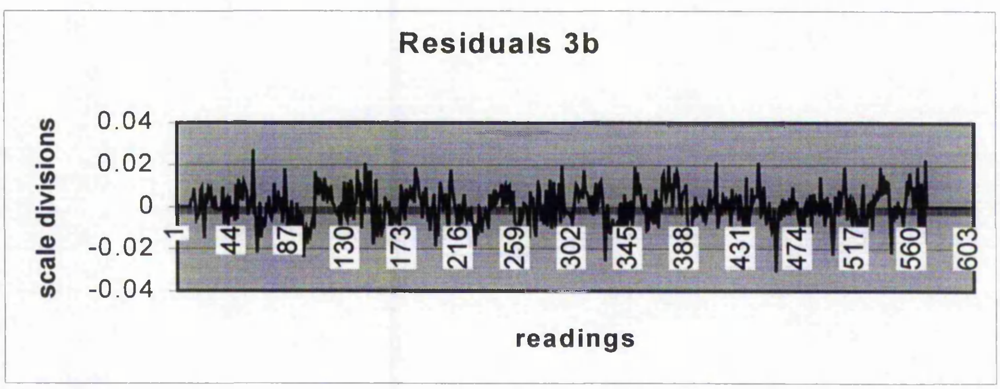


Table 3b shows that the effect of neglecting the parameters λ_2 and θ'_2 results in improvement of the ratios between the values of most other parameters and their standard deviations. However, let us, put $\lambda_3 = 0$ in equation (5-26) to see the effect of neglecting λ_3 upon the other parameters, equation (5-26) becomes:

$$
\begin{aligned}
(5\text{-}27)\quad & A_1 e^{-\lambda_1 t} \cos[(\theta_1 + \theta'_1 t)(t - B_1)] + A_2 \cos(\theta_2 (t - B_2)) \\
& + A_3 \cos[(\theta_3 + \theta'_3 t)(t - B_3)] + \boldsymbol{Q} - \Delta = 0
\end{aligned}
$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 3b. The values of t are in frames, $t = t_1, t_2.....t_n$, see table B1, appendix B, and n is the number of observations

Table 3c and figure 27 summarise the results of values of the parameters of equation (5-27) and their standard deviations computed by a least squares adjustment.

5.6.3.3 Results from using the observation equation (5-27):

Table 3c

Parameters A_1, θ_1, θ'_1, B_1, λ_1, $\boldsymbol{Q}$, A_2, θ_2, B_2, A_3, θ_3, θ'_3, B_3 and their standard deviations using the first set of data

Parameters	Values	sigma
A_1	10.99328	0.004214
θ_1	0.000592	5E-08
θ'_1	2.74E-12	3.26E-13
B_1	9263.026	2.03358
λ_1	1E-07	4.53E-09
$\boldsymbol{Q}$	-0.10445	0.000381
A_2	0.030138	0.002247
θ_2	0.000655	2.16E-06
B_2	4137.216	263.8212
A_3	0.027563	0.002893
θ_3	0.000533	5.36E-06
θ'_3	1.48E-11	2.56E-11
B_3	21968.72	365.1492
$\hat{\sigma}_0$	0.0089	

Figure 27

Residuals 3c of the first set of time observations

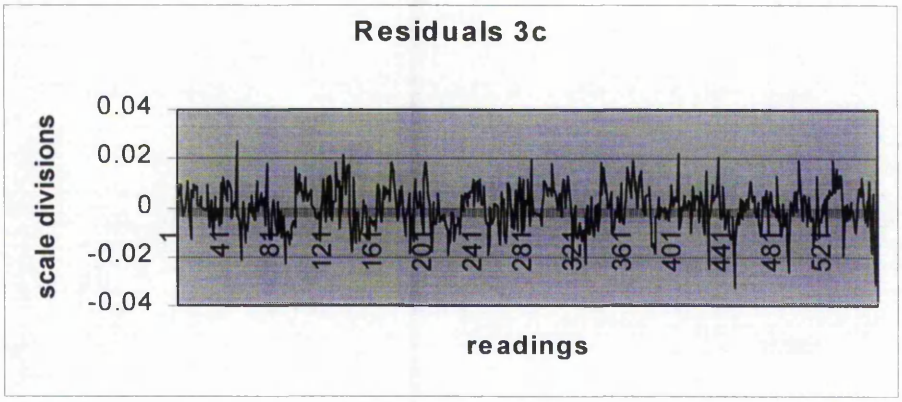


Table 3c shows that the effect of neglecting the parameter λ_3 results in improvement of the ratios between the values of most other parameters and their standard deviations. Finally, let us, put $\theta'_3 = 0$ in equation (5-27) to see the effect of neglecting θ'_3 upon the other parameters, equation (5-27) becomes:

$$(5\text{-}28)\quad \begin{aligned} &A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + A_2 \cos(\theta_2 (t - B_2)) \\ &+ A_3 \cos(\theta_3 (t - B_3)) + \boldsymbol{Q} - \Delta = 0 \end{aligned}$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 3c. The values of t are in frames, $t = t_1, t_2 \ldots . t_n$, see table B1, appendix B, and n is the number of observations

Table 3d and figure 28 summarise the results of values of the parameters of equation (5-28) and their standard deviations computed by a least squares adjustment.

5.6.3.4 Results from using the observation equation (5-28):

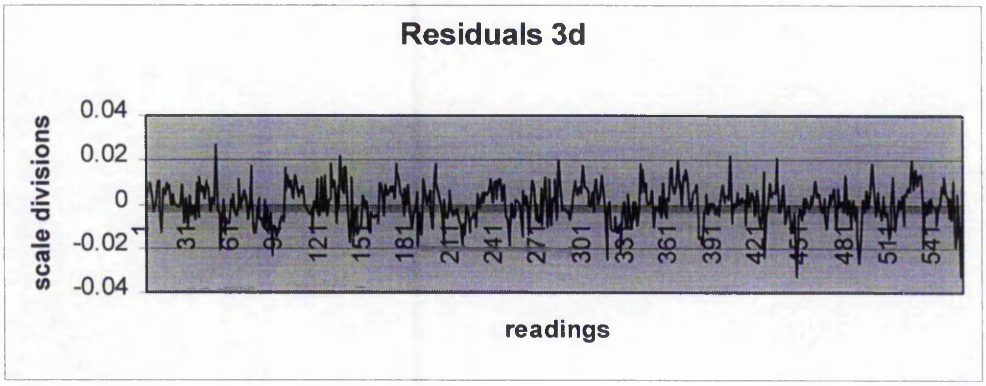
Table 3d

Parameters A_1, θ_1, θ'_1, B_1, λ_1, Q, A_2, θ_2, B_2, A_3, θ_3, B_3 and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.9941	0.004016
θ_1	0.000592	4.48E-08
θ'_1	2.8E-12	2.94E-13
B_1	9262.664	1.779515
λ_1	1.01E-07	4.56E-09
Q	-0.10446	0.000381
A_2	0.030424	0.002083
θ_2	0.000655	2.06E-06
B_2	4114.494	250.2048
A_3	0.028061	0.002646
θ_3	0.000535	2.47E-06
B_3	22060.64	286.5536
$\hat{\sigma}_0$	0.0089	

Figure 28

Residuals 3d of the first set of time observations



Tables 3, 3a, 3b, 3c and 3d show that the change in the value of $\hat{\sigma}_0$ is very little and changes from 0.0087 to 0.0089. However, neglecting the parameters λ_2, θ'_2, λ_3 and θ'_3 improved the precision of most other parameters in equation (5-24). The most important result, which may be drawn so far, is that including the parameters λ and θ' in equation (5-2), further than $i = 1$ has insignificant effect. Therefore, table 3d gives the best results for the parameters in equation (5-2).

Table 3d shows that the value of $\hat{\sigma}_0$ improved from 0.0145 in the second period to 0.0089. The standard deviation of the midpoint of swing, $\boldsymbol{Q}$ improved from 0.000618 scale divisions (0″.37) to 0.000381 scale divisions (0″.23). The third oscillation period, ($2\pi/\theta_3$), is about eight minutes and the corresponding amplitude is 0.028 scale divisions (16″.8). The amplitude is much smaller than the corresponding one obtained for the second period.

Now put $i = 4$ in equation (5-2) to obtain the fourth model of the observation equation (5-29) for the fourth set of oscillations:

$$(5\text{-}29)\quad \begin{aligned} &A_1 e^{-\lambda_1 t}\cos[(\theta_1+\theta_1' t)(t-B_1)] + A_2\cos(\theta_2(t-B_2)) + A_3\cos(\theta_3(t-B_3)) \\ &+ A_4\cos(\theta_4(t-B_4)) + \boldsymbol{Q} - \Delta = 0 \end{aligned}$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 3d. The values of t are in frames, $t = t_1, t_2 \ldots\ldots t_n$, see table B1, appendix B, and n is the number of observations

Table 4 and figure 29 summarise the results of values of the parameters of equation (5-29) and their standard deviations computed by a least squares adjustment.

5.6.4 Results from using the observation equation (5-29):

Table 4

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q (i = 1, 2, 3, 4) and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.99415	0.004035
θ_1	0.000592	4.5E-08
θ'_1	2.8E-12	2.96E-13
B_1	9262.638	1.786141
λ_1	1.01E-07	4.58E-09
Q	-0.10446	0.000381
A_2	0.030455	0.002092
θ_2	0.000654	2.06E-06
B_2	4112.509	250.6992
A_3	0.028098	0.002661
θ_3	0.000535	2.48E-06
B_3	10324.09	341.2761
A_4	0.00064	0.000537
θ_4	0.005498	2.14E-05
B_4	7161.258	303.6418
$\hat{\sigma}_0$	0.0089	

Figure 29

Residuals 4 of the first set of time observations

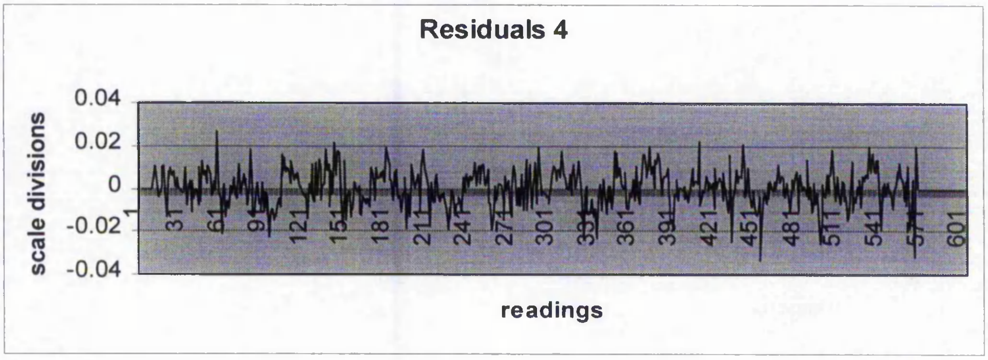


Table 4 shows that the fourth period of oscillations, ($2\pi/\theta_4$), is about 45 seconds, the corresponding amplitude is 0.00064 scale divisions (0''.38). This amplitude is too small to be seen, even with electronic registration instruments. The precision of the midpoint of swing did not change. However, we are going to compute all terms of the mathematical model to the end, in order to find whether there are any significant components of amplitudes and periods of oscillations.

Now put $i = 5$ in equation (5-2) to obtain the fifth model of the observation equation (5-30) for the fifth set of oscillations:

$$\begin{aligned}(5\text{-}30)\quad & A_1 e^{-\lambda_1 t}\cos[(\theta_1+\theta_1' t)(t-B_1)] + A_2\cos(\theta_2(t-B_2)) + A_3\cos(\theta_3(t-B_3)) \\ & + A_4\cos(\theta_4(t-B_4)) + A_5\cos(\theta_5(t-B_5)) + \boldsymbol{Q} - \Delta = 0\end{aligned}$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 4. The values of t are in frames, $t = t_1, t_2 \ldots . t_n$, see table B1, appendix B, and n is the number of observations

Table 5 and figure 30 summarise the results of values of the parameters of equation (5-30) and their standard deviations computed by a least squares adjustment.

5.6.5 Results from using the observation equation (5-30):

Table 5

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q (i = 1, 2,..5) and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.994	0.003936
θ_1	0.000592	4.42E-08
θ'_1	2.83E-12	2.91E-13
B_1	9262.498	1.750326
λ_1	1.01E-07	4.48E-09
Q	-0.10432	0.000372
A_2	0.030444	0.002066
θ_2	0.000654	2.03E-06
B_2	4093.863	246.3591
A_3	0.028479	0.002602
θ_3	0.000535	2.4E-06
B_3	10324.83	328.7829
A_4	0.000768	0.000524
θ_4	0.005492	1.74E-05
B_4	7140.605	246.9908
A_5	0.001192	0.000521
θ_5	0.004989	1.13E-05
B_5	9942.246	171.8059
$\hat{\sigma}_0$	0.0086	

Figure 30

Residuals 5 of the first set of time observations

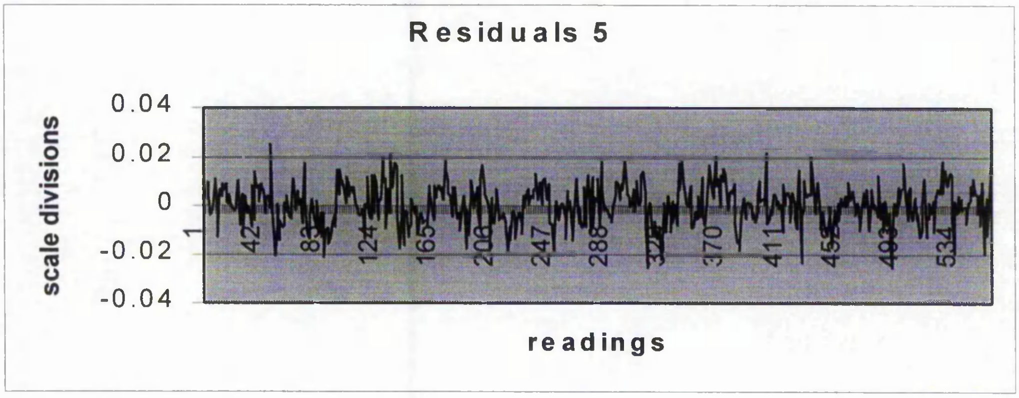


Table 5 shows that the fifth period of oscillations, ($2\,\pi/\theta_5$), is about 50 seconds, the corresponding amplitude is 0.0012 scale divisions (0".72). The precision of the midpoint of swing improved very little. Also, the improvement in the value of $\hat{\sigma}_0$ is small.

Now put $i = 6$ in equation (5-2) to obtain the sixth model of the observation equation (5-31) for the sixth set of oscillations:

$$\text{(5-31)}\quad A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + \sum_{i=2}^{6} A_i \cos(\theta_i (t - B_i)) + Q - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 5. The values of t are in frames, $t = t_1, t_2 \ldots . t_n$, see table B1, appendix B, and n is the number of observations

Table 6 and figure 31 summarise the results of values of the parameters of equation (5-31) and their standard deviations computed by a least squares adjustment.

5.6.6 Results from using the observation equation (5-31):

Table 6

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q (i = 1, 2,..6) and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.99251	0.002576
θ_1	0.000592	3.12E-08
θ'_1	2.16E-12	2.05E-13
B_1	9266.728	1.297892
λ_1	9.92E-08	2.95E-09
Q	-0.10434	0.000345
A_2	0.025733	0.001361
θ_2	0.000659	1.92E-06
B_2	4602.817	229.8256
A_3	0.02327	0.001527
θ_3	0.00053	2.06E-06
B_3	9652.699	287.3798
A_4	0.000783	0.000486
θ_4	0.005493	1.58E-05
B_4	7157.089	224.7367
A_5	0.001245	0.000483
θ_5	0.004989	1E-05
B_5	9941.915	152.6644
A_6	0.004896	0.000518
θ_6	0.000786	3.03E-06
B_6	8346.771	288.4146
$\hat{\sigma}_0$	0.0080	

Figure 31

Residuals 6 of the first set of time observations

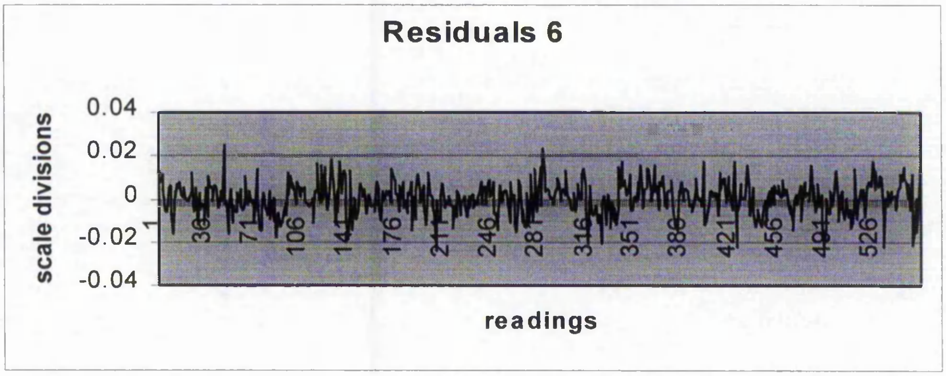


Table 6 shows that the sixth period of oscillations, ($2\,\pi/\theta_6$), is about 320 seconds. The corresponding amplitude is 0.0049 scale divisions (2″.9). The precision of the midpoint of swing improved very little. Also, the improvement in the precision of the value of $\hat{\sigma}_0$ is small.

Now put $i = 7$ in equation (5-2) to obtain the seventh model of the observation equation (5-32) for the seventh set of oscillations:

$$(5\text{-}32)\quad A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + \sum_{i=2}^{7} A_i \cos(\theta_i (t - B_i)) + \boldsymbol{Q} - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 6. The values of t are in frames, $t = t_1, t_2 \ldots . t_n$, see table B1, appendix B, and n is the number of observations

Table 7 and figure 32 summarise the results of values of the parameters of equation (5-32) and their standard deviations computed by a least squares adjustment.

5.6.7 Results from using the observation equation (5-32):

Table 7

Parameters A_i, θ_i, θ'_1, B_i, λ_1, $\boldsymbol{Q}$ ($i = 1, 2,..7$) and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.9923	0.002633
θ_1	0.000592	3.25E-08
θ'_1	2.22E-12	2.14E-13
B_1	9266.392	1.345589
λ_1	9.89E-08	3.02E-09
$\boldsymbol{Q}$	-0.10434	0.00034
A_2	0.026187	0.001414
θ_2	0.000658	1.9E-06
B_2	4559.217	228.2145
A_3	0.023631	0.001606
θ_3	0.00053	2.08E-06
B_3	9693.231	290.7822
A_4	0.000783	0.000479
θ_4	0.005493	1.56E-05
B_4	7157.433	221.4268
A_5	0.00122	0.000476
θ_5	0.004989	1.01E-05
B_5	9947.968	153.4587
A_6	0.004779	0.000519
θ_6	0.000783	3.15E-06
B_6	8171.162	305.0142
A_7	0.002141	0.000485
θ_7	0.000874	6.57E-06
B_7	10488.73	571.1728
$\hat{\sigma}_0$	0.0079	

Figure 32

Residuals 7 of the first set of time observations

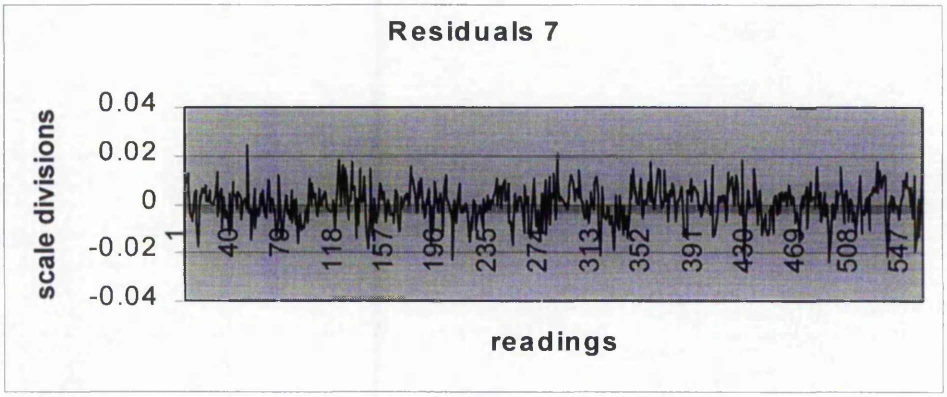


Table 7 shows that the seventh period of oscillations, ($2\ \pi / \theta_7$), is about 287 seconds. The corresponding amplitude is 0.0021 scale divisions (1″.26). The precision of the midpoint of swing improved very little. Also, the improvement in the value of $\hat{\sigma}_0$ is small.

Now put i = 8 in equation (5-2) to obtain the eighth model of the observation equation (5-33) for the eighth set of oscillations:

$$(5\text{-}33)\quad A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + \sum_{i=2}^{8} A_i \cos(\theta_i (t - B_i)) + Q - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 7. The values of t are in frames, $t = t_1, t_2 \ldots. t_n$, see table B1, appendix B, and n is the number of observations

Table 8 and figure 33 summarise the results of values of the parameters of equation (5-33) and their standard deviations computed by a least squares adjustment.

5.6.8 Results from using the observation equation (5-33):

Table 8

Parameters A_i, θ_i, θ'_1, B_i, λ_1, $\boldsymbol{Q}$ ($i = 1, 2,..8$) and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.99321	0.002615
θ_1	0.000592	5.88E-08
θ'_1	2.14E-12	3.84E-13
B_1	9266.792	2.472393
λ_1	9.99E-08	3E-09
$\boldsymbol{Q}$	-0.10437	0.000338
A_2	0.025659	0.002568
θ_2	0.000659	3.36E-06
B_2	4659.981	400.6382
A_3	0.023251	0.002687
θ_3	0.00053	3.18E-06
B_3	9665.204	442.7895
A_4	0.000794	0.000475
θ_4	0.005493	1.53E-05
B_4	7160.485	216.6394
A_5	0.001228	0.000473
θ_5	0.004989	9.95E-06
B_5	9947.454	151.254
A_6	0.004795	0.000623
θ_6	0.000785	5.96E-06
B_6	8238.855	611.7966
A_7	0.001949	0.000512
θ_7	0.000876	7.35E-06
B_7	10664.74	640.872
A_8	0.001777	0.000531
θ_8	0.000742	1.3E-05
B_8	6188.041	1766.129
$\hat{\sigma}_0$	0.0078	

Figure 33

Residuals 8 of the first set of time observations

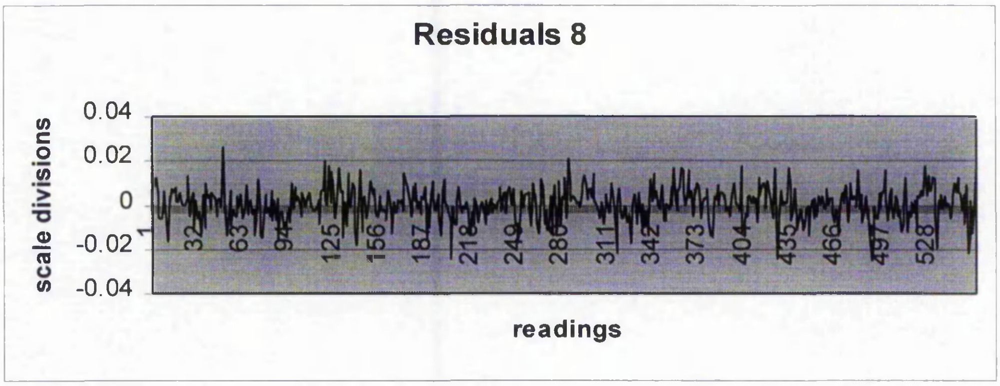


Table 8 shows that the eighth period of oscillations, $(2\pi/\theta_8)$, is about 339 seconds. The corresponding amplitude is 0.0018 scale divisions (1″.07). The precision of the midpoint of swing improved very little. Also, the improvement in the value of $\hat{\sigma}_0$ is small.

Now put $i = 9$ in equation (5-2) to obtain the ninth model of the observation equation (5-34) for the ninth set of oscillations:

$$(5\text{-}34)\quad A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + \sum_{i=2}^{9} A_i \cos(\theta_i (t - B_i)) + Q - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 8. The values of t are in frames, $t = t_1, t_2 \ldots . t_n$, see table B1, appendix B, and n is the number of observations

Table 9 and figure 34 summarise the results of values of the parameters of equation (5-34) and their standard deviations computed by a least squares adjustment.

5.6.9 Results from using the observation equation (5-34):

Table 9

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q ($i = 1, 2,..9$) and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.9932	0.00239
θ_1	0.000592	5.35E-08
θ'_1	1.97E-12	3.5E-13
B_1	9267.872	2.272653
λ_1	9.99E-08	2.76E-09
Q	-0.10439	0.000335
A_2	0.024545	0.002258
θ_2	0.00066	3.26E-06
B_2	4827.704	387.1396
A_3	0.022187	0.002332
θ_3	0.000529	3.06E-06
B_3	9470.922	427.8994
A_4	0.000796	0.000471
θ_4	0.005493	1.51E-05
B_4	7159.61	214.2411
A_5	0.001217	0.000468
θ_5	0.004989	9.95E-06
B_5	9945.114	151.2847
A_6	0.005068	0.000768
θ_6	0.000783	6.39E-06
B_6	8059.247	666.2624
A_7	0.002009	0.000508
θ_7	0.000877	7.02E-06
B_7	10635.44	613.1868
A_8	0.001758	0.000655
θ_8	0.000742	1.4E-05
B_8	6577.401	1974.41
A_9	0.001701	0.000475
θ_9	0.001091	7.41E-06
B_9	7883.404	522.7772
$\hat{\sigma}_0$	0.0078	

Figure 34

Residuals 9 of the first set of time observations

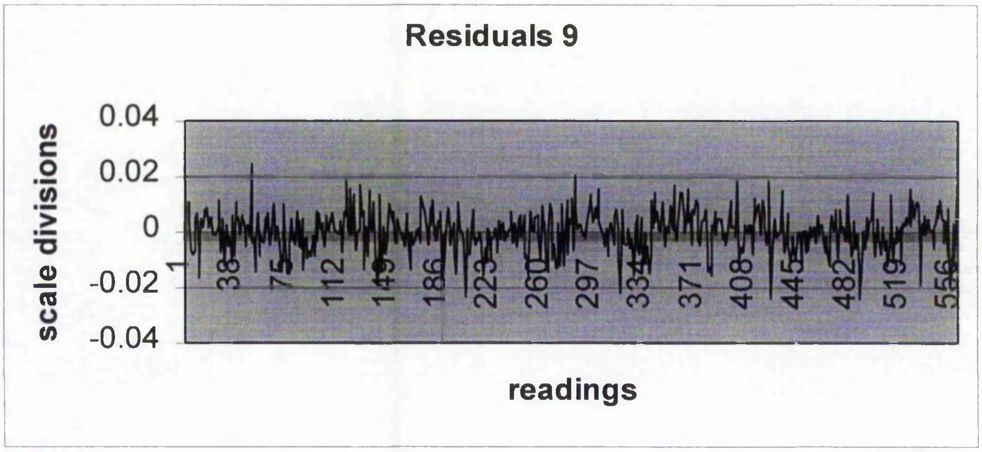


Table 9 shows that the ninth period of oscillations, ($2\ \pi / \theta_9$), is about 230 seconds. The corresponding amplitude is 0.0017 scale divisions ($1''.02$). The precision of the midpoint of swing improved very little. The value of $\hat{\sigma}_0$ did not change.

Finally, put i = 10 in equation (5-2) to obtain the tenth model of the observation equation (5-35) for the tenth set of oscillations:

$$(5\text{-}35) \quad A_1 e^{-\lambda_1 t} \cos[(\theta_1 + \theta_1' t)(t - B_1)] + \sum_{i=2}^{10} A_i \cos(\theta_i (t - B_i)) + \boldsymbol{Q} - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 9. The values of t are in frames, $t = t_1, t_2.....t_n$, see table B1, appendix B, and n is the number of observations

Table 10 and figure 35 summarise the results of values of the parameters of equation (5-35) and their standard deviations computed by a least squares adjustment.

5.6.10 Results from using the observation equation (5-35):

Table 10

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q ($i = 1, 2, \ldots 10$) and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.99447	0.002463
θ_1	0.000592	6.08E-08
θ'_1	2.02E-12	3.98E-13
B	9267.6	2.580415
λ_1	1.01E-07	2.84E-09
Q	-0.10438	0.000329
A_2	0.024696	0.00253
θ_2	0.000661	3.73E-06
B_2	4867.033	442.0997
A_3	0.022625	0.002685
θ_3	0.000529	3.34E-06
B_3	9550.117	464.9574
A_4	0.000825	0.000461
θ_4	0.005493	1.43E-05
B_4	7164.202	202.5521
A_5	0.001247	0.000459
θ_5	0.004988	9.52E-06
B_5	9928.548	144.8325
A_6	0.00522	0.001193
θ_6	0.000778	7.81E-06
B_6	7653.342	825.3774
A_7	0.002084	0.000501
θ_7	0.000881	7.04E-06
B_7	10956.17	609.9519
A_8	0.001949	0.001114
θ_8	0.000739	1.6E-05
B_8	6776.634	2294.958
A_9	0.001468	0.00047
θ_9	0.001091	8.68E-06
B_9	7968.57	612.4939
A_{10}	0.002414	0.000483
θ_{10}	0.000978	5.69E-06
B_{10}	12107.36	425.9293
$\hat{\sigma}_0$	0.0076	

Figure 35

Residuals 10 of the first set of time observations

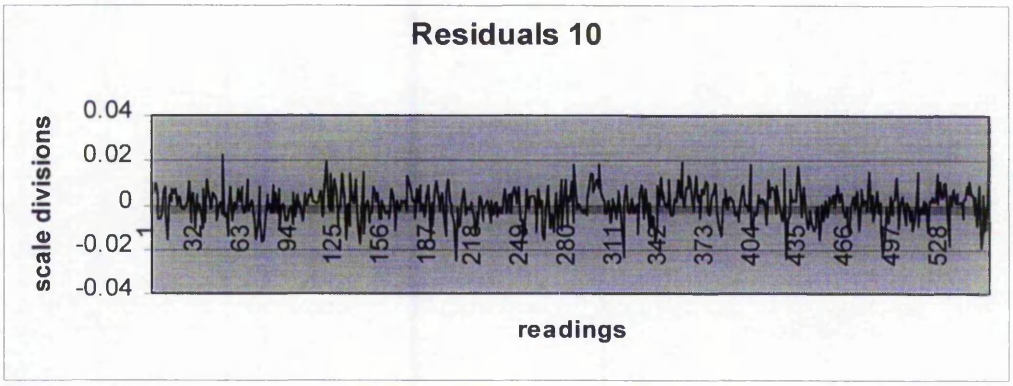


Table 10 shows that the tenth period of oscillations, $(2\pi/\theta_{10})$, is about 257 seconds. The corresponding amplitude is 0.0024 scale divisions (1″.45). The precision of the midpoint of swing improved very little. Also the improvement of the value of $\hat{\sigma}_0$ is small.

The maximum numbers of parameters, which is allowed, theoretically in terms of equation (5-2) is $i = 10$. However, we tried to find further parameters in the term $i = 11$ as a test of justification. Put $i = 11$ in terms of equation (5-2) to obtain the equation (5-35′):

$$(5\text{-}35') \quad A_1 e^{-\lambda_1 t} \cos[(\theta_1 + \theta_1' t)(t - B_1)] + \sum_{i=2}^{11} A_i \cos(\theta_i (t - B_i)) + \boldsymbol{Q} - \Delta = 0$$

We computed the adjustment by least squares using as approximate values of the parameters the values of table 10. The values of t are in frames, $t = t_1, t_2 \ldots\ldots t_n$, see table B1, appendix B, and n is the number of observations.

Table 10a and figure 35a summarise the results of values of the parameters of equation (5-35′) and their standard deviations computed by a least squares adjustment.

5.6.10.1 Results from using the observation equation (5-35′):

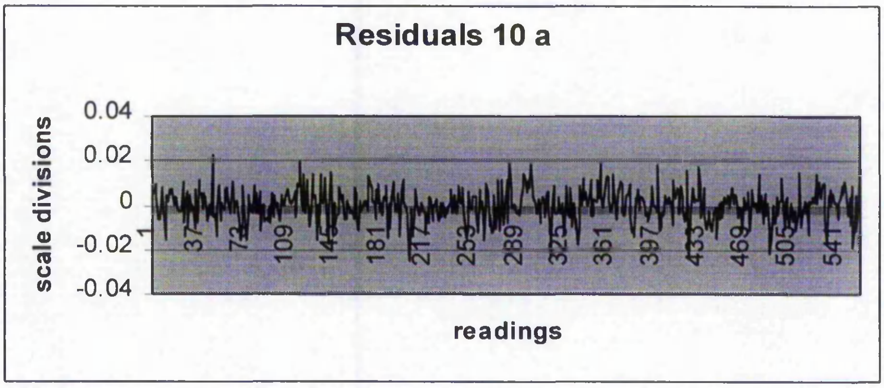
Table 10a

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q (i = 1, 2,....11) and their standard deviations using the first set of data

Parameters	Values	σ
A_1	10.9943993	0.00245082
θ_1	0.00059224	6.0174E-08
θ'_1	2.0047E-12	3.9377E-13
B_1	9267.69168	2.55537617
λ_1	1.0124E-07	2.8286E-09
Q	-0.10441088	0.00033404
A_2	0.02461678	0.00249635
θ_2	0.00066078	3.6917E-06
B_2	4878.10205	437.689886
A_3	0.02252202	0.00265208
θ_3	0.00052928	3.3292E-06
B_3	9536.16782	463.845344
A_4	0.00082339	0.00046228
θ_4	0.00549344	1.4336E-05
B_4	7163.93418	203.348187
A_5	0.00124632	0.00045971
θ_5	0.00498813	9.5473E-06
B_5	9928.94596	145.282313
A_6	0.00523253	0.00122206
θ_6	0.00077847	7.8994E-06
B_6	7657.97113	834.986684
A_7	0.00207499	0.00050305
θ_7	0.00088034	7.1481E-06
B_7	10937.8214	617.161381
A_8	0.00195841	0.00113902
θ_8	0.00073998	1.6092E-05
B_8	6828.62197	2303.54663
A_9	0.00147222	0.00047221
θ_9	0.00109115	8.7542E-06
B_9	8042.45075	615.776389
A_{10}	0.00239908	0.00048493
θ_{10}	0.0009773	5.7768E-06
B_{10}	12094.3293	431.089758
A_{11}	-0.00034121	0.00047258
θ_{11}	0.00124744	3.6669E-05
B_{11}	20392.0278	1939.73513
$\hat{\sigma}_0$	0.0076	

Figure 35a

Residuals 10a of the first set of time observations



From table 10a, it is clear that the standard deviation of the parameter A_{11} is much greater than its value 0″.2. This suggests that the term A_{11} does not really exist and therefore all terms with subscript 11 also do not exist.

The conclusions, which may be drawn from the results obtained from paragraphs (5-6), are summarised as follows:

Least squares adjustment shows that the midpoint of swing may be determined at precision of 0″.2 by using a video camera and video imagery with frames analysis. This research is the first, to our knowledge, to use least squares adjustments for Gyrotheodolite observations obtained with this technique. Obtaining ten periods of oscillations resulted in a better understanding of the motion of the gyroscope. However, we are really only concerned with precise determination of the midpoint of swing. That precision improves very little beyond the fourth period of the oscillations. Therefore, based upon the above experimental data the best model that may express the motion of the moving mark of Wild GAK1 suspended gyroscope is equation (5-28):

$$A_1 e^{-\lambda_1 t}\cos[(\theta_1+\theta_1' t)(t-B_1)] + A_2\cos(\theta_2(t-B_2)) +$$
$$A_3\cos(\theta_3(t-B_3)) + \boldsymbol{Q} - \Delta = 0$$

5.7 Results from using the second set of time observations:

The second set of observations took about three hours, the battery was fully charged at start. The values of these time observations are shown in "table B2, appendix B". The same procedures are followed, as for the first set of observations, to obtain the parameters and the standard deviations of their values in equation (5-2). In all the tables below, the parameters A, θ, θ', B, λ and $\boldsymbol{Q}$ have the same units as described in paragraph (5.6).

A least squares adjustment was performed in a step by step manner for the mathematical model (5-2) $\sum_{i=1}^{10} A_i e^{-\lambda_i t}\cos((\theta_i + \theta_i' t)(t - B_i)) + \boldsymbol{Q} - \Delta = 0$. This represents the equation of motion of the moving mark of the Gyrotheodolite. The model is in the form of a damped harmonic motion with ten periods for the oscillations.

Now put i = 1 in equation (5-2) to obtain the first model of the observation equation (5-22) for the first set of oscillations:

(5-22) $A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + \boldsymbol{Q} - \Delta = 0$

The provisional values of the parameters of this equation (5-22) are found following the steps explained in paragraph (5.5). t, is the time in frames in table B2, appendix B. Table 11 and figure 36 summarise the results for values of these parameters and their standard deviations computed by a least squares adjustment.

5.7.1 Results from using the observation equation (5-22):

Table 11

Parameters A, θ, θ', B, λ, Q and their standard deviations using the second set of data

Parameters	Values	σ
A	12.33664	0.009132
θ	0.000597	1.26E-08
θ'	3.9E-12	4.52E-14
B	9056.718	1.159053
λ	9.3E-08	4.6E-09
Q	-0.93581	0.002431
$\hat{\sigma}_0$	0.082	

Figure 36

Residuals 1 of the second set of time observations

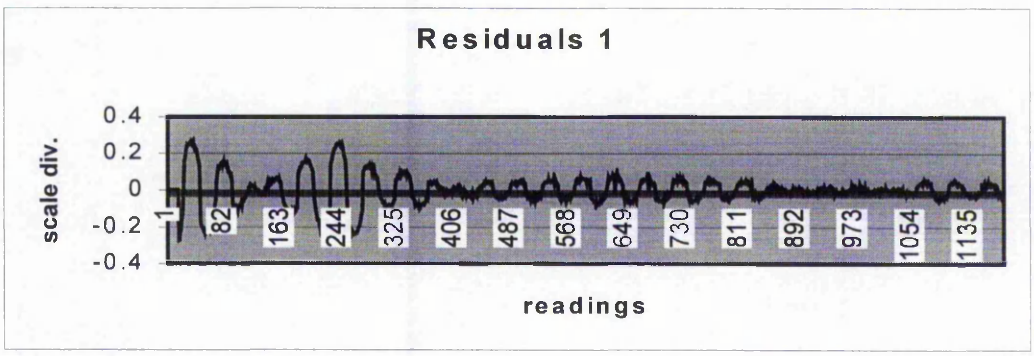


Table 11 shows that the main period of oscillations ($2\pi/\theta$), is about 421 seconds, which is approximately the same as for the first set of observations. The standard deviation of θ', the change in the frequency of the main oscillation, as for the first set of observations, is much less than the value of θ'. The damping coefficients, which are shown in tables 11 and 1, and obtained for the first period of oscillations for both sets of observations, are very close. The standard deviation of the midpoint of swing, $\boldsymbol{Q}$, is about 1″.46 arc seconds. The value of $\hat{\sigma}_0$ is 0.082. Figure 36 shows the residuals of the observations, values of which are shown in appendix C.

Now put $i = 2$ in equation (5-2), but take account of the reasoning in paragraph (5.6.3.4) about ignoring the parameters λ and θ' from the second period of oscillations. The observation equation for the second model for the second set of oscillations is:

(5-23′) $A_1 e^{-\lambda_1 t} \cos[(\theta_1 + \theta_1' t)(t - B_1)] + A_2 \cos[\theta_2 (t - B_2)] + \boldsymbol{Q} - \Delta = 0$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 11. The values of t are in frames, $t = t_1, t_2 \ldots . t_n$, see table B1, appendix B, and n is the number of observations

Table 12 and figure 37 summarise the results of values of the parameters of equation (5-23′) and their standard deviations computed by a least squares adjustment.

5.7.2 Results from using the observation equation (5-23′):

Table 12

Parameters A_1, θ_1, θ'_1, B_1, λ_1, Q, A_2, θ_2, B_2 and their standard deviations using the second set of data

Parameters	Values	σ
A_1	12.3366	0.009141
θ_1	0.000597	1.27E-08
θ'_1	3.9E-12	4.53E-14
B_1	9056.691	1.160211
λ_1	9.3E-08	4.6E-09
Q	-0.9357	0.002436
A_2	0.003885	0.003399
θ_2	0.001306	1.14E-05
B_2	10887.61	1318.179
$\hat{\sigma}_0$	0.082	

Figure 37

Residuals 2 of the second set of time observations

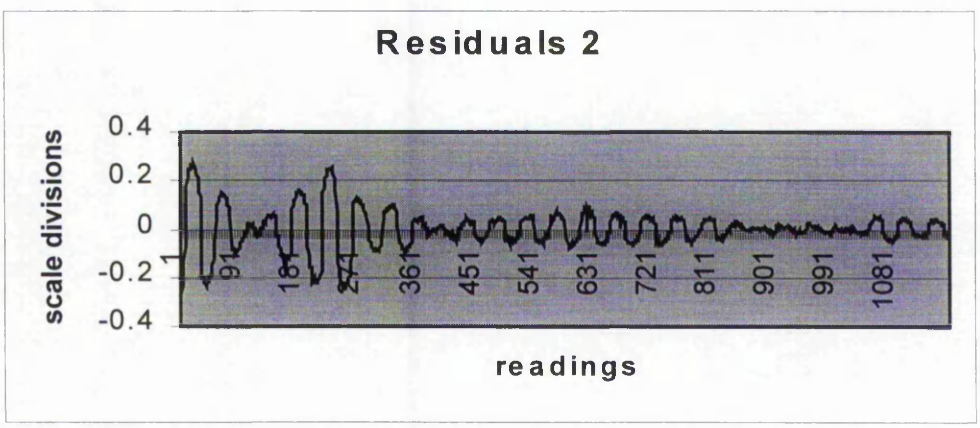


Table 12 shows that the second period of oscillations ($2\pi/\theta_2$) is about 192 seconds, 46% of the main period of oscillations, the corresponding amplitude is 2″.3 arc seconds. The standard deviation of the midpoint of swing, $\boldsymbol{Q}$, did not change. Also, the value of $\hat{\sigma}_0$ did not improve.

Now put $i = 3$ in equation (5-2) to obtain the third model for the third set of oscillations, the observation equation (5-28):

$$A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + A_2\cos(\theta_2(t - B_2)) + A_3\cos(\theta_3(t - B_3)) + \boldsymbol{Q} - \Delta = 0 \qquad (5\text{-}28)$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 12. The values of t are in frames, $t = t_1, t_2 \ldots . t_n$, see table B1, appendix B, and n is the number of observations

Table 13 and figure 38 summarise the results of values of the parameters of equation (5-28) and their standard deviations computed by a least squares adjustment.

5.7.3 Results from using the observation equation (5-28):

Table 13

Parameters A_1, θ_1, θ'_1, B_1, λ_1, Q, A_2, θ_2, B_2, A_3, θ_3, B_3

and their standard deviations using the second set of data

Parameters	Values	σ
A_1	12.33694	0.009187
θ_1	0.000597	1.27E-08
θ'_1	3.9E-12	4.57E-14
B_1	9056.522	1.167436
λ_1	9.33E-08	4.63E-09
Q	-0.9357	0.002437
A_2	0.003892	0.003399
θ_2	0.001307	1.14E-05
B_2	10881.49	1314.364
A_3	0.006502	0.003406
θ_3	0.000427	6.93E-06
B_3	10637.74	2452.808
$\hat{\sigma}_0$	0.082	

Figure 38

Residuals 3 of the second set of time observations

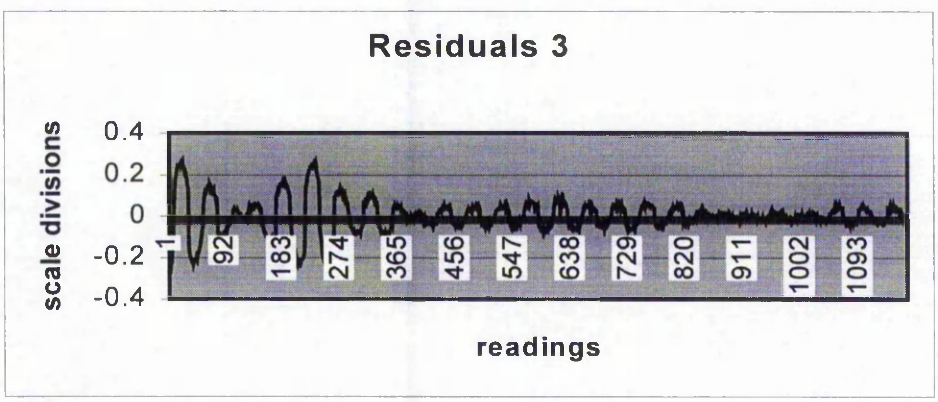


Table 13 shows that the third period of oscillations ($2\pi/\theta_3$) is about 588 seconds, the corresponding amplitude is 3″.9 arc seconds. The standard deviation of the midpoint of swing, ***Q***, did not change. Also, the value of $\hat{\sigma}_0$ did not improve.

Now put $i = 4$ in equation (5-2) to obtain the fourth model for the fourth set of oscillations, the observation equation (5-29):

$$(5\text{-}29)\quad A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + A_2\cos(\theta_2(t - B_2)) + A_3\cos(\theta_3(t - B_3)) + A_4\cos(\theta_4(t - B_4)) + \boldsymbol{Q} - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 13. The values of t are in frames, $t = t_1, t_2.....t_n$, see table B1, appendix B, and n is the number of observations

Table 14 and figure 39 summarise the results of values of the parameters of equation (5-29) and their standard deviations computed by a least squares adjustment.

5.7.4 Results from using the observation equation (5-29):

Table 14

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q (i = 1, 2, 3, 4) and their standard deviations using the second set of data

Parameters	Values	σ
A_1	12.33613	0.009147
θ_1	0.000597	1.27E-08
θ'_1	3.9E-12	4.53E-14
B_1	9056.309	1.160395
λ_1	9.27E-08	4.6E-09
Q	-0.93589	0.002431
A_2	0.004634	0.003388
θ_2	0.001285	9.55E-06
B_2	10821.27	1121.058
A_3	0.006904	0.00346
θ_3	0.00043	6.74E-06
B_3	9895.815	2340.092
A_4	0.012601	0.003463
θ_4	0.000391	3.68E-06
B_4	10449.69	1399.952
$\hat{\sigma}_0$	0.081	

Figure 39

Residuals 4 of the second set of time observations

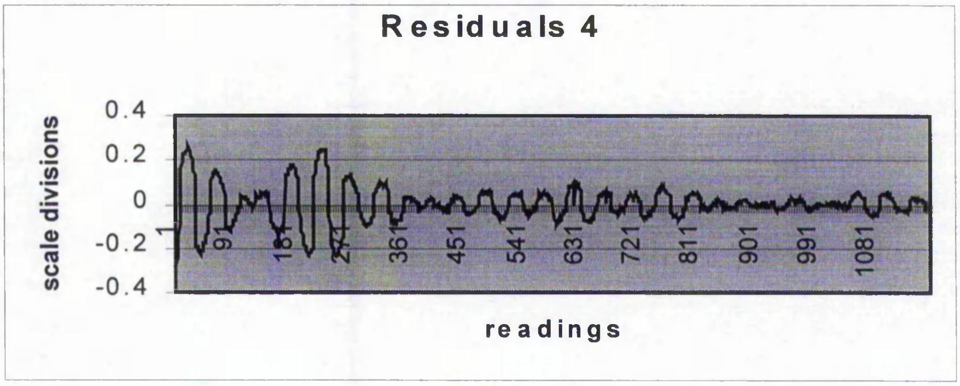


Table 14 shows that the fourth period of oscillations ($2\pi/\theta_4$) is about 643 seconds, the corresponding amplitude is 7″.5 arc seconds. The standard deviation of the midpoint of swing, $\boldsymbol{Q}$, changed very little. Also, the value of $\hat{\sigma}_0$ improved very little.

Now put $i = 5$ in equation (5-2) to obtain the fifth model for the fifth set of oscillations, the observation equation (5-30):

$$\text{(5-30)} \quad \begin{aligned} &A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + A_2\cos(\theta_2(t - B_2)) + A_3\cos(\theta_3(t - B_3)) \\ &+ A_4\cos(\theta_4(t - B_4)) + A_5\cos(\theta_5(t - B_5)) + \boldsymbol{Q} - \Delta = 0 \end{aligned}$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 14. The values of t are in frames, $t = t_1, t_2 \ldots. t_n$, see table B1, appendix B, and n is the number of observations

Table 15 and figure 40 summarise the results of values of the parameters of equation (5-30) and their standard deviations computed by a least squares adjustment.

5.6.5 Results from using the observation equation (5-30):

Table 15

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q (i = 1, 2,..5) and their standard deviations using the second set of data

Parameters	Values	σ
A_1	12.32736	0.008204
θ_1	0.000597	1.18E-08
θ'_1	3.87E-12	4.19E-14
B_1	9055.182	1.100592
λ_1	8.91E-08	4.14E-09
Q	-0.93568	0.002175
A_2	0.00534	0.003034
θ_2	0.00128	7.48E-06
B_2	10601.9	880.0046
A_3	0.052502	0.003073
θ_3	0.000664	8.17E-07
B_3	78262.35	120.2787
A_4	0.011313	0.003036
θ_4	0.000383	3.57E-06
B_4	9783.19	1410.636
A_5	0.007514	0.003043
θ_5	0.000266	5.36E-06
B_5	104139.8	1693.834
$\hat{\sigma}_0$	0.073	

Figure 40

Residuals 5 of the second set of time observations

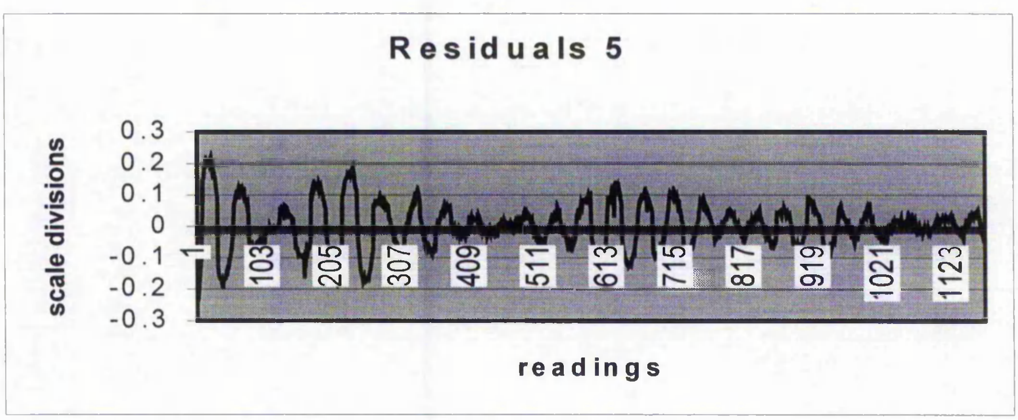


Table 15 shows that the fifth period of oscillations ($2\pi/\theta_5$) is 945 seconds, the corresponding amplitude is $4''.5$ arc seconds. The standard deviation of the midpoint of swing, $\boldsymbol{Q}$, improved from $1''.46$ arc seconds to $1''.30$ arc seconds. Also, the value of $\hat{\sigma}_0$ improved from 0.081 to 0.073.

Now put $i = 6$ in equation (5-2) to obtain the sixth model for the sixth set of oscillations, the observation equation (5-31):

$$(5\text{-}31)\quad A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + \sum_{i=2}^{6} A_i \cos(\theta_i (t - B_i)) + \boldsymbol{Q} - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 15. The values of t are in frames, $t = t_1, t_2 \ldots . t_n$, see table B1, appendix B, and n is the number of observations

Table 16 and figure 41 summarise the results of values of the parameters of equation (5-31) and their standard deviations computed by a least squares adjustment.

5.7.6 Results from using the observation equation (5-31):

Table 16

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q ($i = 1, 2,..6$) and their standard deviations using the second set of data

Parameters	Values	σ
A_1	12.35175	0.007335
θ_1	0.000597	1.85E-08
θ'_1	3.43E-12	6.3E-14
B_1	9065.596	1.689128
λ_1	9.92E-08	3.91E-09
Q	-0.93648	0.001657
A_2	0.003605	0.002314
θ_2	0.001279	8.44E-06
B_2	10499.88	995.5869
A_3	0.076816	0.003091
θ_3	0.000562	7.98E-07
B_3	58581.56	145.9844
A_4	0.00955	0.002324
θ_4	0.00039	3.23E-06
B_4	10053.47	1255.743
A_5	0.00582	0.002321
θ_5	0.000267	5.24E-06
B_5	103207.7	1666.777
A_6	0.050048	0.002369
θ_6	0.000662	6.92E-07
B_6	21163.52	148.697
$\hat{\sigma}_0$	0.055	

Figure 41

Residuals 6 of the second set of time observations

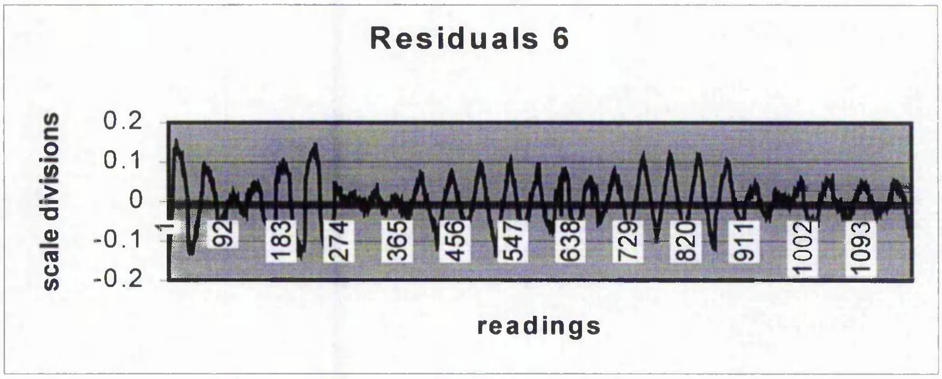


Table 16 shows that the sixth period of oscillations ($2\pi/\theta_6$) is about 380 seconds, the corresponding amplitude is about 30″ arc seconds, which is too large to be neglected. The standard deviation of the midpoint of swing, $\boldsymbol{Q}$, improved from 1″.30 arc seconds to 1″ arc seconds. Also, the value of $\hat{\sigma}_0$ improved from 0.073 to 0.055.

Now put i = 7 in equation (5-2) to obtain the seventh model for the seventh set of oscillations, the observation equation (5-32):

$$(5\text{-}32)\quad A_1 e^{-\lambda_1 t}\cos[(\theta_1+\theta_1' t)(t-B_1)]+\sum_{i=2}^{7} A_i\cos(\theta_i(t-B_i))+\boldsymbol{Q}-\Delta=0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 16. The values of t are in frames, $t = t_1, t_2.....t_n$, see table B1, appendix B, and n is the number of observations

Table 17 and figure 42 summarise the results of values of the parameters of equation (5-32) and their standard deviations computed by least squares adjustment.

5.7.7 Results from using the observation equation (5-32):

Table 17

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q ($i = 1, 2,..7$) and their standard deviations using the second set of data

Parameters	Values	σ
A_1	12.35147	0.007402
θ_1	0.000597	1.89E-08
θ'_1	3.43E-12	6.42E-14
B_1	9065.537	1.72151
λ_1	9.9E-08	3.95E-09
Q	-0.9365	0.001659
A_2	0.003622	0.002317
θ_2	0.001279	8.42E-06
B_2	10498.54	993.0766
A_3	0.076759	0.003127
θ_3	0.000562	8.09E-07
B_3	58584.11	148.0386
A_4	0.009904	0.002431
θ_4	0.000391	3.36E-06
B_4	10171.82	1279.893
A_5	0.005793	0.002335
θ_5	0.000267	5.28E-06
B_5	103189.2	1675.208
A_6	0.049999	0.002372
θ_6	0.000662	6.97E-07
B_6	21167.8	149.5173
A_7	0.002545	0.002429
θ_7	0.000431	1.31E-05
B_7	22854.72	4160.438
$\hat{\sigma}_0$	0.055	

Figure 42

Residuals 7 of the second set of time observations

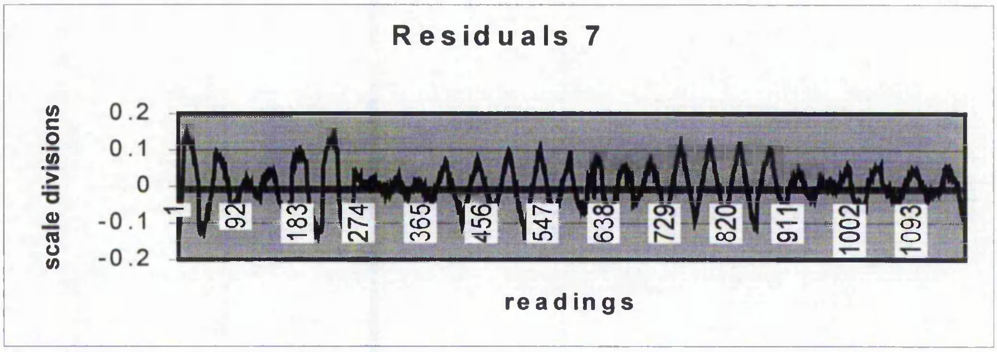


Table 17 shows that the seventh period of oscillations ($2\pi/\theta_7$) is about 583 seconds, the corresponding amplitude is about 1″.5 arc seconds. The standard deviation of the midpoint of swing, $\boldsymbol{Q}$, changed very little. The value of $\hat{\sigma}_0$ did not change significantly.

Now put $i = 8$ in equation (5-2) to obtain the eighth model for the eighth set of oscillations, the observation equation (5-33):

$$(5\text{-}33)\quad A_1 e^{-\lambda_1 t}\cos[(\theta_1+\theta_1' t)(t-B_1)]+\sum_{i=2}^{8} A_i\cos(\theta_i(t-B_i))+\boldsymbol{Q}-\Delta=0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 17. The values of t are in frames, $t = t_1, t_2 \ldots\ldots t_n$, see table B1, appendix B, and n is the number of observations

Table 18 and figure 43 summarise the results of values of the parameters of equation (5-33) and their standard deviations computed by a least squares adjustment.

5.7.8 Results from using the observation equation (5-33):

Table 18

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q (i = 1, 2,..8) and their standard deviations using the second set of data

Parameters	Values	σ
A_1	12.35106	0.007428
θ_1	0.000597	1.89E-08
θ'_1	3.43E-12	6.44E-14
B_1	9065.548	1.726627
λ_1	9.88E-08	3.97E-09
Q	-0.93657	0.001663
A_2	0.003617	0.002318
θ_2	0.001279	8.46E-06
B_2	10484.55	998.5067
A_3	0.07679	0.003137
θ_3	0.000562	8.11E-07
B_3	58588.72	148.3942
A_4	0.009894	0.002437
θ_4	0.00039	3.37E-06
B_4	10139.19	1286.75
A_5	0.005956	0.002353
θ_5	0.000267	5.21E-06
B_5	103148.8	1638.732
A_6	0.049964	0.002374
θ_6	0.000662	6.98E-07
B_6	21170.13	149.7461
A_7	0.002556	0.002439
θ_7	0.000431	1.3E-05
B_7	22638.5	4178.326
A_8	0.003347	0.002348
θ_8	0.000177	9.18E-06
B_8	10738.4	7894.361
$\hat{\sigma}_0$	0.055	

Figure 43

Residuals 8 of the second set of time observations

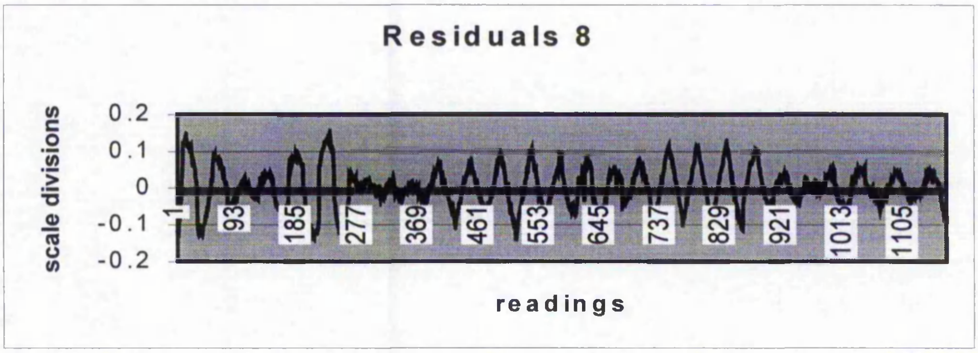


Table 18 shows that the eighth period of oscillations ($2\ \pi / \theta_8$) is a very long at about 1420 seconds, the corresponding amplitude is 2″ arc seconds. The standard deviation of the midpoint of swing, $\boldsymbol{Q}$, changed very little. The value of $\hat{\sigma}_0$ did not change.

Now put i = 9 in equation (5-2) to obtain the ninth model for the ninth set of oscillations, the observation equation (5-34):

$$(5\text{-}34) \quad A_1 e^{-\lambda_1 t} \cos[(\theta_1 + \theta_1' t)(t - B_1)] + \sum_{i=2}^{9} A_i \cos(\theta_i (t - B_i)) + \boldsymbol{Q} - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 18. The values of t are in frames, $t = t_1, t_2 \ldots.. t_n$, see table B1, appendix B, and n is the number of observations

Table 19 and figure 44 summarise the results of values of the parameters of equation (5-34) and their standard deviations computed by a least squares adjustment.

5.7.9 Results from using the observation equation (5-34):

Table 19

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q ($i = 1, 2,..9$) and their standard deviations using the second set of data

Parameters	Values	σ
A_1	12.35081	0.007461
θ_1	0.000597	1.9E-08
θ'_1	3.43E-12	6.46E-14
B_1	9065.557	1.73114
λ_1	9.87E-08	3.99E-09
Q	-0.93661	0.00167
A_2	0.003618	0.00232
θ_2	0.001278	8.54E-06
B_2	10482.32	1010.032
A_3	0.076804	0.003146
θ_3	0.000562	8.14E-07
B_3	58591.69	148.9161
A_4	0.009872	0.002441
θ_4	0.00039	3.38E-06
B_4	10122.44	1292.974
A_5	0.005987	0.002362
θ_5	0.000267	5.21E-06
B_5	103127.4	1636.478
A_6	0.049949	0.002377
θ_6	0.000662	6.99E-07
B_6	21171.69	150.0542
A_7	0.002531	0.002445
θ_7	0.000431	1.32E-05
B_7	22607.63	4230.685
A_8	0.003419	0.002437
θ_8	0.000176	9.42E-06
B_8	10164.16	8124.802
A_9	0.001996	0.002447
θ_9	0.00013	1.59E-05
B_9	106587.9	10098.07
$\hat{\sigma}_0$	0.055	

Figure 44

Residuals 9 of the second set of time observations

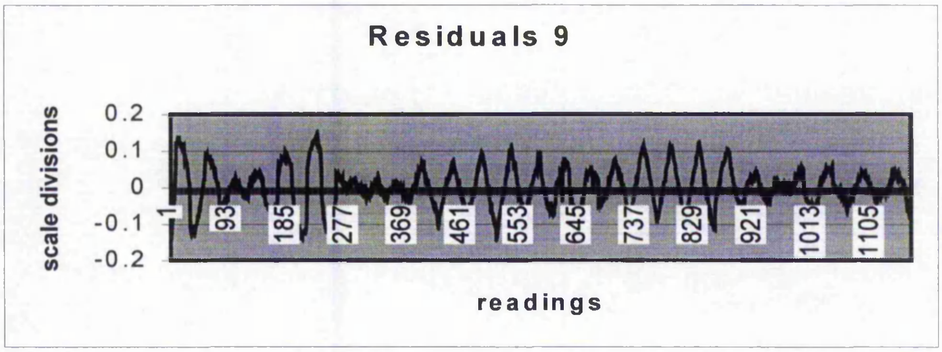


Table 19 shows that the ninth period of oscillations ($2\pi/\theta_9$) is about 1933 seconds, the longest period so far, the corresponding amplitude is about 1″.2 arc seconds. The standard deviation of the midpoint of swing, $\boldsymbol{Q}$, changed very little. The value of $\hat{\sigma}_0$ did not change.

Finally, put $i = 10$ in equation (5-2) to obtain the tenth model for the tenth set of oscillations, the observation equation (5-35):

$$\text{(5-35)} \quad A_1 e^{-\lambda_1 t} \cos[(\theta_1 + \theta_1' t)(t - B_1)] + \sum_{i=2}^{10} A_i \cos(\theta_i (t - B_i)) + \boldsymbol{Q} - \Delta = 0$$

To improve the mathematical model, we computed the adjustment by least squares using as approximate values of the parameters the values of table 19. The values of t are in frames, $t = t_1, t_2 \ldots . t_n$, see table B1, appendix B, and n is the number of observations

Table 20 and figure 45 summarise the results of values of the parameters of equation (5-35) and their standard deviations computed by a least squares adjustment.

5.7.10 Results from using the observation equation (5-35):

Table 20

Parameters A_i, θ_i, θ'_1, B_i, λ_1, Q ($i = 1, 2, \ldots 10$) and their standard deviations using the second set of data

Parameters	Values	σ
A_1	12.34984	0.007534
θ_1	0.000597	1.91E-08
θ'_1	3.43E-12	6.5E-14
B_1	9065.717	1.741303
λ_1	9.81E-08	4.03E-09
Q	-0.93657	0.001669
A_2	0.003643	0.002319
θ_2	0.001278	8.47E-06
B_2	10495.15	1002.139
A_3	0.077101	0.003168
θ_3	0.000562	8.18E-07
B_3	58603.83	149.4967
A_4	0.009882	0.002439
θ_4	0.00039	3.51E-06
B_4	10113.06	1355.505
A_5	0.006373	0.002366
θ_5	0.000267	5.04E-06
B_5	103041	1558.807
A_6	0.049872	0.002375
θ_6	0.000662	7.02E-07
B_6	21169.74	150.7232
A_7	0.002523	0.002444
θ_7	0.000431	1.34E-05
B_7	22403.58	4336.511
A_8	0.003559	0.002433
θ_8	0.000176	9.09E-06
B_8	10170.49	7805.207
A_9	0.002102	0.002445
θ_9	0.000131	1.52E-05
B_9	106077	9565.012
A_{10}	0.005566	0.002345
θ_{10}	0.000339	5.82E-06
B_{10}	115898.5	1386.915
$\hat{\sigma}_0$	0.055	

Figure 45

Residuals 10 of the second set of time observations

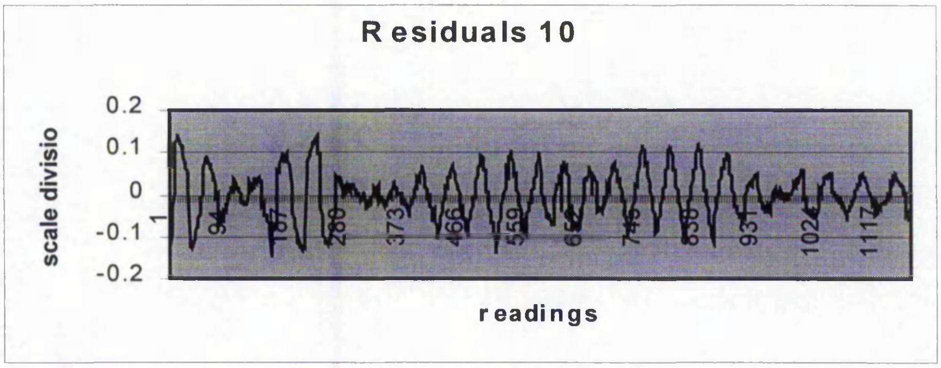


Table 20 shows that the tenth period of oscillations ($2\pi/\theta_{10}$) is about 741 seconds, the corresponding amplitude is about 3″.3 arc seconds. The standard deviation of the midpoint of swing, ***Q***, changed very little. The value of $\hat{\sigma}_0$ did not change.

Figure 45, shows a number of waveforms. This suggests that the gyroscope may be affected by some systematic errors. It is very hard to analysis these errors because we do not know what other unmodelled forces there may have been within the internal gyro environment on this occasion. One possibility might be that there were step changes in the voltage or current output of the battery.

From the results obtained in the above paragraphs (5-7), we may deduce that at least three terms of the mathematical model (5-2) cannot be ignored. This is because the values of the amplitudes are significant. The precision of the midpoint of swing improved with the introduction of the fifth and sixth period of oscillations. Therefore, the best model that may express the motion of the moving mark, in this case, will be the equation:

$$A_1 e^{-\lambda_1 t}\cos[(\theta_1 + \theta_1' t)(t - B_1)] + A_5 \cos(\theta_5(t - B_5)) +$$
$$A_6 \cos(\theta_6(t - B_6)) + \boldsymbol{Q} - \Delta = 0$$

5-8 Correlation between the parameters:

The diagonal elements of the parameters in Variance–Covariance matrix, equation (5–19) represent a measure of dispersion or uncertainty in the individual parameter estimates. Linear correlation coefficients among these parameters are computed from the equation:

$$(5\text{-}36)\quad \rho_{x_1x_2} = \frac{\sigma_{x_1x_2}}{\sqrt{\sigma_{x_1}{}^2\sigma_{x_2}{}^2}}$$

Where:

$\sigma_{x_1}{}^2$ and $\sigma_{x_2}{}^2$ are the Variances of any two parameters x_1 and x_2.

$\sigma_{x_1x_2}$ is the Covariance between x_1 and x_2.

In terms of equation (5-2), if $i = 1$ the unknown parameters will be A, θ, θ', B, λ and $\boldsymbol{Q}$. There are fifteen linear correlation coefficients presented in the form of a symmetric matrix. The matrix has ones along the diagonal and all off–diagonal elements are between −1 and +1 (see tables 21 and 22). Provided that the mathematical model is an accurate representation, the linear correlation coefficients are interpreted as a measure of independence between any two parameters such that when a coefficient is near 1 in absolute value, the parameters are highly correlated. This means that the pair is linearly dependent under the given model.

Tables 21 and 22 show that the correlation between the pairs of parameters (θ, θ'), (A, λ) and (θ, B) is very strong. There is a significant correlation between the two parameters θ' and B. However, the correlation between the other pairs of parameters is small.

Table 21

Correlation between parameters A, θ, θ', B, λ and Q

Using the first set of time observations

Parameters	A	θ	θ'	B	λ	Q
A	1	0.021205	-0.02097	0.020246	0.882757	0.027449
θ		1	-0.96828	0.82316	0.022608	0.045306
θ'			1	-0.69175	-0.02369	-0.04574
B				1	0.019576	0.041905
λ					1	0.0179
Q						1

Table 22

Correlation between parameters A, θ, θ', B, and Q

Using the second set of time observations

Parameters	A	θ	θ'	B	λ	Q
A	1	0.019961	-0.01962	0.0191	0.872975	-0.07436
θ		1	-0.96951	0.857997	0.021891	-0.0066
θ'			1	-0.73791	-0.02305	0.002178
B				1	0.018763	-0.0126
λ					1	0.000851
Q						1

5.9 Numerical comparisons:

In all conventional methods of observations, the midpoint of swing is determined uniquely, without any degrees of freedom, using the minimum data available from the Gyrotheodolite. The individual results obtained from such methods cannot be assessed from the observations since the standard errors of the parameters cannot be found. In this research, the midpoint of swing is determined using large sets of observations. The precision of time observations to one frame (0.04 seconds) is obtained with the aid of a video camera and video imagery with frame analysis. Using least squares adjustments, the midpoint of swing is determined with a precision of less than one second of arc.

Jeudy (Jeudy et al. 1981), in a model of the motion with five periods of oscillations, could not find the values of the parameters of the fifth period and some parameters of the fourth period of oscillations because he used too few observations. In the fourth term of the model, he was left with insufficient data.

Table 23

Comparison with Jeudy's model

	Jeudy's work with a MoM Gyrotheodolite		This research with a GAK1 Gyrotheodolite	
i	A (amplitude) Arc seconds	T (period) Seconds	A (amplitude) Arc seconds	T (period) Seconds
1	11534	412	6597	424
2	70	206	15	380
3	2.5	0.02	14	475
4	0.7	0.5	0.5	46
5			0.75	50
6			3	323
7			1.25	285
8			1.2	340
9			0.9	230
10			1.5	256

Table 23 shows the periods of oscillations and the corresponding amplitudes obtained by Jeudy and by this research (the first set of observations). The first periods of oscillations in both models are very close and the other terms are different.

Jeudy (Jeudy, 1981 and 1982) assumed that the angular velocity of the spinner was constant. However, this assumption in practice is not true. In our model, we assumed the form of a linear change with time for the frequency. To see the effect of including the term $\theta' t$ on the values of the midpoint of swing and their standard deviations, we performed least squares adjustments for the mathematical model with including the term $\theta' t$ on one hand, and without this parameter on the other hand.

Tables 24, 25, 26, 27 and 28 summarise the results of values of the parameters A, θ, θ', B, λ and $\boldsymbol{Q}$ and their standard deviations with the term $\theta' t$ included and excluded. The observations used in the comparison are time t, in frames, values of which are shown in table B1, appendix B.

The units of parameters A, θ, θ', B, λ and $\boldsymbol{Q}$ and their standard deviations that appear in the tables below are as follows:

A and $\boldsymbol{Q}$ the amplitude and the midpoint of swing respectively, are in scale division units, one scale unit is about 600″ arc seconds.

θ and θ', the frequency of the oscillations and the rate of change in the frequency are in radians per frame and radians per $(\text{frame})^2$ respectively.

B, the phase is in frames.

λ, the coefficient of damping is unitless.

Table 24

The effect of including the term $\theta' t$ in the model of the first period of oscillations

$A_1 e^{-\lambda_1 t}\cos(\theta_1(t-B_1))+\boldsymbol{Q}-\Delta=0$			$A_1 e^{-\lambda_1 t}\cos[(\theta_1+\theta_1' t)(t-B_1)]+\boldsymbol{Q}-\Delta=0$		
Parameters		σ	Parameters		σ
A_1	10.98556381	0.005481049	A_1	10.98296	0.004172
θ_1	0.000592533	3.91789E-09	θ_1	0.000592	1.23E-08
			θ'_1	9.29E-13	8.31E-14
B_1	9278.415568	0.527370213	B_1	9273.193	0.55987
λ_1	8.9445E-08	5.73711E-09	λ_1	8.53E-08	4.38E-09
$\boldsymbol{Q}$	-0.103916614	0.001470229	$\boldsymbol{Q}$	-0.1028	0.001126
$\hat{\sigma}_0$	0.0358		$\hat{\sigma}_0$	0.0272	

Table 24 shows that including the term $\theta' t$ in the model of simple damped harmonic motion has improved the precision of the midpoint of swing from about 0.0015 scale divisions to about 0.0011 scale divisions. Also, the value of $\hat{\sigma}_0$, improved from 0.0358 to 0.0272.

Table 25

The effect of including the term $\theta' t$ in the model of the second period of oscillations

$A_1 e^{-\lambda_1 t}\cos(\theta_1(t-B_1)) + A_2 e^{-\lambda_2 t}\cos(\theta_2(t-B_2)) + \boldsymbol{Q} - \Delta = 0$			$A_1 e^{-\lambda_1 t}\cos[(\theta_1+\theta'_1 t)(t-B_1)] + A_2 e^{-\lambda_2 t}\cos[(\theta_2+\theta'_2 t)(t-B_2)] + \boldsymbol{Q} - \Delta = 0$		
Parameters		σ	Parameters		σ
A_1	11.00088654	0.004425684	A_1	11.00503	0.003392
θ_1	0.000592538	3.27955E-09	θ_1	0.000592	1.61E-08
			θ'_1	4.48E-13	1.18E-13
B_1	9279.038587	0.436770645	B_1	9277.126	0.637706
λ_1	1.06675E-07	4.63624E-09	λ_1	1.1E-07	3.72E-09
$\boldsymbol{Q}$	-0.104171342	0.001158162	$\boldsymbol{Q}$	-0.10486	0.000618
A_2	0.03197437	0.001682284	A_2	0.044418	0.00472
θ_2	0.000678704	1.32046E-06	θ_2	0.000677	6.42E-06
			θ'_2	-1.7E-11	5.47E-11
B_2	6794.11545	160.4282427	B_2	6547.588	189.8212
λ_2	5.33239E-06	2.08906E-06	λ_2	8.87E-06	1.78E-06
$\hat{\sigma}_0$	0.0282		$\hat{\sigma}_0$	0.0145	

Table 25 shows that including the term $\theta' t$ in the model of the second period of oscillations has improved the precision of the midpoint of swing from about 0.0012 scale divisions to about 0.0006 scale divisions. Also, the value of $\hat{\sigma}_0$, improved from 0.0282 to 0.0145.

Table 26

The effect of including the term $\theta' t$ in the model of the third period of oscillations

$A_1 e^{-\lambda_1 t}\cos(\theta_1(t-B_1)) +$ $A_2\cos(\theta_2(t-B_2)) +$ $A_3\cos(\theta_3(t-B_3)) + \boldsymbol{Q} - \Delta = 0$			$A_1 e^{-\lambda_1 t}\cos[(\theta_1+\theta_1' t)(t-B_1)] +$ $A_2\cos(\theta_2(t-B_2)) +$ $A_3\cos(\theta_3(t-B_3)) + \boldsymbol{Q} - \Delta = 0$		
Parameters		σ	Parameters		σ
A_1	10.98628428	0.004171234	A_1	10.9941	0.004016
θ_1	0.00059254	3.00843E-09	θ_1	0.000592	4.48E-08
			θ'_1	2.8E-12	2.94E-13
B_1	9279.216499	0.402215764	B_1	9262.664	1.779515
λ_1	8.94842E-08	4.4194E-09	λ_1	1.01E-07	4.56E-09
$\boldsymbol{Q}$	-0.10420133	0.001039673	$\boldsymbol{Q}$	-0.10446	0.000381
A_2	0.024359096	0.001640498	A_2	0.030424	0.002083
θ_2	0.000677214	1.61105E-06	θ_2	0.000655	2.06E-06
B_2	6741.230918	194.5842175	B_2	4114.494	250.2048
A_3	0.019933854	0.001656441	A_3	0.028061	0.002646
θ_3	0.000504654	1.99012E-06	θ_3	0.000535	2.47E-06
B_3	6434.062531	321.061865	B_3	22060.64	286.5536
$\hat{\sigma}_0$	0.0253		$\hat{\sigma}_0$	0.0089	

Table 26 shows that including the term $\theta' t$ in the model of the third period of oscillations has improved the precision of the midpoint of swing from about 0.0010 scale divisions to about 0.0004 scale divisions. Also, the value of $\hat{\sigma}_0$, improved from 0.0253 to 0.0089.

Table 27

The effect of including the term $\theta' t$ in the model of the fourth period of oscillations

$A_1 e^{-\lambda_1 t}\cos(\theta_1(t-B_1))+ A_2\cos(\theta_2(t-B_2))+ A_3\cos(\theta_3(t-B_3))+ A_4\cos(\theta_4(t-B_4))+\boldsymbol{Q}-\Delta=0$			$A_1 e^{-\lambda_1 t}\cos[(\theta_1+\theta_1' t)(t-B_1)]+ A_2\cos(\theta_2(t-B_2))+ A_3\cos(\theta_3(t-B_3))+ A_4\cos(\theta_4(t-B_4))+\boldsymbol{Q}-\Delta=0$		
Parameters		σ	Parameters		σ
A_1	10.98633278	0.004174171	A_1	10.99415	0.004035
θ_1	0.00059254	3.01007E-09	θ_1	0.000592	4.5E-08
			θ'_1	2.8E-12	2.96E-13
B_1	9279.217861	0.402434759	B_1	9262.638	1.786141
λ_1	8.95359E-08	4.42237E-09	λ_1	1.01E-07	4.58E-09
$\boldsymbol{Q}$	-0.104198033	0.00104026	$\boldsymbol{Q}$	-0.10446	0.000381
A_2	0.024365453	0.00164147	A_2	0.030455	0.002092
θ_2	0.000677216	1.61146E-06	θ_2	0.000654	2.06E-06
B_2	6742.020879	194.6348392	B_2	4112.509	250.6992
A_3	0.019905835	0.001657492	A_3	0.028098	0.002661
θ_3	0.000504647	1.99403E-06	θ_3	0.000535	2.48E-06
B_3	6434.090711	321.7005262	B_3	10324.09	341.2761
A_4	0.002256462	0.001469662	A_4	0.00064	0.000537
θ_4	0.005488696	1.55897E-05	θ_4	0.005498	2.14E-05
B_4	7163.158769	231.4299844	B_4	7161.258	303.6418
$\hat{\sigma}_0$	0.0253		$\hat{\sigma}_0$	0.0089	

Table 27 shows that including the term $\theta' t$ in the model of the fourth period of oscillations has no effect upon the precision of the midpoint of swing. The same precision in both cases remained as in the previous model, from about 0.0010 scale divisions to about 0.0004 scale divisions. Also, the value of $\hat{\sigma}_0$, remained the same for both cases, as in the previous model 0.0253 and 0.0089. This suggests that the effect of including the term $\theta' t$ is significant in the models for the first three periods of the oscillations. To justify this conclusion, let us carry on with this comparison with a fifth set of oscillations.

Table 28

The effect of including the term $\theta' t$ in the model of the fifth period of oscillations

$A_1 e^{-\lambda_1 t}\cos(\theta_1(t-B_1))+$ $A_2\cos(\theta_2(t-B_2))+$ $A_3\cos(\theta_3(t-B_3))+$ $A_4\cos(\theta_4(t-B_4))+$ $A_5\cos(\theta_5(t-B_5))+Q-\Delta=0$			$A_1 e^{-\lambda_1 t}\cos[(\theta_1+\theta_1' t)(t-B_1)]+$ $A_2\cos(\theta_2(t-B_2))+$ $A_3\cos(\theta_3(t-B_3))+$ $A_4\cos(\theta_4(t-B_4))+$ $A_5\cos(\theta_5(t-B_5))+Q-\Delta=0$		
Parameters		σ	Parameters		σ
A_1	10.98632714	0.004182245	A_1	10.994	0.003936
θ_1	0.00059254	3.01593E-09	θ_1	0.000592	4.42E-08
			θ_1'	2.83E-12	2.91E-13
B_1	9279.218563	0.403206177	B_1	9262.498	1.750326
λ_1	8.95236E-08	4.43101E-09	λ_1	1.01E-07	4.48E-09
Q	-0.104200214	0.001042233	Q	-0.10432	0.000372
A_2	0.024351036	0.001644682	A_2	0.030444	0.002066
θ_2	0.000677221	1.61556E-06	θ_2	0.000654	2.03E-06
B_2	6743.135698	195.1275634	B_2	4093.863	246.3591
A_3	0.019904261	0.001660686	A_3	0.028479	0.002602
θ_3	0.00050465	1.99807E-06	θ_3	0.000535	2.4E-06
B_3	6435.118031	322.3574828	B_3	10324.83	328.7829
A_4	0.00225562	0.001473417	A_4	0.000768	0.000524
θ_4	0.005488487	1.56374E-05	θ_4	0.005492	1.74E-05
B_4	7159.740225	232.1017255	B_4	7140.605	246.9908
A_5	0.001325166	0.001465108	A_5	0.001192	0.000521
θ_5	0.004968315	2.67992E-05	θ_5	0.004989	1.13E-05
B_5	9780.576531	428.7964224	B_5	9942.246	171.8059
$\hat{\sigma}_0$	0.0254		$\hat{\sigma}_0$	0.0086	

Table 28 shows that including the term $\theta' t$ in the model of the fifth period of oscillations has improved the precision of the midpoint of swing very little. Also, the value of $\hat{\sigma}_0$ has improved very little from 0.0254 to 0.0086, which is approximately the same difference as for the previous case. Therefore, the effect of the term $\theta' t$ is significant in the models of the first three sets of oscillations.

VI. EVALUATION OF RESULTS AND CONCLUSIONS

6.1 Summary:

In Chapter 5, the values for most of the parameters in terms of equation:

$$(5\text{-}2) \quad \sum_{i=1}^{10} A_i e^{-\lambda_i t} \cos((\theta_i + \theta'_i t)(t - B_i)) + \boldsymbol{Q} - \Delta = 0$$

are found.

The following values for the model of ten oscillations are summarised for the first set of time observations:

Table 29

Values of parameters A_i, θ_i, θ'_1, λ_1 and the periods T_i for the first set of observations

i	A_i arc seconds	θ_i $\frac{\text{radians}}{\text{sec.}}$	θ'_i $\frac{\text{radians}}{(\text{sec.})^2}$	λ_i unitless	period in seconds $T_i = \frac{2\pi}{\theta_i}$ when $t = 0$
1	6596.68	0.0148	5.05E-11	1.01E-07	424.54
2	14.82	0.016525			380.22
3	13.58	0.013225			475.10
4	0.46	0.137325			45.75
5	0.75	0.1247			50.39
6	3.13	0.01945			323.04
7	1.25	0.022025			285.27
8	1.17	0.018475			340.09
9	0.88	0.027275			230.36
10	1.49	0.02445			256.98

From a review of the results obtained in the previous Chapter, three periods of oscillations, apart from the main period, have significant effect upon the values of the midpoints of swing and their respective standard deviations. Table 29 shows that the corresponding amplitudes of the second, third and sixth periods of oscillation are too large to be neglected.

The following values for the model with ten oscillations are summarised for the second set of time observations:

Table 30

Values of parameters A_i, θ_i, θ'_1, λ_1 and the period T_i for the second set of observations

i	A_i arc seconds	θ_i $\frac{\text{radians}}{\text{sec.}}$	θ'_i $\frac{\text{radians}}{(\text{sec.})^2}$	λ_i unitless	period in seconds $T_i = \frac{2\pi}{\theta_i}$ when $t = 0$
1	7409.91	0.014921	8.57E-11	9.81E-08	421.10
2	2.18	0.031957			196.61
3	46.26	0.014058			446.94
4	5.93	0.009744			644.84
5	3.82	0.006672			941.74
6	29.92	0.016539			379.90
7	1.51	0.010779			582.91
8	2.13	0.004405			1426.49
9	1.26	0.003271			1920.91
10	3.34	0.008485			740.54

From a review of the results obtained in the previous Chapter, only three oscillations, apart from the main period, have significant effect upon the values of the midpoints of swing and their respective standard deviations. Table 30 shows that the corresponding amplitudes of the third, fourth and sixth periods of oscillation are too large to be neglected. Tables 29 and 30 are re-arranged to take into account of only the significant terms. Table 31 shows the significant parameters A_i, θ_i, θ'_1, λ_1 and the periods T_i of the best models for both sets of observations.

Table 31

Values of parameters A_i, θ_i, θ'_1, λ_1 and the periods T_i of the best practical models for both sets of observations

First set i	A_i arc seconds	θ_i $\frac{\text{radians}}{\text{sec.}}$	θ'_i $\frac{\text{radians}}{(\text{sec.})^2}$	λ_i unitless	Periods T_i in seconds
1	6597	0.0148	5.05E-11	1.01E-07	424
2	15	0.0165			380
3	14	0.0132			475
6	3	0.0195			323
Second set i	A_i arc seconds	θ_i $\frac{\text{radians}}{\text{sec.}}$	θ'_i $\frac{\text{radians}}{(\text{sec.})^2}$	λ_i unitless	Periods T_i in seconds
1	7410	0.0149	8.57E-11	9.81E-08	421
3	46	0.0141			447
4	6	0.0097			645
6	30	0.0165			380

6.2 Evaluation of results:

From tables 29 and 30, we find that the values of parameters θ_1, θ'_1, λ_1 and the period T_1 for the first and second set of time observations are approximately the same. There are large differences in the values of the other terms of θ_i and T_i for the first and second set of time observations. However, there is no significant effect of these terms on the computed standard deviations of $\boldsymbol{Q}$, the midpoint of swing.

Table 31 shows that the first term of the magnitudes of oscillations for both sets of observations is the largest. The second and third terms A_2 and A_3, for the first set of observations, and the third and sixth terms A_3 and A_6, for the second set of observations, have significant values. Tables 29, 30 and 31 show that the values of the other terms for both sets of observations are much smaller, but they are not negligible. The amplitudes of these terms range from 0.46 to 3.13 arc seconds for the first set of observations and they range from 1.26 to 5.93 arc seconds for the second set of observations. From the practical point of view, these terms may be ignored. However, they should not be ignored because their corresponding periods of oscillations are significant.

The damping coefficients for each set of observations are very similar for the first term of equation (5-2). However, the results of the previous Chapter show that the effect of this parameter is negligible in the terms $i = 2$ to 10 of equation (5-2).

The results in tables 2, 3, 3a, 3b and 3c, in Chapter 5, show that the standard deviations of some parameters are much greater than the values of these parameters. This is due to the effect of including parameters λ_2, λ_3, θ'_2 and θ'_3 in the observation equations (5-23), (5-24), (5-25), (5-26) and (5-27). Eliminating one of these parameters at a time improves the values of other parameters and their standard deviations.

From results obtained in Chapter 5, the values of $\boldsymbol{Q}$, the midpoint of swing, and their standard deviations, σ_Q, are the most interesting parameters because the azimuth determination is concerned with $\boldsymbol{Q}$ observed on the gyro scale. Table 32 shows the values of $\boldsymbol{Q}$ and their respective standard deviations for both sets of observations.

Table 32

Values of $\boldsymbol{Q}$ and σ_Q in scale divisions for both sets of observations

First set of observations			Second set of observations		
i	$\boldsymbol{Q}$	σ_Q	i	$\boldsymbol{Q}$	σ_Q
1	-0.1028	0.00113	1	-0.9358	0.00243
2	-0.1049	0.00062	2	-0.9357	0.00244
3	-0.1044	0.00038	3	-0.9357	0.00244
4	-0.1045	0.00038	4	-0.9359	0.00243
5	-0.1043	0.00037	5	-0.9357	0.00218
6	-0.1043	0.00034	6	-0.9365	0.00166
7	-0.1043	0.00034	7	-0.9365	0.00166
8	-0.1044	0.00034	8	-0.9366	0.00166
9	-0.1044	0.00034	9	-0.9366	0.00167
10	-0.1044	0.00033	10	-0.9366	0.00167

From table 32 and figures 46 and 47, we can see that the values of Q for the first set of observations had changed for the first three terms, that is, i = 1, 2 and 3, and also the quality of their standard deviations improved. However, the values of Q and their standard deviations for the other terms, that is, i = 4, 5, 6, 7, 8, 9, and 10, remained approximately the same or changed very little. For the second set of observations the values of Q also changed for three terms, that is, i = 1, 5 and 6, and the quality of their standard deviations improved. However, the values of Q and their standard deviations for the other terms remained approximately the same or changed very little. The reason of that values of Q and their standard deviations changed very little is likely to be due to the fact that the correlation between Q and the other parameters of equation (5-2) is very weak. This correlation is shown in tables 21 and 22 for both sets of observations.

Figure 46

Standard deviations of midpoints of swing

for the first set of observations

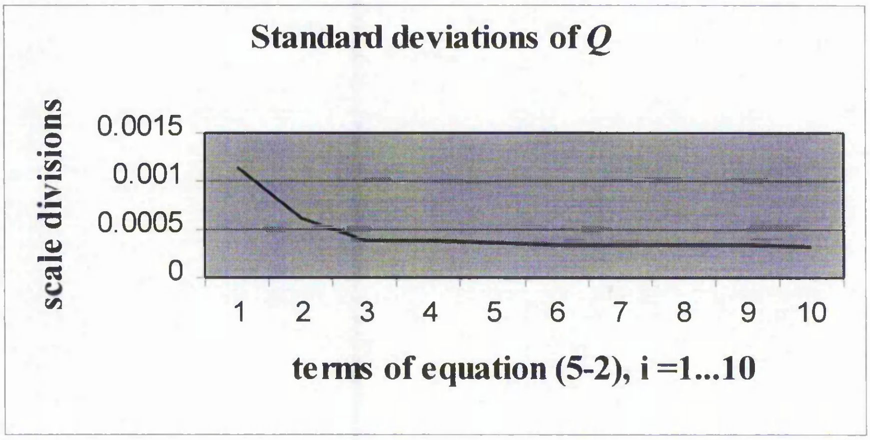


Figure 46 shows that the standard deviations of the midpoint of swing, for the first set of observations, improved for the first three models. Additional terms in other models of equation (5-2) have insignificant effect on the values of Q and their respective standard deviations.

Figure 47

Standard deviations of midpoints of swing

for the second set of observations

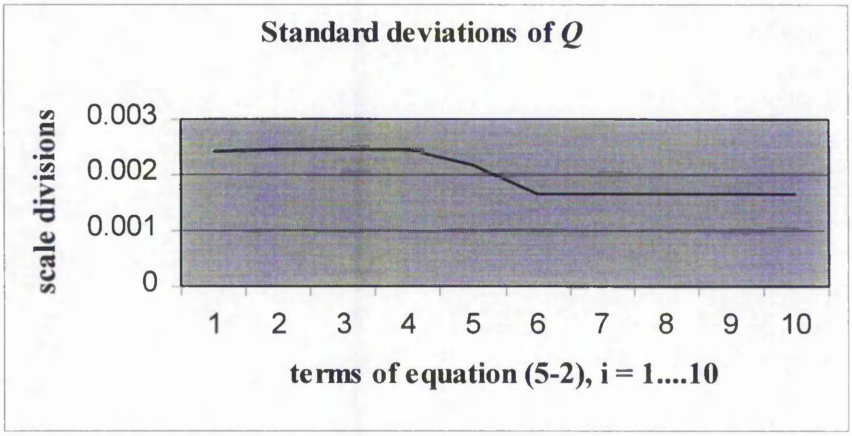


Figure 47 shows that the standard deviations of the midpoint of swing, for the second set of observations, have improved in the fourth, fifth and sixth models. Additional terms in other models of equation (5-2) have insignificant effect on the values of $\boldsymbol{Q}$ and their respective standard deviations.

Table 32 is re-arranged to take account of the values of the midpoint of swing, $\boldsymbol{Q}$ and their respective standard deviations in the best models for both sets of observations. Table 33 shows the values of $\boldsymbol{Q}$ and their respective standard deviations, σ_Q in arc seconds. Figures 48 and 49 show the improvement in the precision of the determination of the midpoint of swing for the first and second set of observations, respectively.

Table 33

Values of Q and their standard deviations σ_Q in arc seconds of the best models for both sets of observations

First set of observations			Second set of observations		
i	Q	σ_Q	i	Q	σ_Q
1	-61.68	0.68	1	-561.49	1.46
2	-62.92	0.37	4	-561.53	1.45
3	-62.65	0.23	5	-561.41	1.30
6	-62.61	0.21	6	-560.89	0.99

Figure 48

Standard deviations of midpoints of swing for the best models for the first set of observations

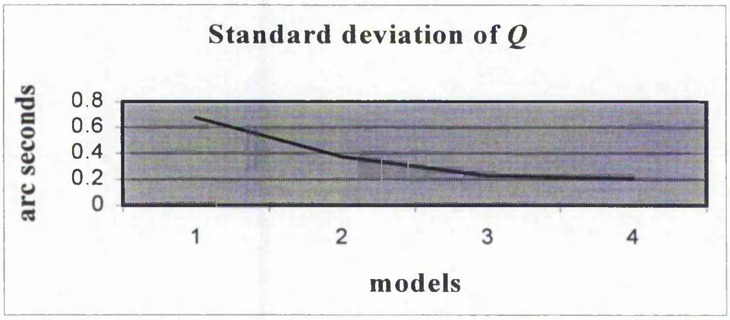
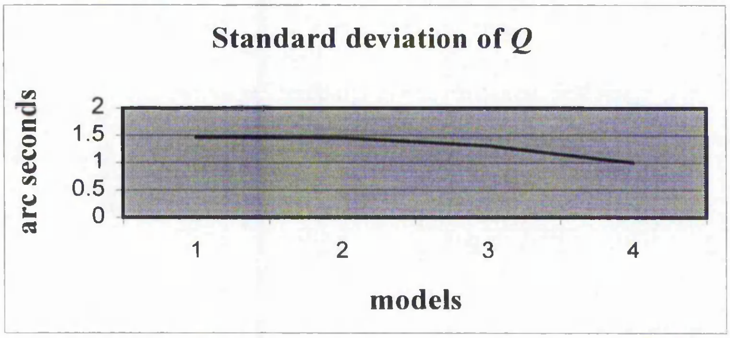


Figure 49

Standard deviations of midpoints of swing for the best models for the second set of observations



From table 33 and figures 48 and 49, it can be seen that the precision of the midpoint of swing may be determined to less than one second of arc.

6.3 Conclusions:

The conclusions, which may be drawn from this research, are summarised as follows:

- Using a video camera and video imagery with frame analysis for Gyrotheodolite observations leads to a great increase in the quantity of data capture compared with conventional methods. The internal precision of time observed by this method is at least one frame (0.04 seconds), which is five times greater than the precision that may be obtained by manual stopwatch methods of observations. This research is believed to be the first to use video techniques in Gyrotheodolite applications.

- Least squares adjustment analysis showed that the precision of the midpoint of swing might be determined to less than one second of arc. The precise determination of the midpoint of swing leads to a precise determination of azimuth. The azimuth may be obtained with a precision of ± 3″ or ± 6″ assuming the knowledge of the instrument constant to about ± 1″ or ± 5″ respectively. The precision of the midpoint of swing and azimuth determination cannot be assessed from the observations in conventional methods.

- Modelling the linear change with time of frequency, of the main period of oscillation of the moving mark, has improved the precision of the computed midpoint of swing. The term $\theta' t$ should be added to all terms of the equation of motion of the moving mark whenever the value of θ' is greater than its standard deviation.

- This research takes account of all significant terms, which affect the gyroscope. The mathematical model used for the motion of the moving mark is in the form of ten oscillations. The complete equation is expressed as:

$$\sum_{i=1}^{10} A_i e^{-\lambda_i t} \cos((\theta_i + \theta_i' t)(t - B_i)) + Q - \Delta = 0$$

- At least three terms of the above equation have significant effect on the values of the midpoint of swing and their respective standard deviations. The best mathematical model, which may express, in practice, the motion of the moving mark for the suspended gyroscope may be written in the form:

$$A_1 e^{-\lambda_1 t} \cos[(\theta_1 + \theta_1' t)(t - B_1)] +$$
$$A_2 \cos(\theta_2 (t - B_2)) +$$
$$A_3 \cos(\theta_3 (t - B_3)) +$$
$$A_4 \cos(\theta_4 (t - B_4)) + \boldsymbol{Q} - \Delta = 0$$

- The observational method used in this research can be used safely in many industrial applications, for example, railway tunnels. However, The method must be made acceptable for mine safety for electrical equipment if used in a mine.

- Although a new automated Gyrotheodolite from Bochum University was used on the Channel Tunnel it was not used on the Jubilee Line extension because of cost. The purchase price is understood to be £50,000-70,000. The Wild GAK1 is 1970's technology but the solutions from existing instruments could be improved by application of the method used in this research, to a comparable precision at little cost compared with the cost of purchase a Bochum instrument.

6.4 Further improvement and research:

Using a video camera and video imagery with video frame analysis for the Gyrotheodolite observations has greatly improved the observational method, especially in terms of the quantity of observations and the precision of the determination of the midpoint of swing. Correspondingly, the azimuth determination has also improved.

In this work the effect of dislevelment of the vertical has been neglected. From design, the axis of the spinner of the Gyrotheodolite should remain perpendicular to the vertical axis of the theodolite. However, if the theodolite was not levelled precisely, the mast would still hang vertically but in a different position with respect to the zero position of the gyro scale. This dislevelment will cause error each time the instrument is set up. If the instrument is set up on a pillar and levelled as precisely as possible, this may reduce the effect of the problem.

Further improvements upon this research may be made in these areas:

- The manual method of dropping the mast to release the gyro for swinging is not a simple operation and needs experience. This operation causes to the gyroscope to exhibit some rapid but small vibrations. It should be possible to improve this situation with automated mast dropping, possibly with the use of a small external hydraulic motor. The system must be acceptable for mine safety for electrical equipment. A design for such a device was produced but has not been included in this research.

- In this research, it was assumed that the gyro scale readings are perfectly engraved and therefore, are errorless. They were not considered as observed values. However, Caspary (Caspary et al, 1981) suggests that azimuths may be obtained by considering both time and scale readings as observed values. Even if the scale divisions of the gyro are assumed to be equally spaced, there is still a constant error associated with each scale reading. Therefore, the weight matrix in this case will not be a unit matrix for only time observations and the second design matrix will be of dimension $n \times n$ matrix, where n is the number of observations.

- The proof of the mathematical model was tested in the laboratory for two sets of observations, using many sets of gyro oscillations. The instrument used in this research was Wild GAK1 suspended gyroscope. Further research should consider other types of suspended gyroscopes such those manufactured by MoM and Sokkia companies.

- The video method of measuring time used in this research is an improvement over hand held stopwatch methods. The effect of most systematic timing errors should be eliminated or at least reduced by this method. However, there are still some unknown systematic errors that affect the gyroscope. An example of these errors may be seen in figure 45, Chapter 5. The approach to dealing with systematic errors may need to be “methodical rather than mathematical”. From a review of the results obtained in this research, further research may be required to study all the systematic errors that affect the behaviour of the gyroscope. That research may take into account the effects of temperature and the behaviour of different types of batteries.

REFERENCES

1. Bencze, W. J. Xiao, Y. Hipkins, D. N. Franklin, G. F. and Parkinson, B. W. "Gyroscope Spin Axis Direction Control for the Gravity B Satellite". IEEE Conference on Decision and Control, Vol. 1, Japan, 1996.

2. Bennett, G. G. "Theoretical and Practical Study of a Gyroscopic Attachment for a Theodolite". UNISURV Report No. 16, University of New South Wales, 1970.

3. Breach, M. C. "Astronomical Azimuth by the Method of Multiple Timed Observations of a Gyrotheodolite without Electronic Registration". MSc thesis, 1983.

4. Brown, H. B. and Xu, Y. " A Single-Wheel Gyroscopically Stabilized Robot". IEEE International Conference on Robotics and Automation, No. 4, USA, 1996.

5. Caspary, W. F. and Schwintzer, P. A. W. "An Extension of Chronometric Gyroscope Observation Methods". The Canadian Surveyor, Vol. 35, No.4, December 1981.

6. Chester, W. "Mechanics". George Allen and Unwin, London, 1979.

7. Deimel, R. F. "Mechanics of the Gyroscope". Dover, New York, 1950.

8. EMR "Standard Operating Procedure". Survey and Mapping Branch, Geodetic Survey, EMR International Report, 1975.

9. Eyre, J. M. and Wetherelt A. "Taking Azimuths Underground". Mining Magazine, Vol. 172 Part 3, Camborne School of Mines, University of Exeter, 1995.

10. Goldstein, H. "Classical Mechanics". Addison-Wesley Publishing Company, Inc. 1959.

11. Gregerson, L. F. Vanicek, P. and Symonds, G. "Report from Experiments with a Gyroscope Equipped with Electronic Registration". IAG Congress, Moscow, 1971.

12. Gregerson, L. F. "Investigation of the Nature and Changes of K-Factor in Gyroscopic Determinations". Budapest, 1972.

13. Gregerson, L. F. and Robertson, W. J. The "Gyroscope as a Tool for Azimuth Determinations". The Canadian Surveyor, March 1974.

14. Halmos, F. "High Precision Measurement and Evaluation Method for Azimuth Determination with Gyrotheodolites". Manuscripta Geodaetica, Vol. 2, pp 213-231, 1977.

15. Heiberg, C. J. Bailey, D. and Wie, B. "Precision Control Moment Gyroscope Spacecraft Control with Disturbance". Proceedings-Spie the International Society for Optical Engineering, USA, 1997.

16. Hodges, D. J. "Underground and Surface Orientation Measurements with Gyrotheodolite Attachments". The Mining Engineer, Vol.132, No.145, October 1972.

17. Hodges, D. J. Kimber, M. J. and Sleeman, T. W. "Gyrotheodolite Calibration". Mining Technology -Manchester then Doncaster-, Vol. 78, No. 896, UK, April 1996.

18. Hodges, D. J. Kimber, M. J. and Sleeman, T. W. "Establishing a Gyrotheodolite Baseline". Surveying Australia, Vol. 16, No. 2, IEMS Australia, 1994.

Also at web site:
http://www.nottingham.ac.uk/~enzsurv/papers.html, last seen, April 99.

19. Jaroszewicz, L. R. and Szelmanowski, A. "Application of Fibre Optic Gyroscope for Quality Research of Angular Rotation in Slow-Speed Platforms". Proceedings-Spie the International Society for Optical Engineering, USA, 1998.

20. Jeudy, L. M. A. "Theorie du Gyrocompas Suspendu I. Equations du Mouvement". Manuscripta Geodaetica, Vol 6 pp75-122, 1981.

21. Jeudy, L. M. A. "Theorie du Gyrocompas Suspendu II. Equations des Petites Oscillations". Manuscripta Geodaetica, Vol 7 pp37-76, 1982.

22. Jeudy, L. M. A. and Gagnon P. "Spectral Analysis as Applied to Gyrocompass Transit Times". Proceedings of the 2nd International Symposium on Inertial Technology for Surveying and Geodesy, Banff, Canada, June 1-5 1981.

23. King, B. A. "A Rigorous Determination of Azimuth from Gyroscopic Observation". The Australian Surveyor, Vo.33, No.5, March 1987.

24. Lin, J. D. Wu, J. T. and Guo, K. H. "Optimal Design of a Resonator Fibre-Optic Gyroscope System". International Symposium on Test and Measurement, China, August 1995.

25. Martusewicz, J. "Optimal Location of Extra Gyroscopic Azimuth in Underground Traverses". Journal of Surveying Engineering, Vol.119, No.4, November 1993.

26. Mikhail, E. M. and Gracie, G. "Analysis and Adjustment of Survey Measurements". Van Nostrand Reinhold Company Inc., New York, 1981.

27. Nassar, M. M. "Matrix Treatment of Adjustment Computations in Surveying". Ain Shams University, Cairo, Egypt, 1985.

28. Plakhtienko, N. P. and Dmitriev, A. Y. "Measuring the Effect of the Anomaly of the Earth's Gravitational Field on the Reading of a Gyrotheodolite". International Applied Mechanics C/C of Prikladnaia Mekhanika, Vol. 32 No. 4, USA, 1996.

29. Rellensmann, O. "Theodolite Gyro". Clausthal-Zellerfeld Mining Academy, Switzerland, 1959/1960.

30. Schultz, P. "Gleichrichtmomente bei Nordweisenden Kreiseln Kurzer Nordsuchzeit Infolge Oszllatorischer Storbewegungen". DGK, Reihe C, Heft Nr. 205, Munchen, 1975.

31. Smith, R. C. H. "A Modified GAK1 Attachment". Survey Review XXIV, 183, January 1977.

32. Strasser, G. J. and Schwendener, H. R. "North Seeking Gyro Attachment for the Theodolite, as a New Aid to Surveyors". Wild of Heerbrugg Ltd., Heerbrugg, Switzerland, 1964.

33. Stroud, K. A. "Engineering Mathematics: Programmes and Problems". Fourth Edition, Basingstoke: Macmillan, 1995.

34. Tanaka, H. and Wakatsuki, N. "Electromechanical Coupling Coefficients for a New H-Type LiTaO~3 Piezoelectric Gyroscope". Japanese Journal of Applied Physics, Vol. 37, No. 5B, Japan, November 1998.

35. Thomas, T. L. "Theory of the Suspended Gyroscope". Journal of Chartered Surveyor Land Hydrographic and Minerals Quarterly, No. 3, Vol. 3 Part 2, April 1976.

36. Thomas, T. L. “The Six Methods of Finding North Using a Suspended Gyroscope”. Survey Review Vo.26, 203, January 1982.

37. Tierney, J. A. “Differential Equations”. Second Edition, Allyn and Bacon, London, 1985.

38. Vanicek, P. “Dynamical Aspects of the Suspended Gyrocompass”. The Canadian Surveyor, Vol. 26, No. 1, March 1972.

39. Vanicek, P. and Krakiwsky, E. “Geodesy; The Concepts”. North Holland, 1982.

40. Williams, H. S. “The Spinning-Mass Gyroscope and Azimuth Determination for Geodetic Purposes”. Papers, Commission 5, XVIII Congress FIG, Toronto, Canada, 1986.

APPENDIX A

NOTATIONS

The following notations have been used:

$O_{\overline{xyz}}$ is a system of reference fixed with respect to the earth's surface. Figure 3.

$G_{X'Y'Z'}$ is a reference system having its axes parallel to those of $O_{\overline{xyz}}$. Figure 4.

G_{xyz} is a system of reference fixed with respect to the carriage. Figures 4 and 5.

C is the centre of gravity of the carriage. Figure 5.

G is the centre of gravity of the gyroscope. Figures 4 and 5.

T is the centre of gravity of the whole system gyroscope and carriage. Figure 5.

O is the centre of gravity of the earth. Figure 4.

I is a fixing point of the tape attached to the carriage. Figures 4 and 5.

A is a fixed end of the tape in the system $O_{\overline{xyz}}$. Figure 4.

$\vec{S}$ is the tension of the suspension tape at point, I. Equation (2-75).

$\overline{x}_I$, $\overline{y}_I$ and $\overline{z}_I$ are the co-ordinates of point I in the system $O_{\overline{xyz}}$. Figure 4.

ℓ is the length of the tape. Figure 4.

l is the distance from I to **T** ($l = IT$). Figure 5.

l_C is the distance from I to **C** ($l_C = IC$). Figure 5.

l_G is the distance from I to **G** ($l_G = IG$). Figure 5.

α, β and γ are the Euler angles of the system G_{xyz} with respect to the system $G_{X'Y'Z'}$. Figures 4 and 11.

Λ and φ are angles allowing the marking of the position of the point I. Figure 4.

ϕ is astronomical latitude. Figure 4.

$\vec{\Gamma}_C$ is the absolute angular velocity of the carriage. Equation (2-2).

$\vec{M}_{C_I}$ denotes the kinetic moment of the carriage with respect to the point I. Equation (2-1′).

I_{C_G} is the inertial moment of the carriage with respect to point G in the system G_{xyz}. Equation (2-3).

P' is the inertial moment of the carriage with respect to the axis G_x. Equation (2-4).

Q' is the inertial moment of the carriage with respect to the axis G_y. Equation (2-4).

R' is the inertial moment of the carriage with respect to the axis G_z. Equation (2-4).

$\vec{\Gamma}_G$ is the absolute angular velocity of the gyroscope. Equation (2-5).

$\vec{M}_{G_I}$ denotes the kinetic moment of the gyroscope with respect to point I. Equation (2-5).

I_G is the inertial moment of the gyroscope with respect to point G in the system G_{xyz}. Equation (2-6).

P is the inertial moment of the gyroscope with respect to its axis of rotation G_x. Equation (2-7).

Q is the inertial moment of the gyroscope with respect to any axis perpendicular to G_x and passing through G, for example, G_y and G_z. Equation (2-7).

$\vec{M}_I$ is the total kinetic moment of the whole system, gyroscope and carriage; $(\vec{M}_I = \vec{M}_{CI} + \vec{M}_{GI})$ Equation (2-8).

$\vec{M}_{G_G}$ is the kinetic moment of the gyroscope with respect to point G. Equation (2-83).

$\vec{M}_{E_G}$ is the moment of external forces acting on the gyroscope. Equation (2-85).

$\vec{M}_E$ is the moment of external forces in the total system (gyroscope and carriage). Equation (2-10).

w is the mass of an infinitely small element. Equation (2-1′).

m_C is the mass of the carriage. Equation (2-1′).

m_G is the mass of the gyroscope. Equation (2-6).

m is the mass of the gyroscope and the carriage $(m = m_C + m_G)$. Equation (2-10).

$\vec{\varpi}$ is the angular velocity vector of the earth with respect to inertial space. Figure 3.

D, E, and F are components of $\vec{\varpi}$ in G_{xyz}. Equation (2-41′)

$\vec{\Gamma}$ is the instantaneous angular velocity of the system G_{xyz} with respect to $O_{\overline{xyz}}$. Equation (2-31).

d, e, and f are components of $\vec{\Gamma}$ in G_{xyz}. Equation (2-41′).

p, q, and r are the components of $\vec{\Gamma}$ in $G_{X'Y'Z'}$. Equation (2-73).

$\vec{V}_P$ is the absolute velocity of the point P. Equation (2-1′).

$\vec{V}_{I_O}$ is the velocity of point I with respect to the system $O_{\overline{xyz}}$. Equation (2-32).

$\vec{\psi}_P$ in a general way, describes the absolute acceleration of the point P. Equation (2-9).

$*$ is the symbol of the vectorial product. Equation (2-1′).

K is the constant of suspension tape torsion. Equation (3-12).

λ is the constant of damping forces. (3-51).

C is the constant equal to $m_C(l_G - l)(l_G - l_C)$. Equation (3-19).

L is the constant defined by: $L = -l_G(m_C l_G - m_C l_C - ml)$. Equation (2-51).

J is an arbitrary constant. $J = D + d + \omega_1$. Equation (3-9).

$\vec{\omega}$ is the angular velocity of the gyroscope with respect to the carriage. Figure 5.

ω_1 is the component of $\vec{\omega}$ on the axis G_x. Equation (2-63).

δ, τ and μ are residual parameters used in linearised equations. Equations (3-26), (3-27) and (3-28).

γ_0 is the value of angle γ when the gyroscope is in the position of apparent balance, no torsion about the tape. Equation (3-13).

APPENDIX B

TIME OBSERVATIONS DATA

The first set of data involves some 552-time observations versus scale divisions.

Table B1

The first set of data

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
5	0	4	57	3		7428
6	0	5	4	7	179	7607
7	0	5	11	24	192	7799
8	0	5	20	14	215	8014
9	0	5	30	15	251	8265
10	0	5	43	13	323	8588
10	0	6	37	19	1356	9944
9	0	6	50	20	326	10270
8	0	7	0	19	249	10519
7	0	7	9	5	211	10730
6	0	7	16	21	191	10921
5	0	7	24	4	183	11104
4	0	7	30	23	169	11273
3	0	7	37	12	164	11437
2	0	7	43	20	158	11595
1	0	7	50	1	156	11751
0	0	7	56	4	153	11904
-1	0	8	2	8	154	12058
-2	0	8	8	14	156	12214
-3	0	8	14	21	157	12371
-4	0	8	21	8	162	12533
-5	0	8	28	1	168	12701
-6	0	8	35	3	177	12878
-7	0	8	42	18	190	13068
-8	0	8	51	4	211	13279
-9	0	9	0	15	236	13515
-10	0	9	12	20	305	13820
-11	0	9	35	5	560	14380
-11	0	9	51	11	406	14786
-10	0	10	13	1	540	15326
-9	0	10	25	6	305	15631
-8	0	10	34	20	239	15870
-7	0	10	43	4	209	16079
-6	0	10	50	20	191	16270
-5	0	10	57	19	174	16444
-4	0	11	4	15	171	16615

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
-3	0	11	10	24	159	16774
-2	0	11	17	8	159	16933
-1	0	11	23	13	155	17088
0	0	11	29	18	155	17243
1	0	11	35	21	153	17396
2	0	11	42	2	156	17552
3	0	11	48	11	159	17711
4	0	11	55	1	165	17876
5	0	12	1	20	169	18045
6	0	12	8	24	179	18224
7	0	12	16	18	194	18418
8	0	12	25	1	208	18626
9	0	12	35	8	257	18883
10	0	12	48	9	326	19209
10	0	13	41	20	1336	20545
9	0	13	55	1	331	20876
8	0	14	5	4	253	21129
7	0	14	13	15	211	21340
6	0	14	21	10	195	21535
5	0	14	28	14	179	21714
4	0	14	35	9	170	21884
3	0	14	41	24	165	22049
2	0	14	48	7	158	22207
1	0	14	54	14	157	22364
0	0	15	0	17	153	22517
-1	0	15	6	21	154	22671
-2	0	15	13	2	156	22827
-3	0	15	19	12	160	22987
-4	0	15	25	22	160	23147
-5	0	15	32	16	169	23316
-6	0	15	39	19	178	23494
-7	0	15	47	8	189	23683
-8	0	15	55	18	210	23893
-9	0	16	5	6	238	24131
-10	0	16	17	11	305	24436
-11	0	16	41	8	597	25033
-11	0	16	54	14	331	25364
-10	0	17	17	16	577	25941
-9	0	17	29	18	302	26243
-8	0	17	39	6	238	26481
-7	0	17	47	14	208	26689
-6	0	17	55	6	192	26881
-5	0	18	2	8	177	27058
-4	0	18	9	2	169	27227
-3	0	18	15	12	160	27387
-2	0	18	21	22	160	27547

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
-1	0	18	28	1	154	27701
0	0	18	34	7	156	27857
1	0	18	40	10	153	28010
2	0	18	46	19	159	28169
3	0	18	53	1	157	28326
4	0	18	59	14	163	28489
5	0	19	6	10	171	28660
6	0	19	13	14	179	28839
7	0	19	21	8	194	29033
8	0	19	29	22	214	29247
9	0	19	39	22	250	29497
10	0	19	53	0	328	29825
10	0	20	46	17	1342	31167
9	0	20	59	13	321	31488
8	0	21	9	15	252	31740
7	0	21	18	3	213	31953
6	0	21	25	22	194	32147
5	0	21	33	1	179	32326
4	0	21	39	20	169	32495
3	0	21	46	10	165	32660
2	0	21	52	18	158	32818
1	0	21	58	23	155	32973
0	0	22	5	3	155	33128
-1	0	22	11	7	154	33282
-2	0	22	17	13	156	33438
-3	0	22	23	20	157	33595
-4	0	22	30	7	162	33757
-5	0	22	37	0	168	33925
-6	0	22	44	2	177	34102
-7	0	22	51	17	190	34292
-8	0	23	0	2	210	34502
-9	0	23	9	15	238	34740
-10	0	23	21	23	308	35048
-11	0	23	44	23	575	35623
-11	0	23	57	19	321	35944
-10	0	24	21	18	599	36543
-9	0	24	33	22	304	36847
-8	0	24	43	9	237	37084
-7	0	24	51	22	213	37297
-6	0	24	59	9	187	37484
-5	0	25	6	12	178	37662
-4	0	25	13	8	171	37833
-3	0	25	19	19	161	37994
-2	0	25	26	2	158	38152
-1	0	25	32	5	153	38305
0	0	25	38	11	156	38461

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
1	0	25	44	15	154	38615
2	0	25	50	24	159	38774
3	0	25	57	5	156	38930
4	0	26	3	20	165	39095
5	0	26	10	11	166	39261
6	0	26	17	18	182	39443
7	0	26	25	12	194	39637
8	0	26	33	24	212	39849
9	0	26	43	24	250	40099
10	0	26	57	7	333	40432
10	0	27	50	6	1324	41756
9	0	28	3	14	333	42089
8	0	28	13	19	255	42344
7	0	28	22	3	209	42553
6	0	28	29	24	196	42749
5	0	28	37	4	180	42929
4	0	28	43	22	168	43097
3	0	28	50	11	164	43261
2	0	28	56	20	159	43420
1	0	29	3	3	158	43578
0	0	29	9	7	154	43732
-1	0	29	15	11	154	43886
-2	0	29	21	17	156	44042
-3	0	29	27	23	156	44198
-4	0	29	34	10	162	44360
-5	0	29	41	5	170	44530
-6	0	29	48	7	177	44707
-7	0	29	55	22	190	44897
-8	0	30	4	7	210	45107
-9	0	30	13	23	241	45348
-10	0	30	26	1	303	45651
-11	0	30	50	9	608	46259
-11	0	31	1	18	284	46543
-10	0	31	25	18	600	47143
-9	0	31	37	24	306	47449
-8	0	31	47	12	238	47687
-7	0	31	55	23	211	47898
-6	0	32	3	13	190	48088
-5	0	32	10	15	177	48265
-4	0	32	17	9	169	48434
-3	0	32	23	21	162	48596
-2	0	32	30	5	159	48755
-1	0	32	36	8	153	48908
0	0	32	42	13	155	49063
1	0	32	48	18	155	49218
2	0	32	55	0	157	49375

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
3	0	33	1	9	159	49534
4	0	33	7	24	165	49699
5	0	33	14	17	168	49867
6	0	33	21	23	181	50048
7	0	33	29	15	192	50240
8	0	33	38	8	218	50458
9	0	33	48	9	251	50709
10	0	34	1	16	332	51041
10	0	34	54	11	1320	52361
9	0	35	7	15	329	52690
8	0	35	17	18	253	52943
7	0	35	26	3	210	53153
6	0	35	34	0	197	53350
5	0	35	41	3	178	53528
4	0	35	48	1	173	53701
3	0	35	54	16	165	53866
2	0	36	0	24	158	54024
1	0	36	7	4	155	54179
0	0	36	13	9	155	54334
-1	0	36	19	14	155	54489
-2	0	36	25	18	154	54643
-3	0	36	32	0	157	54800
-4	0	36	38	16	166	54966
-5	0	36	45	11	170	55136
-6	0	36	52	12	176	55312
-7	0	37	0	2	190	55502
-8	0	37	8	11	209	55711
-9	0	37	18	2	241	55952
-10	0	37	30	2	300	56252
-11	0	38	5	11	884	57136
-11	0	38	6	2	16	57152
-10	0	38	29	18	591	57743
-9	0	38	41	21	303	58046
-8	0	38	51	12	241	58287
-7	0	38	59	24	212	58499
-6	0	39	7	15	191	58690
-5	0	39	14	17	177	58867
-4	0	39	21	11	169	59036
-3	0	39	27	23	162	59198
-2	0	39	34	7	159	59357
-1	0	39	40	13	156	59513
0	0	39	46	19	156	59669
1	0	39	52	22	153	59822
2	0	39	59	4	157	59979
3	0	40	5	11	157	60136
4	0	40	12	0	164	60300

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
5	0	40	18	22	172	60472
6	0	40	26	1	179	60651
7	0	40	33	20	194	60845
8	0	40	42	10	215	61060
9	0	40	52	13	253	61313
10	0	41	5	20	332	61645
10	0	41	58	11	1316	62961
9	0	42	11	13	327	63288
8	0	42	21	16	253	63541
7	0	42	30	7	216	63757
6	0	42	38	0	193	63950
5	0	42	45	6	181	64131
4	0	42	52	1	170	64301
3	0	42	58	15	164	64465
2	0	43	4	24	159	64624
1	0	43	11	6	157	64781
0	0	43	17	9	153	64934
-1	0	43	23	15	156	65090
-2	0	43	29	21	156	65246
-3	0	43	36	3	157	65403
-4	0	43	42	17	164	65567
-5	0	43	49	11	169	65736
-6	0	43	56	12	176	65912
-7	0	44	4	3	191	66103
-8	0	44	12	15	212	66315
-9	0	44	22	4	239	66554
-10	0	44	34	10	306	66860
-11	0	44	57	20	585	67445
-11	0	45	9	19	299	67744
-10	0	45	33	12	593	68337
-9	0	45	45	21	309	68646
-8	0	45	55	13	242	68888
-7	0	46	3	22	209	69097
-6	0	46	11	13	191	69288
-5	0	46	18	16	178	69466
-4	0	46	25	11	170	69636
-3	0	46	32	0	164	69800
-2	0	46	38	5	155	69955
-1	0	46	44	12	157	70112
0	0	46	50	16	154	70266
1	0	46	56	23	157	70423
2	0	47	3	5	157	70580
3	0	47	9	12	157	70737
4	0	47	15	24	162	70899
5	0	47	22	20	171	71070
6	0	47	30	3	183	71253

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
7	0	47	37	21	193	71446
8	0	47	46	14	218	71664
9	0	47	56	17	253	71917
10	0	48	9	18	326	72243
10	0	49	2	1	1308	73551
9	0	49	15	12	336	73887
8	0	49	25	17	255	74142
7	0	49	34	3	211	74353
6	0	49	42	0	197	74550
5	0	49	49	7	182	74732
4	0	49	55	24	167	74899
3	0	50	2	15	166	75065
2	0	50	8	24	159	75224
1	0	50	15	7	158	75382
0	0	50	21	9	152	75534
-1	0	50	27	18	159	75693
-2	0	50	33	22	154	75847
-3	0	50	40	4	157	76004
-4	0	50	46	17	163	76167
-5	0	50	53	11	169	76336
-6	0	51	0	14	178	76514
-7	0	51	8	5	191	76705
-8	0	51	16	16	211	76916
-9	0	51	26	8	242	77158
-10	0	51	38	18	310	77468
-11	0	52	5	17	674	78142
-11	0	52	9	9	92	78234
-10	0	52	37	6	697	78931
-9	0	52	49	17	311	79242
-8	0	52	59	10	243	79485
-7	0	53	7	21	211	79696
-6	0	53	15	13	192	79888
-5	0	53	22	14	176	80064
-4	0	53	29	10	171	80235
-3	0	53	35	23	163	80398
-2	0	53	42	7	159	80557
-1	0	53	48	14	157	80714
0	0	53	54	19	155	80869
1	0	54	0	23	154	81023
2	0	54	7	6	158	81181
3	0	54	13	14	158	81339
4	0	54	20	2	163	81502
6	0	54	34	5	81855	81855
7	0	54	41	23	193	82048
8	0	54	50	14	216	82264

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
9	0	55	0	19	255	82519
10	0	55	14	6	337	82856
10	0	56	5	17	1286	84142
9	0	56	19	10	343	84485
8	0	56	29	15	255	84740
7	0	56	38	4	214	84954
6	0	56	45	24	195	85149
5	0	56	53	5	181	85330
4	0	57	0	2	172	85502
3	0	57	6	15	163	85665
2	0	57	13	0	160	85825
1	0	57	19	7	157	85982
0	0	57	25	10	153	86135
-1	0	57	31	18	158	86293
-2	0	57	38	0	157	86450
-3	0	57	44	7	157	86607
-4	0	57	50	20	163	86770
-5	0	57	57	17	172	86942
-6	0	58	4	19	177	87119
-7	0	58	12	9	190	87309
-8	0	58	20	23	214	87523
-9	0	58	30	15	242	87765
-10	0	58	43	3	313	88078
-11	0	59	9	15	662	88740
-11	0	59	13	7	92	88832
-10	0	59	41	2	695	89527
-9	0	59	53	18	316	89843
-8	1	0	3	10	242	90085
-7	1	0	11	24	214	90299
-6	1	0	19	15	191	90490
-5	1	0	26	20	180	90670
-4	1	0	33	15	170	90840
-3	1	0	40	3	163	91003
-2	1	0	46	11	158	91161
-1	1	0	52	20	159	91320
0	1	0	59	0	155	91475
1	1	1	5	4	154	91629
2	1	1	11	11	157	91786
3	1	1	17	21	160	91946
4	1	1	24	9	163	92109
5	1	1	31	7	173	92282
6	1	1	38	10	178	92460
7	1	1	46	8	198	92658
8	1	1	55	1	218	92876
9	1	2	5	3	252	93128
10	1	2	18	14	336	93464

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
10	1	3	10	9	1295	94759
9	1	3	23	18	334	95093
8	1	3	33	20	252	95345
7	1	3	42	11	216	95561
6	1	3	50	7	196	95757
5	1	3	57	15	183	95940
4	1	4	4	9	169	96109
3	1	4	10	24	165	96274
2	1	4	17	7	158	96432
1	1	4	23	16	159	96591
0	1	4	29	21	155	96746
-1	1	4	36	2	156	96902
-2	1	4	42	8	156	97058
-3	1	4	48	15	157	97215
-4	1	4	55	3	163	97378
-5	1	5	1	24	171	97549
-6	1	5	9	3	179	97728
-7	1	5	16	20	192	97920
-8	1	5	25	8	213	98133
-9	1	5	34	24	241	98374
-10	1	5	47	7	308	98682
-11	1	6	16	17	735	99417
-11	1	6	18	8	41	99458
-10	1	6	45	12	679	100137
-9	1	6	57	24	312	100449
-8	1	7	7	17	243	100692
-7	1	7	16	6	214	100906
-6	1	7	23	22	191	101097
-5	1	7	31	1	179	101276
-4	1	7	37	21	170	101446
-3	1	7	44	9	163	101609
-2	1	7	50	19	160	101769
-1	1	7	57	0	156	101925
0	1	8	3	6	156	102081
1	1	8	9	11	155	102236
2	1	8	15	17	156	102392
3	1	8	22	1	159	102551
4	1	8	28	17	166	102717
5	1	8	35	13	171	102888
6	1	8	42	19	181	103069
7	1	8	50	15	196	103265
8	1	8	59	3	213	103478
9	1	9	9	16	263	103741
10	1	9	23	3	337	104078
10	1	10	14	5	1277	105355
9	1	10	27	17	337	105692

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
8	1	10	37	24	257	105949
7	1	10	46	16	217	106166
6	1	10	54	11	195	106361
5	1	11	1	18	182	106543
4	1	11	8	14	171	106714
3	1	11	15	4	165	106879
2	1	11	21	12	158	107037
1	1	11	27	22	160	107197
0	1	11	34	1	154	107351
-1	1	11	40	7	156	107507
-2	1	11	46	14	157	107664
-3	1	11	52	23	159	107823
-4	1	11	59	10	162	107985
-5	1	12	6	7	172	108157
-6	1	12	13	10	178	108335
-7	1	12	21	1	191	108526
-8	1	12	29	12	211	108737
-9	1	12	39	9	247	108984
-10	1	12	52	4	320	109304
-11	1	13	20	10	706	110010
-10	1	13	49	16	731	110741
-9	1	14	2	3	312	111053
-8	1	14	11	21	243	111296
-7	1	14	20	10	214	111510
-6	1	14	28	2	192	111702
-5	1	14	35	6	179	111881
-4	1	14	42	1	170	112051
-3	1	14	48	12	161	112212
-2	1	14	54	23	161	112373
-1	1	15	1	7	159	112532
0	1	15	7	12	155	112687
1	1	15	13	16	154	112841
2	1	15	20	0	159	113000
3	1	15	26	11	161	113161
4	1	15	33	0	164	113325
5	1	15	39	21	171	113496
6	1	15	47	4	183	113679
7	1	15	55	1	197	113876
8	1	16	3	19	218	114094
9	1	16	13	23	254	114348
10	1	17	18	1	1251	115951
9	1	17	31	20	344	116295
8	1	17	42	2	257	116552
7	1	17	50	19	217	116769
6	1	17	58	17	198	116967

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
5	1	18	5	22	180	117147
4	1	18	12	20	173	117320
3	1	18	19	9	164	117484
2	1	18	25	19	160	117644
1	1	18	32	2	158	117802
0	1	18	38	7	155	117957
-1	1	18	44	13	156	118113
-2	1	18	50	21	158	118271
-3	1	18	57	4	158	118429
-4	1	19	3	18	164	118593
-5	1	19	10	13	170	118763
-6	1	19	17	16	178	118941
-7	1	19	25	8	192	119133
-8	1	19	33	22	214	119347
-9	1	19	43	16	244	119591
-10	1	19	56	2	311	119902
-10	1	20	53	15	121340	121340
-9	1	21	6	6	316	121656
-8	1	21	16	1	245	121901
-7	1	21	24	13	212	122113
-6	1	21	32	9	196	122309
-5	1	21	39	11	177	122486
-4	1	21	46	5	169	122655
-3	1	21	52	20	165	122820
-2	1	21	59	4	159	122979
-1	1	22	5	11	157	123136
0	1	22	11	17	156	123292
1	1	22	17	20	153	123445
2	1	22	24	5	160	123605
3	1	22	30	14	159	123764
4	1	22	37	6	167	123931
5	1	22	44	2	171	124102
6	1	22	51	7	180	124282
7	1	22	59	5	198	124480
8	1	23	7	22	217	124697
9	1	23	18	7	260	124957
10	1	23	32	4	347	125304
10	1	24	22	2	1248	126552
9	1	24	35	24	347	126899
8	1	24	46	4	255	127154
7	1	24	54	22	218	127372
6	1	25	2	20	198	127570
5	1	25	10	0	180	127750
4	1	25	16	23	173	127923
3	1	25	23	13	165	128088

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
2	1	25	30	0	162	128250
1	1	25	36	5	155	128405
0	1	25	42	9	154	128559
-1	1	25	48	16	157	128716
-2	1	25	54	23	157	128873
-3	1	26	1	6	158	129031
-4	1	26	7	19	163	129194
-5	1	26	14	16	172	129366
-6	1	26	21	20	179	129545
-7	1	26	29	11	191	129736
-8	1	26	37	24	213	129949
-9	1	26	47	17	243	130192
-10	1	27	0	7	315	130507
-10	1	27	57	11	131936	131936
-9	1	28	10	9	323	132259
-8	1	28	20	3	244	132503
-7	1	28	28	16	213	132716
-6	1	28	36	7	191	132907
-5	1	28	43	12	180	133087
-4	1	28	50	6	169	133256
-3	1	28	56	22	166	133422
-2	1	29	3	4	157	133579
-1	1	29	9	11	157	133736
0	1	29	15	16	155	133891
1	1	29	21	22	156	134047
2	1	29	28	4	157	134204
3	1	29	34	12	158	134362
4	1	29	41	4	167	134529
5	1	29	47	24	170	134699
6	1	29	55	7	183	134882
7	1	30	3	3	196	135078
8	1	30	11	20	217	135295
9	1	30	22	1	256	135551
10	1	30	35	22	346	135897
10	1	31	25	21	1249	137146
9	1	31	39	21	350	137496
8	1	31	50	1	255	137751
7	1	31	58	21	220	137971
6	1	32	6	16	195	138166
5	1	32	13	24	183	138349
4	1	32	20	20	171	138520
3	1	32	27	10	165	138685
2	1	32	33	20	160	138845
1	1	32	40	1	156	139001
0	1	32	46	8	157	139158

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
-1	1	32	52	14	156	139314
-2	1	32	58	22	158	139472
-3	1	33	5	6	159	139631
-4	1	33	11	21	165	139796
-5	1	33	18	15	169	139965
-6	1	33	25	19	179	140144
-7	1	33	33	10	191	140335
-8	1	33	42	0	215	140550
-9	1	33	51	19	244	140794
-10	1	34	4	16	322	141116
-10	1	35	1	17	1426	142542
-9	1	35	14	4	312	142854
-8	1	35	23	21	242	143096
-7	1	35	32	12	216	143312
-5	1	35	47	4	170	143679

The second set of data involves some 1165-time observations versus scale divisions.

Table B2

The second set of data

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
5	0	4	52	9		7309
6	0	4	58	21	162	7471
7	0	5	5	14	168	7639
8	0	5	13	2	188	7827
9	0	5	21	11	209	8036
10	0	5	31	20	259	8295
11	0	5	47	2	382	8677
11	0	6	20	9	832	9509
10	0	6	36	6	397	9906
9	0	6	46	9	253	10159
8	0	6	54	21	212	10371
7	0	7	2	6	185	10556
6	0	7	9	3	172	10728
5	0	7	15	10	157	10885
4	0	7	21	10	150	11035
3	0	7	27	6	146	11181
2	0	7	32	22	141	11322

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
1	0	7	38	11	139	11461
0	0	7	43	21	135	11596
-1	0	7	49	9	138	11734
-2	0	7	54	21	137	11871
-3	0	8	0	5	134	12005
-4	0	8	5	17	137	12142
-5	0	8	11	11	144	12286
-6	0	8	17	7	146	12432
-7	0	8	23	9	152	12584
-8	0	8	29	20	161	12745
-9	0	8	36	15	170	12915
-10	0	8	44	4	189	13104
-11	0	8	52	22	218	13322
-11	0	10	15	9	2062	15384
-10	0	10	24	3	219	15603
-9	0	10	31	17	189	15792
-8	0	10	38	13	171	15963
-7	0	10	44	23	160	16123
-6	0	10	51	1	153	16276
-5	0	10	56	21	145	16421
-4	0	11	2	15	144	16565
-3	0	11	8	2	137	16702
-2	0	11	13	14	137	16839
-1	0	11	19	0	136	16975
0	0	11	24	11	136	17111
1	0	11	29	21	135	17246
2	0	11	35	12	141	17387
3	0	11	41	1	139	17526
4	0	11	46	23	147	17673
5	0	11	52	24	151	17824
6	0	11	59	7	158	17982
7	0	12	6	4	172	18154
8	0	12	13	14	185	18339
9	0	12	22	3	214	18553
10	0	12	32	8	255	18808
11	0	12	48	0	392	19200
11	0	13	20	23	823	20023
10	0	13	36	11	388	20411
9	0	13	46	17	256	20667
8	0	13	55	5	213	20880
7	0	14	2	14	184	21064
6	0	14	9	10	171	21235
5	0	14	15	20	160	21395
4	0	14	21	22	152	21547
3	0	14	27	18	146	21693

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
2	0	14	33	7	139	21832
1	0	14	38	23	141	21973
0	0	14	44	8	135	22108
-1	0	14	49	20	137	22245
-2	0	14	55	5	135	22380
-3	0	15	0	17	137	22517
-4	0	15	6	5	138	22655
-5	0	15	11	24	144	22799
-6	0	15	17	18	144	22943
-7	0	15	23	20	152	23095
-8	0	15	30	6	161	23256
-9	0	15	37	2	171	23427
-10	0	15	44	19	192	23619
-11	0	15	53	8	214	23833
-11	0	17	15	19	2061	25894
-10	0	17	24	11	217	26111
-9	0	17	31	24	188	26299
-8	0	17	38	20	171	26470
-7	0	17	45	5	160	26630
-6	0	17	51	9	154	26784
-5	0	17	57	5	146	26930
-4	0	18	2	24	144	27074
-3	0	18	8	11	137	27211
-2	0	18	13	22	136	27347
-1	0	18	19	9	137	27484
0	0	18	24	20	136	27620
1	0	18	30	5	135	27755
2	0	18	35	21	141	27896
3	0	18	41	10	139	28035
4	0	18	47	6	146	28181
5	0	18	53	6	150	28331
6	0	18	59	16	160	28491
7	0	19	6	12	171	28662
8	0	19	13	23	186	28848
9	0	19	22	10	212	29060
10	0	19	32	20	260	29320
11	0	19	48	5	385	29705
11	0	20	20	15	810	30515
10	0	20	36	15	400	30915
9	0	20	47	0	260	31175
8	0	20	55	11	211	31386
7	0	21	2	22	186	31572
6	0	21	9	18	171	31743
5	0	21	16	2	159	31902
4	0	21	22	3	151	32053

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
3	0	21	28	0	147	32200
2	0	21	33	15	140	32340
1	0	21	39	4	139	32479
0	0	21	44	16	137	32616
-1	0	21	50	2	136	32752
-2	0	21	55	13	136	32888
-3	0	22	0	23	135	33023
-4	0	22	6	12	139	33162
-5	0	22	12	5	143	33305
-6	0	22	18	1	146	33451
-7	0	22	24	3	152	33603
-8	0	22	30	14	161	33764
-9	0	22	37	12	173	33937
-10	0	22	45	2	190	34127
-11	0	22	53	18	216	34343
-11	0	24	15	23	2055	36398
-10	0	24	24	16	218	36616
-9	0	24	32	5	189	36805
-8	0	24	38	24	169	36974
-7	0	24	45	13	164	37138
-6	0	24	51	17	154	37292
-5	0	24	57	11	144	37436
-4	0	25	3	5	144	37580
-3	0	25	8	19	139	37719
-2	0	25	14	3	134	37853
-1	0	25	19	14	136	37989
0	0	25	25	1	137	38126
1	0	25	30	14	138	38264
2	0	25	36	2	138	38402
3	0	25	41	19	142	38544
4	0	25	47	14	145	38689
5	0	25	53	15	151	38840
6	0	26	0	0	160	39000
7	0	26	6	22	172	39172
8	0	26	14	7	185	39357
9	0	26	22	20	213	39570
10	0	26	33	4	259	39829
11	0	26	48	14	385	40214
11	0	27	20	23	809	41023
10	0	27	36	22	399	41422
9	0	27	47	6	259	41681
8	0	27	55	16	210	41891
7	0	28	3	3	187	42078
6	0	28	10	0	172	42250
5	0	28	16	10	160	42410

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
4	0	28	22	11	151	42561
3	0	28	28	7	146	42707
2	0	28	33	24	142	42849
1	0	28	39	13	139	42988
0	0	28	45	0	137	43125
-1	0	28	50	9	134	43259
-2	0	28	55	12	128	43387
-3	0	29	1	8	146	43533
-4	0	29	6	21	138	43671
-5	0	29	12	17	146	43817
-6	0	29	18	12	145	43962
-7	0	29	24	16	154	44116
-8	0	29	31	0	159	44275
-9	0	29	37	22	172	44447
-10	0	29	45	13	191	44638
-11	0	29	54	3	215	44853
-11	0	31	16	3	2050	46903
-10	0	31	24	20	217	47120
-9	0	31	32	13	193	47313
-8	0	31	39	9	171	47484
-7	0	31	45	22	163	47647
-6	0	31	51	24	152	47799
-5	0	31	57	19	145	47944
-4	0	32	3	12	143	48087
-3	0	32	9	2	140	48227
-2	0	32	14	14	137	48364
-1	0	32	20	1	137	48501
0	0	32	25	10	134	48635
1	0	32	30	23	138	48773
2	0	32	36	13	140	48913
3	0	32	42	3	140	49053
4	0	32	47	24	146	49199
5	0	32	54	2	153	49352
6	0	33	0	12	160	49512
7	0	33	7	9	172	49684
8	0	33	14	21	187	49871
9	0	33	23	9	213	50084
10	0	33	33	16	257	50341
11	0	33	49	22	406	50747
11	0	34	21	7	785	51532
10	0	34	37	3	396	51928
9	0	34	47	12	259	52187
8	0	34	56	0	213	52400
7	0	35	3	13	188	52588
6	0	35	10	9	171	52759

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
5	0	35	16	21	162	52921
4	0	35	22	21	150	53071
3	0	35	28	17	146	53217
2	0	35	34	7	140	53357
1	0	35	39	24	142	53499
0	0	35	45	10	136	53635
-1	0	35	50	20	135	53770
-2	0	35	56	8	138	53908
-3	0	36	1	20	137	54045
-4	0	36	7	8	138	54183
-5	0	36	13	2	144	54327
-6	0	36	18	24	147	54474
-7	0	36	25	1	152	54626
-8	0	36	31	11	160	54786
-9	0	36	38	8	172	54958
-10	0	36	45	24	191	55149
-11	0	36	54	20	221	55370
-11	0	38	16	11	2041	57411
-10	0	38	25	6	220	57631
-9	0	38	32	22	191	57822
-8	0	38	39	19	172	57994
-7	0	38	46	5	161	58155
-6	0	38	52	8	153	58308
-5	0	38	58	5	147	58455
-4	0	39	3	23	143	58598
-3	0	39	9	11	138	58736
-2	0	39	14	24	138	58874
-1	0	39	20	10	136	59010
0	0	39	25	21	136	59146
1	0	39	31	8	137	59283
2	0	39	36	23	140	59423
3	0	39	42	13	140	59563
4	0	39	48	9	146	59709
5	0	39	54	9	150	59859
6	0	40	0	21	162	60021
7	0	40	7	19	173	60194
8	0	40	15	8	189	60383
9	0	40	23	19	211	60594
10	0	40	34	1	257	60851
11	0	40	50	4	403	61254
11	0	41	21	6	777	62031
10	0	41	37	12	406	62437
9	0	41	48	0	263	62700
8	0	41	56	11	211	62911
7	0	42	4	0	189	63100

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
6	0	42	10	22	172	63272
5	0	42	17	9	162	63434
4	0	42	23	10	151	63585
3	0	42	29	9	149	63734
2	0	42	35	0	141	63875
1	0	42	40	15	140	64015
0	0	42	46	3	138	64153
-1	0	42	51	15	137	64290
-2	0	42	57	3	138	64428
-3	0	43	2	16	138	64566
-4	0	43	8	5	139	64705
-5	0	43	13	24	144	64849
-6	0	43	19	22	148	64997
-7	0	43	26	0	153	65150
-8	0	43	32	13	163	65313
-9	0	43	39	13	175	65488
-10	0	43	47	5	192	65680
-11	0	43	56	1	221	65901
-11	0	45	17	13	2037	67938
-10	0	45	26	9	221	68159
-9	0	45	34	2	193	68352
-8	0	45	41	0	173	68525
-7	0	45	47	13	163	68688
-6	0	45	53	17	154	68842
-5	0	45	59	16	149	68991
-4	0	46	5	9	143	69134
-3	0	46	10	23	139	69273
-2	0	46	16	10	137	69410
-1	0	46	22	0	140	69550
0	0	46	27	12	137	69687
1	0	46	32	23	136	69823
2	0	46	38	16	143	69966
3	0	46	44	6	140	70106
4	0	46	50	5	149	70255
5	0	46	56	5	150	70405
6	0	47	2	19	164	70569
7	0	47	9	15	171	70740
8	0	47	17	5	190	70930
9	0	47	25	18	213	71143
10	0	47	36	6	263	71406
11	0	47	52	15	409	71815
11	0	48	22	19	754	72569
10	0	48	39	3	409	72978
9	0	48	49	16	263	73241
8	0	48	58	6	215	73456

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
7	0	49	5	20	189	73645
6	0	49	12	16	171	73816
5	0	49	19	2	161	73977
4	0	49	25	3	151	74128
3	0	49	31	1	148	74276
2	0	49	36	17	141	74417
1	0	49	42	6	139	74556
0	0	49	47	19	138	74694
-1	0	49	53	7	138	74832
-2	0	49	58	19	137	74969
-3	0	50	4	6	137	75106
-4	0	50	9	20	139	75245
-5	0	50	15	14	144	75389
-6	0	50	21	11	147	75536
-7	0	50	27	13	152	75688
-8	0	50	34	2	164	75852
-9	0	50	41	1	174	76026
-10	0	50	48	18	192	76218
-11	0	50	57	16	223	76441
-11	0	52	18	18	2027	78468
-10	0	52	27	15	222	78690
-9	0	52	35	5	190	78880
-8	0	52	42	2	172	79052
-7	0	52	48	15	163	79215
-6	0	52	54	20	155	79370
-5	0	53	0	15	145	79515
-4	0	53	6	11	146	79661
-3	0	53	12	0	139	79800
-2	0	53	17	11	136	79936
-1	0	53	22	24	138	80074
0	0	53	28	10	136	80210
1	0	53	33	22	137	80347
2	0	53	39	13	141	80488
3	0	53	45	5	142	80630
4	0	53	51	0	145	80775
5	0	53	57	5	155	80930
6	0	54	3	13	158	81088
7	0	54	10	10	172	81260
8	0	54	18	0	190	81450
9	0	54	26	13	213	81663
10	0	54	37	4	266	81929
11	0	54	53	13	409	82338
11	0	55	23	2	739	83077
10	0	55	39	16	414	83491
9	0	55	50	8	267	83758

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
8	0	55	58	23	215	83973
7	0	56	6	11	188	84161
6	0	56	13	7	171	84332
5	0	56	19	20	163	84495
4	0	56	25	22	152	84647
3	0	56	31	18	146	84793
2	0	56	37	10	142	84935
1	0	56	43	1	141	85076
0	0	56	48	12	136	85212
-1	0	56	54	0	138	85350
-2	0	56	59	14	139	85489
-3	0	57	5	2	138	85627
-4	0	57	10	17	140	85767
-5	0	57	16	9	142	85909
-6	0	57	22	6	147	86056
-7	0	57	28	11	155	86211
-8	0	57	34	23	162	86373
-9	0	57	41	23	175	86548
-10	0	57	49	18	195	86743
-11	0	57	58	10	217	86960
-11	0	59	19	13	2028	88988
-10	0	59	28	8	220	89208
-9	0	59	36	1	193	89401
-8	0	59	43	0	174	89575
-7	0	59	49	14	164	89739
-6	0	59	55	19	155	89894
-5	1	0	1	16	147	90041
-4	1	0	7	9	143	90184
-3	1	0	12	23	139	90323
-2	1	0	18	12	139	90462
-1	1	0	24	0	138	90600
0	1	0	29	11	136	90736
1	1	0	34	23	137	90873
2	1	0	40	16	143	91016
3	1	0	46	8	142	91158
4	1	0	52	6	148	91306
5	1	0	58	8	152	91458
6	1	1	4	20	162	91620
7	1	1	11	17	172	91792
8	1	1	19	8	191	91983
9	1	1	27	24	216	92199
10	1	1	38	13	264	92463
11	1	1	55	10	422	92885
11	1	2	23	16	706	93591
10	1	2	40	12	421	94012

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
9	1	2	51	5	268	94280
8	1	2	59	23	218	94498
7	1	3	7	11	188	94686
6	1	3	14	11	175	94861
5	1	3	20	21	160	95021
4	1	3	26	24	153	95174
3	1	3	32	22	148	95322
2	1	3	38	14	142	95464
1	1	3	44	5	141	95605
0	1	3	49	19	139	95744
-1	1	3	55	4	135	95879
-2	1	4	0	18	139	96018
-3	1	4	6	7	139	96157
-4	1	4	11	21	139	96296
-5	1	4	17	15	144	96440
-6	1	4	23	13	148	96588
-7	1	4	29	17	154	96742
-8	1	4	36	6	164	96906
-9	1	4	43	4	173	97079
-10	1	4	50	24	195	97274
-11	1	4	59	20	221	97495
-11	1	6	20	11	2016	99511
-10	1	6	29	8	222	99733
-9	1	6	37	3	195	99928
-8	1	6	44	0	172	100100
-7	1	6	50	15	165	100265
-6	1	6	56	17	152	100417
-5	1	7	2	14	147	100564
-4	1	7	8	10	146	100710
-3	1	7	13	24	139	100849
-2	1	7	19	12	138	100987
-1	1	7	25	0	138	101125
0	1	7	30	13	138	101263
1	1	7	36	1	138	101401
2	1	7	41	16	140	101541
3	1	7	47	9	143	101684
4	1	7	53	7	148	101832
5	1	7	59	7	150	101982
6	1	8	5	22	165	102147
7	1	8	12	17	170	102317
8	1	8	20	8	191	102508
9	1	8	29	1	218	102726
10	1	8	39	21	270	102996
11	1	8	57	0	429	103425
11	1	9	24	5	680	104105

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
10	1	9	41	8	428	104533
9	1	9	51	24	266	104799
8	1	10	0	17	218	105017
7	1	10	8	7	190	105207
6	1	10	15	6	174	105381
5	1	10	21	17	161	105542
4	1	10	27	19	152	105694
3	1	10	33	18	149	105843
2	1	10	39	11	143	105986
1	1	10	45	2	141	106127
0	1	10	50	14	137	106264
-1	1	10	56	3	139	106403
-2	1	11	1	14	136	106539
-3	1	11	7	2	138	106677
-4	1	11	12	18	141	106818
-5	1	11	18	12	144	106962
-6	1	11	24	9	147	107109
-7	1	11	30	14	155	107264
-8	1	11	37	0	161	107425
-9	1	11	44	0	175	107600
-10	1	11	51	20	195	107795
-11	1	12	0	13	218	108013
-11	1	13	21	3	2015	110028
-10	1	13	30	0	222	110250
-9	1	13	37	19	194	110444
-8	1	13	44	19	175	110619
-7	1	13	51	6	162	110781
-6	1	13	57	10	154	110935
-5	1	14	3	9	149	111084
-4	1	14	9	3	144	111228
-3	1	14	14	17	139	111367
-2	1	14	20	4	137	111504
-1	1	14	25	17	138	111642
0	1	14	31	5	138	111780
1	1	14	36	19	139	111919
2	1	14	42	9	140	112059
3	1	14	48	0	141	112200
4	1	14	53	22	147	112347
5	1	15	0	0	153	112500
6	1	15	6	13	163	112663
7	1	15	13	10	172	112835
8	1	15	20	23	188	113023
9	1	15	29	15	217	113240
10	1	15	40	9	269	113509
11	1	15	57	0	416	113925

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
11	1	16	25	0	700	114625
10	1	16	42	0	425	115050
9	1	16	52	18	268	115318
8	1	17	1	11	218	115536
7	1	17	8	24	188	115724
6	1	17	15	22	173	115897
5	1	17	22	8	161	116058
4	1	17	28	12	154	116212
3	1	17	34	8	146	116358
2	1	17	40	0	142	116500
1	1	17	45	17	142	116642
0	1	17	51	4	137	116779
-1	1	17	56	17	138	116917
-2	1	18	2	5	138	117055
-3	1	18	7	19	139	117194
-4	1	18	13	10	141	117335
-5	1	18	19	4	144	117479
-6	1	18	25	0	146	117625
-7	1	18	31	5	155	117780
-8	1	18	37	19	164	117944
-9	1	18	44	16	172	118116
-10	1	18	52	12	196	118312
-11	1	19	1	8	221	118533
-11	1	20	21	17	2009	120542
-10	1	20	30	13	221	120763
-9	1	20	38	9	196	120959
-8	1	20	45	8	174	121133
-7	1	20	51	19	161	121294
-6	1	20	58	0	156	121450
-5	1	21	3	22	147	121597
-4	1	21	9	18	146	121743
-3	1	21	15	7	139	121882
-2	1	21	20	21	139	122021
-1	1	21	26	6	135	122156
0	1	21	31	21	140	122296
1	1	21	37	9	138	122434
2	1	21	43	0	141	122575
3	1	21	48	15	140	122715
4	1	21	54	14	149	122864
5	1	22	0	16	152	123016
6	1	22	7	4	163	123179
7	1	22	14	2	173	123352
8	1	22	21	15	188	123540
9	1	22	30	8	218	123758
10	1	22	41	0	267	124025

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
11	1	22	58	9	434	124459
11	1	23	25	9	675	125134
10	1	23	42	13	429	125563
9	1	23	53	5	267	125830
8	1	24	1	23	218	126048
7	1	24	9	13	190	126238
6	1	24	16	11	173	126411
5	1	24	22	22	161	126572
4	1	24	29	1	154	126726
3	1	24	34	24	148	126874
2	1	24	40	18	144	127018
1	1	24	46	6	138	127156
0	1	24	51	19	138	127294
-1	1	24	57	6	137	127431
-2	1	25	2	19	138	127569
-3	1	25	8	8	139	127708
-4	1	25	13	23	140	127848
-5	1	25	19	19	146	127994
-6	1	25	25	13	144	128138
-7	1	25	31	20	157	128295
-8	1	25	38	10	165	128460
-9	1	25	45	8	173	128633
-10	1	25	53	1	193	128826
-11	1	26	1	23	222	129048
-11	1	27	22	3	2005	131053
-10	1	27	31	2	224	131277
-9	1	27	38	21	194	131471
-8	1	27	45	20	174	131645
-7	1	27	52	10	165	131810
-6	1	27	58	13	153	131963
-5	1	28	4	9	146	132109
-4	1	28	10	4	145	132254
-3	1	28	15	19	140	132394
-2	1	28	21	8	139	132533
-1	1	28	26	20	137	132670
0	1	28	32	10	140	132810
1	1	28	37	21	136	132946
2	1	28	43	10	139	133085
3	1	28	49	4	144	133229
4	1	28	55	2	148	133377
5	1	29	1	4	152	133529
6	1	29	7	18	164	133693
7	1	29	14	14	171	133864
8	1	29	22	6	192	134056
9	1	29	30	23	217	134273

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
10	1	29	41	16	268	134541
11	1	29	58	17	426	134967
11	1	30	26	3	686	135653
10	1	30	42	24	421	136074
9	1	30	53	20	271	136345
8	1	31	2	12	217	136562
7	1	31	10	0	188	136750
6	1	31	17	0	175	136925
5	1	31	23	11	161	137086
4	1	31	29	13	152	137238
3	1	31	35	13	150	137388
2	1	31	41	4	141	137529
1	1	31	46	20	141	137670
0	1	31	52	8	138	137808
-1	1	31	57	21	138	137946
-2	1	32	3	8	137	138083
-3	1	32	8	21	138	138221
-4	1	32	14	12	141	138362
-5	1	32	20	5	143	138505
-6	1	32	26	1	146	138651
-7	1	32	32	9	158	138809
-8	1	32	38	23	164	138973
-9	1	32	45	21	173	139146
-10	1	32	53	15	194	139340
-11	1	33	2	16	226	139566
-11	1	34	22	16	2000	141566
-10	1	34	31	14	223	141789
-9	1	34	39	7	193	141982
-8	1	34	46	7	175	142157
-7	1	34	52	22	165	142322
-6	1	34	59	1	154	142476
-5	1	35	4	23	147	142623
-4	1	35	10	19	146	142769
-3	1	35	16	7	138	142907
-2	1	35	21	23	141	143048
-1	1	35	27	10	137	143185
0	1	35	32	23	138	143323
1	1	35	38	11	138	143461
2	1	35	44	1	140	143601
3	1	35	49	17	141	143742
4	1	35	55	16	149	143891
5	1	36	1	18	152	144043
6	1	36	8	7	164	144207
7	1	36	15	6	174	144381
8	1	36	22	21	190	144571

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
9	1	36	31	16	220	144791
10	1	36	42	11	270	145061
11	1	36	59	9	423	145484
11	1	37	25	23	664	146148
10	1	37	43	9	436	146584
9	1	37	54	8	274	146858
8	1	38	2	22	214	147072
7	1	38	10	11	189	147261
6	1	38	17	11	175	147436
5	1	38	23	22	161	147597
4	1	38	30	0	153	147750
3	1	38	36	0	150	147900
2	1	38	41	17	142	148042
1	1	38	47	6	139	148181
0	1	38	52	8	127	148308
-1	1	38	58	10	152	148460
-2	1	39	3	19	134	148594
-3	1	39	9	12	143	148737
-4	1	39	14	24	137	148874
-5	1	39	20	19	145	149019
-6	1	39	26	16	147	149166
-7	1	39	32	22	156	149322
-8	1	39	39	10	163	149485
-9	1	39	46	10	175	149660
-10	1	39	54	2	192	149852
-11	1	40	3	0	223	150075
-11	1	41	22	20	1995	152070
-10	1	41	31	20	225	152295
-9	1	41	39	17	197	152492
-8	1	41	46	13	171	152663
-7	1	41	53	5	167	152830
-6	1	41	59	9	154	152984
-5	1	42	5	8	149	153133
-4	1	42	11	2	144	153277
-3	1	42	16	16	139	153416
-2	1	42	22	5	139	153555
-1	1	42	27	19	139	153694
0	1	42	33	6	137	153831
1	1	42	38	20	139	153970
2	1	42	44	11	141	154111
3	1	42	50	3	142	154253
4	1	42	56	0	147	154400
5	1	43	2	3	153	154553
6	1	43	8	15	162	154715
7	1	43	15	18	178	154893

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
8	1	43	23	5	187	155080
9	1	43	31	21	216	155296
10	1	43	42	17	271	155567
11	1	44	0	9	442	156009
11	1	44	25	23	639	156648
10	1	44	43	11	438	157086
9	1	44	54	8	272	157358
8	1	45	3	3	220	157578
7	1	45	10	19	191	157769
6	1	45	17	16	172	157941
5	1	45	24	4	163	158104
4	1	45	30	7	153	158257
3	1	45	36	5	148	158405
2	1	45	41	23	143	158548
1	1	45	47	14	141	158689
0	1	45	53	4	140	158829
-1	1	45	58	13	134	158963
-2	1	46	4	3	140	159103
-3	1	46	9	9	131	159234
-4	1	46	15	7	148	159382
-5	1	46	21	0	143	159525
-6	1	46	26	24	149	159674
-7	1	46	33	3	154	159828
-8	1	46	39	17	164	159992
-9	1	46	46	17	175	160167
-10	1	46	54	13	196	160363
-11	1	47	3	12	224	160587
-11	1	48	23	6	1994	162581
-10	1	48	32	3	222	162803
-9	1	48	39	23	195	162998
-8	1	48	46	23	175	163173
-7	1	48	53	11	163	163336
-6	1	48	59	17	156	163492
-5	1	49	5	15	148	163640
-4	1	49	11	9	144	163784
-3	1	49	17	0	141	163925
-2	1	49	22	13	138	164063
-1	1	49	28	1	138	164201
0	1	49	33	12	136	164337
1	1	49	39	4	142	164479
2	1	49	44	18	139	164618
3	1	49	50	11	143	164761
4	1	49	56	9	148	164909
5	1	50	2	11	152	165061
6	1	50	9	1	165	165226

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
7	1	50	16	0	174	165400
8	1	50	23	14	189	165589
9	1	50	32	10	221	165810
10	1	50	43	5	270	166080
11	1	51	0	21	441	166521
11	1	51	25	15	619	167140
10	1	51	43	15	450	167590
9	1	51	54	14	274	167864
8	1	52	3	9	220	168084
7	1	52	10	24	190	168274
6	1	52	17	22	173	168447
5	1	52	24	10	163	168610
4	1	52	30	13	153	168763
3	1	52	36	13	150	168913
2	1	52	42	4	141	169054
1	1	52	47	21	142	169196
0	1	52	53	8	137	169333
-1	1	52	58	21	138	169471
-2	1	53	4	11	140	169611
-3	1	53	9	23	137	169748
-4	1	53	15	13	140	169888
-5	1	53	21	10	147	170035
-6	1	53	27	7	147	170182
-7	1	53	33	13	156	170338
-8	1	53	40	2	164	170502
-9	1	53	47	1	174	170676
-10	1	53	54	22	196	170872
-11	1	54	3	20	223	171095
-11	1	55	23	7	1987	173082
-10	1	55	32	7	225	173307
-9	1	55	40	4	197	173504
-8	1	55	47	2	173	173677
-7	1	55	53	17	165	173842
-6	1	55	59	22	155	173997
-5	1	56	5	22	150	174147
-4	1	56	11	15	143	174290
-3	1	56	17	6	141	174431
-2	1	56	22	20	139	174570
-1	1	56	28	8	138	174708
0	1	56	33	20	137	174845
1	1	56	39	8	138	174983
2	1	56	45	0	142	175125
3	1	56	50	17	142	175267
4	1	56	56	17	150	175417
5	1	57	2	20	153	175570

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
6	1	57	9	8	163	175733
7	1	57	16	8	175	175908
8	1	57	23	23	190	176098
9	1	57	32	18	220	176318
10	1	57	43	16	273	176591
11	1	58	1	19	453	177044
11	1	58	25	19	600	177644
10	1	58	43	18	449	178093
9	1	58	54	18	275	178368
8	1	59	3	11	218	178586
7	1	59	11	4	193	178779
6	1	59	18	2	173	178952
5	1	59	24	16	164	179116
4	1	59	30	18	152	179268
3	1	59	36	18	150	179418
2	1	59	42	10	142	179560
1	1	59	48	2	142	179702
0	1	59	53	15	138	179840
-1	1	59	59	2	137	179977
-2	2	0	4	16	139	180116
-3	2	0	10	4	138	180254
-4	2	0	15	21	142	180396
-5	2	0	21	15	144	180540
-6	2	0	27	14	149	180689
-7	2	0	33	18	154	180843
-8	2	0	40	8	165	181008
-9	2	0	47	10	177	181185
-10	2	0	55	5	195	181380
-11	2	1	4	6	226	181606
-11	2	2	23	10	1979	183585
-10	2	2	32	10	225	183810
-9	2	2	40	6	196	184006
-8	2	2	47	8	177	184183
-7	2	2	53	21	163	184346
-6	2	3	0	1	155	184501
-5	2	3	5	24	148	184649
-4	2	3	11	20	146	184795
-3	2	3	17	11	141	184936
-2	2	3	22	24	138	185074
-1	2	3	28	13	139	185213
0	2	3	33	24	136	185349
1	2	3	39	14	140	185489
2	2	3	45	6	142	185631
3	2	3	50	23	142	185773
4	2	3	56	21	148	185921

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
5	2	4	3	0	154	186075
6	2	4	9	14	164	186239
7	2	4	16	14	175	186414
8	2	4	24	4	190	186604
9	2	4	32	22	218	186822
10	2	4	43	20	273	187095
11	2	5	1	23	453	187548
11	2	5	25	15	592	188140
10	2	5	43	21	456	188596
9	2	5	54	19	273	188869
8	2	6	3	12	218	189087
7	2	6	11	5	193	189280
6	2	6	18	5	175	189455
5	2	6	24	19	164	189619
4	2	6	30	21	152	189771
3	2	6	36	20	149	189920
2	2	6	42	14	144	190064
1	2	6	48	5	141	190205
0	2	6	53	18	138	190343
-1	2	6	59	6	138	190481
-2	2	7	4	21	140	190621
-3	2	7	10	9	138	190759
-4	2	7	16	1	142	190901
-5	2	7	21	19	143	191044
-6	2	7	27	19	150	191194
-7	2	7	33	23	154	191348
-8	2	7	40	12	164	191512
-9	2	7	47	12	175	191687
-10	2	7	55	13	201	191888
-11	2	8	4	11	223	192111
-11	2	9	23	14	1978	194089
-10	2	9	32	12	223	194312
-9	2	9	40	9	197	194509
-8	2	9	47	8	174	194683
-7	2	9	53	24	166	194849
-6	2	10	0	4	155	195004
-5	2	10	6	2	148	195152
-4	2	10	11	22	145	195297
-3	2	10	17	14	142	195439
-2	2	10	23	1	137	195576
-1	2	10	28	15	139	195715
0	2	10	34	2	137	195852
1	2	10	39	16	139	195991
2	2	10	45	8	142	196133
3	2	10	51	1	143	196276

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
4	2	10	57	0	149	196425
5	2	11	3	2	152	196577
6	2	11	9	15	163	196740
7	2	11	16	15	175	196915
8	2	11	24	6	191	197106
9	2	11	33	4	223	197329
10	2	11	44	0	271	197600
11	2	12	2	13	463	198063
11	2	12	25	15	577	198640
10	2	12	43	18	453	199093
9	2	12	54	18	275	199368
8	2	13	3	14	221	199589
7	2	13	11	5	191	199780
6	2	13	18	3	173	199953
5	2	13	24	19	166	200119
4	2	13	30	21	152	200271
3	2	13	36	21	150	200421
2	2	13	42	13	142	200563
1	2	13	48	6	143	200706
0	2	13	53	20	139	200845
-1	2	13	59	7	137	200982
-2	2	14	4	20	138	201120
-3	2	14	10	7	137	201257
-4	2	14	16	0	143	201400
-5	2	14	21	19	144	201544
-6	2	14	27	16	147	201691
-7	2	14	33	24	158	201849
-8	2	14	40	13	164	202013
-9	2	14	47	13	175	202188
-10	2	14	55	9	196	202384
-11	2	15	4	10	226	202610
-11	2	16	23	7	1972	204582
-10	2	16	32	7	225	204807
-9	2	16	40	4	197	205004
-8	2	16	47	6	177	205181
-7	2	16	53	20	164	205345
-6	2	17	0	1	156	205501
-5	2	17	6	0	149	205650
-4	2	17	11	21	146	205796
-3	2	17	17	10	139	205935
-2	2	17	22	24	139	206074
-1	2	17	28	12	138	206212
0	2	17	33	24	137	206349
1	2	17	39	14	140	206489
2	2	17	45	6	142	206631

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
3	2	17	51	0	144	206775
4	2	17	56	24	149	206924
5	2	18	3	2	153	207077
6	2	18	9	16	164	207241
7	2	18	16	16	175	207416
8	2	18	24	7	191	207607
9	2	18	33	4	222	207829
10	2	18	44	6	277	208106
11	2	19	2	16	460	208566
11	2	19	24	24	558	209124
10	2	19	43	8	459	209583
9	2	19	54	12	279	209862
8	2	20	3	7	220	210082
7	2	20	11	1	194	210276
6	2	20	18	1	175	210451
5	2	20	24	14	163	210614
4	2	20	30	18	154	210768
3	2	20	36	18	150	210918
2	2	20	42	10	142	211060
1	2	20	48	2	142	211202
0	2	20	53	16	139	211341
-1	2	20	59	5	139	211480
-2	2	21	4	18	138	211618
-3	2	21	10	7	139	211757
-4	2	21	15	23	141	211898
-5	2	21	21	19	146	212044
-6	2	21	27	18	149	212193
-7	2	21	33	24	156	212349
-8	2	21	40	13	164	212513
-9	2	21	47	14	176	212689
-10	2	21	55	13	199	212888
-11	2	22	4	13	225	213113
-11	2	23	23	2	1964	215077
-10	2	23	32	5	228	215305
-9	2	23	40	2	197	215502
-8	2	23	47	3	176	215678
-7	2	23	53	17	164	215842
-6	2	23	59	23	156	215998
-5	2	24	5	23	150	216148
-4	2	24	11	17	144	216292
-3	2	24	17	10	143	216435
-2	2	24	22	23	138	216573
-1	2	24	28	13	140	216713
0	2	24	34	2	139	216852
1	2	24	39	15	138	216990

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
2	2	24	45	6	141	217131
3	2	24	51	0	144	217275
4	2	24	56	23	148	217423
5	2	25	3	2	154	217577
6	2	25	9	17	165	217742
7	2	25	16	16	174	217916
8	2	25	24	9	193	218109
9	2	25	33	7	223	218332
10	2	25	44	9	277	218609
11	2	26	3	11	477	219086
11	2	26	23	21	510	219596
10	2	26	43	5	484	220080
9	2	26	54	10	280	220360
8	2	27	3	5	220	220580
7	2	27	10	23	193	220773
6	2	27	17	24	176	220949
5	2	27	24	13	164	221113
4	2	27	30	17	154	221267
3	2	27	36	16	149	221416
2	2	27	42	10	144	221560
1	2	27	48	3	143	221703
0	2	27	53	15	137	221840
-1	2	27	59	4	139	221979
-2	2	28	4	19	140	222119
-3	2	28	10	8	139	222258
-4	2	28	16	0	142	222400
-5	2	28	21	18	143	222543
-6	2	28	27	19	151	222694
-7	2	28	34	0	156	222850
-8	2	28	40	16	166	223016
-9	2	28	47	19	178	223194
-10	2	28	55	15	196	223390
-11	2	29	4	19	229	223619
-11	2	30	22	23	1954	225573
-10	2	30	31	24	226	225799
-9	2	30	39	24	200	225999
-8	2	30	47	0	176	226175
-7	2	30	53	17	167	226342
-6	2	30	59	22	155	226497
-5	2	31	5	21	149	226646
-4	2	31	11	17	146	226792
-3	2	31	17	9	142	226934
-2	2	31	22	23	139	227073
-1	2	31	28	12	139	227212
0	2	31	34	2	140	227352

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
1	2	31	39	13	136	227488
2	2	31	45	7	144	227632
3	2	31	51	0	143	227775
4	2	31	57	0	150	227925
5	2	32	3	4	154	228079
6	2	32	9	19	165	228244
7	2	32	16	21	177	228421
8	2	32	24	11	190	228611
9	2	32	33	10	224	228835
10	2	32	44	15	280	229115
11	2	33	4	0	485	229600
11	2	33	23	12	487	230087
10	2	33	43	3	491	230578
9	2	33	54	5	277	230855
8	2	34	3	2	222	231077
7	2	34	10	20	193	231270
6	2	34	17	22	177	231447
5	2	34	24	12	165	231612
4	2	34	30	17	155	231767
3	2	34	36	15	148	231915
2	2	34	42	11	146	232061
1	2	34	48	1	140	232201
0	2	34	53	15	139	232340
-1	2	34	59	5	140	232480
-2	2	35	4	19	139	232619
-3	2	35	10	8	139	232758
-4	2	35	16	1	143	232901
-5	2	35	21	20	144	233045
-6	2	35	27	18	148	233193
-7	2	35	34	2	159	233352
-8	2	35	40	18	166	233518
-9	2	35	47	20	177	233695
-10	2	35	55	18	198	233893
-11	2	36	4	21	228	234121
-11	2	37	22	23	1952	236073
-10	2	37	32	1	228	236301
-9	2	37	39	23	197	236498
-8	2	37	46	24	176	236674
-7	2	37	53	16	167	236841
-6	2	37	59	23	157	236998
-5	2	38	5	22	149	237147
-4	2	38	11	17	145	237292
-3	2	38	17	10	143	237435
-2	2	38	22	24	139	237574
-1	2	38	28	12	138	237712

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
0	2	38	34	0	138	237850
1	2	38	39	16	141	237991
2	2	38	45	7	141	238132
3	2	38	51	0	143	238275
4	2	38	57	0	150	238425
5	2	39	3	6	156	238581
6	2	39	9	19	163	238744
7	2	39	16	20	176	238920
8	2	39	24	13	193	239113
9	2	39	33	10	222	239335
10	2	39	44	15	280	239615
11	2	40	4	12	497	240112
11	2	40	23	11	474	240586
10	2	40	42	19	483	241069
9	2	40	54	4	285	241354
8	2	41	3	3	224	241578
7	2	41	10	18	190	241768
6	2	41	17	19	176	241944
5	2	41	24	8	164	242108
4	2	41	30	13	155	242263
3	2	41	36	13	150	242413
2	2	41	42	6	143	242556
1	2	41	47	24	143	242699
0	2	41	53	13	139	242838
-1	2	41	59	2	139	242977
-2	2	42	4	14	137	243114
-3	2	42	10	4	140	243254
-4	2	42	15	21	142	243396
-5	2	42	21	16	145	243541
-6	2	42	27	15	149	243690
-7	2	42	33	21	156	243846
-8	2	42	40	10	164	244010
-9	2	42	47	14	179	244189
-10	2	42	55	11	197	244386
-11	2	43	4	16	230	244616
-11	2	44	22	15	1949	246565
-10	2	44	31	17	227	246792
-9	2	44	39	15	198	246990
-8	2	44	46	17	177	247167
-7	2	44	53	7	165	247332
-6	2	44	59	13	156	247488
-5	2	45	5	13	150	247638
-4	2	45	11	9	146	247784
-3	2	45	16	24	140	247924
-2	2	45	22	13	139	248063

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
-1	2	45	28	3	140	248203
0	2	45	33	15	137	248340
1	2	45	39	6	141	248481
2	2	45	44	22	141	248622
3	2	45	50	18	146	248768
4	2	45	56	16	148	248916
5	2	46	2	20	154	249070
6	2	46	9	11	166	249236
7	2	46	16	11	175	249411
8	2	46	24	5	194	249605
9	2	46	33	2	222	249827
10	2	46	44	12	285	250112
11	2	47	4	18	506	250618
11	2	47	22	16	448	251066
10	2	47	42	10	494	251560
9	2	47	53	15	280	251840
8	2	48	2	13	223	252063
7	2	48	10	9	196	252259
6	2	48	17	8	174	252433
5	2	48	23	23	165	252598
4	2	48	30	3	155	252753
3	2	48	36	3	150	252903
2	2	48	41	21	143	253046
1	2	48	47	14	143	253189
0	2	48	53	5	141	253330
-1	2	48	58	18	138	253468
-2	2	49	4	7	139	253607
-3	2	49	9	22	140	253747
-4	2	49	15	13	141	253888
-5	2	49	21	10	147	254035
-6	2	49	27	9	149	254184
-7	2	49	33	15	156	254340
-8	2	49	40	5	165	254505
-9	2	49	47	10	180	254685
-10	2	49	55	9	199	254884
-11	2	50	4	15	231	255115
-11	2	51	22	3	1938	257053
-10	2	51	31	8	230	257283
-9	2	51	39	6	198	257481
-8	2	51	46	9	178	257659
-7	2	51	53	1	167	257826
-6	2	51	59	7	156	257982
-5	2	52	5	6	149	258131
-4	2	52	11	3	147	258278
-3	2	52	16	20	142	258420

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
-2	2	52	22	9	139	258559
-1	2	52	27	24	140	258699
0	2	52	33	14	140	258839
1	2	52	39	3	139	258978
2	2	52	44	19	141	259119
3	2	52	50	15	146	259265
4	2	52	56	14	149	259414
5	2	53	2	20	156	259570
6	2	53	9	9	164	259734
7	2	53	16	12	178	259912
8	2	53	24	5	193	260105
9	2	53	33	5	225	260330
10	2	53	44	15	285	260615
11	2	54	5	7	517	261132
11	2	54	21	12	405	261537
10	2	54	42	1	514	262051
9	2	54	53	9	283	262334
8	2	55	2	10	226	262560
7	2	55	10	3	193	262753
6	2	55	17	4	176	262929
5	2	55	23	20	166	263095
4	2	55	30	0	155	263250
3	2	55	36	0	150	263400
2	2	55	41	19	144	263544
1	2	55	47	11	142	263686
0	2	55	53	1	140	263826
-1	2	55	58	15	139	263965
-2	2	56	4	4	139	264104
-3	2	56	9	18	139	264243
-4	2	56	15	11	143	264386
-5	2	56	21	7	146	264532
-6	2	56	27	6	149	264681
-7	2	56	33	13	157	264838
-8	2	56	40	6	168	265006
-9	2	56	47	9	178	265184
-10	2	56	55	7	198	265382
-11	2	57	4	11	229	265611
-11	2	58	21	21	1935	267546
-10	2	58	31	3	232	267778
-9	2	58	39	0	197	267975
-8	2	58	46	2	177	268152
-7	2	58	52	21	169	268321
-6	2	58	59	1	155	268476
-5	2	59	5	2	151	268627
-4	2	59	10	23	146	268773

scale divisions, Δ	time					time, t_i in frames
	hours	minutes	seconds	frames	$t_i - t_{i+1}$	
-3	2	59	16	13	140	268913
-2	2	59	22	5	142	269055
-1	2	59	27	18	138	269193
0	2	59	33	8	140	269333
1	2	59	38	22	139	269472
2	2	59	44	15	143	269615
3	2	59	50	9	144	269759
4	2	59	56	8	149	269908
5	3	0	2	14	156	270064
6	3	0	9	3	164	270228
7	3	0	16	8	180	270408
8	3	0	24	0	192	270600
9	3	0	32	23	223	270823
10	3	0	44	6	283	271106
11	3	1	5	4	523	271629
11	3	1	21	2	398	272027
10	3	1	41	14	512	272539
9	3	1	53	1	287	272826
8	3	2	1	24	223	273049
7	3	2	9	18	194	273243
6	3	2	16	22	179	273422
5	3	2	23	12	165	273587
4	3	2	29	16	154	273741

APPENDIX C

LEAST SQUARES ADJUSTMENT FOR GYRO OBSERVATIONS

The mathematical model, equation (5-2):

$$F(x,l) = \sum_{i=1}^{10} A_i e^{-\lambda_i t} \cos((\theta_i + \theta_i' t)(t - B_i)) + Q - \Delta = 0$$

is linearised as a Taylor series. The solution may be written as:

(5-5) $\quad \boldsymbol{A\hat{x}} + \boldsymbol{Bv} + \boldsymbol{b} = 0$

For $i = 1$, equation (5-2) may be written as:

(5-22) $\quad A_1 e^{-\lambda_1 t} \cos((\theta_1 + \theta_1' t)(t - B_1)) + Q - \Delta = 0$

Least squares adjustment computations using the first set of time observation data are summarised below.

A							
$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	$\boldsymbol{B_{ii}}$	b
-0.45382	-33716.054	-196429731	-0.005794037	29038.2986	1	0.005794431	0.08705845
-0.36198	-33532.64	-201061710	-0.006061442	23837.8886	1	0.00606175	0.07843034
-0.27218	-32924.752	-202684775	-0.006257044	18402.5731	1	0.006257268	0.09217033
-0.18053	-31938.119	-201689224	-0.006395777	12520.746	1	0.006395916	0.08549778
-0.08904	-30637.072	-198252490	-0.006476789	6328.04757	1	0.006476844	0.08070724
0.004389	-29024.358	-192402468	-0.006502497	-319.557179	1	0.006502465	0.05459255
0.093061	-27258.828	-184787596	-0.006474226	-6928.74999	1	0.006474114	0.08070821
0.185071	-25210.403	-174859354	-0.00638993	-14098.3269	1	0.006389733	0.07016774
0.276048	-22988.933	-163083488	-0.006249339	-21507.7651	1	0.006249059	0.07097368
0.368461	-20550.642	-149177112	-0.006044152	-29375.6688	1	0.006043788	0.0560057
0.459458	-17985.504	-133596325	-0.005774202	-37483.2473	1	0.005773755	0.05658856
0.550801	-15259.401	-116078262	-0.005425296	-46017.9196	1	0.005424766	0.05338089
0.64187	-12401.435	-96718791	-0.004983437	-54980.0218	1	0.004982824	0.05317395
0.733948	-9379.2642	-75165424	-0.004412533	-64600.1944	1	0.004411837	0.0418808
0.826303	-6222.7038	-51430647	-0.003656344	-75006.9031	1	0.003655564	0.0275531
0.918072	-2969.2558	-25499969	-0.002567122	-86594.0517	1	0.002566259	0.0196532
0.921295	-2848.8352	-28328817	0.00251584	-100618.735	1	-0.00251671	-0.0157388
0.829932	-6090.2351	-62546715	0.003619399	-93612.1013	1	-0.00362018	-0.0123039
0.739143	-9197.6827	-96750424	0.004373622	-85393.0027	1	-0.00437432	-0.0151788
0.649559	-12145.745	-130323846	0.004938964	-76548.6614	1	-0.00493958	-0.0312799
0.559688	-14977.354	-163567680	0.005384465	-67131.6807	1	-0.005385	-0.0442284
0.466853	-17760.373	-197211181	0.005746765	-56934.9791	1	-0.00574722	-0.0246312
0.376229	-20327.239	-229148963	0.006021495	-46581.2806	1	-0.00602187	-0.0293119
0.284678	-22757.49	-260277418	0.006230458	-35758.9829	1	-0.00623075	-0.0238066
0.193929	-24990.486	-289764681	0.006376213	-24696.323	1	-0.00637642	-0.0271129

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.102662	-27042.434	-317775637	0.006465358	-13249.6039	1	-0.00646548	-0.0247314
0.012296	-28862.511	-343579337	0.006499195	-1607.63324	1	-0.00649923	-0.0322513
-0.07876	-30458.852	-367272835	0.006479372	10430.0851	1	-0.00647933	-0.0322086
-0.17032	-31792.677	-388315763	0.006404352	22847.8205	1	-0.00640422	-0.026576
-0.261	-32806.615	-405850633	0.006273672	35461.9831	1	-0.00627346	-0.0306602
-0.35219	-33467.526	-419448505	0.006082002	48478.6226	1	-0.0060817	-0.0291234
-0.44332	-33700.297	-428027478	0.005824147	61840.7036	1	-0.00582376	-0.0282382
-0.53457	-33408.294	-430232006	0.005490189	75608.3446	1	-0.00548972	-0.0260768
-0.62595	-32443.642	-423973517	0.005064716	89839.9244	1	-0.00506416	-0.0223986
-0.7181	-30546.477	-405626670	0.004517378	104730.21	1	-0.00451673	-0.0102966
-0.80783	-27367.954	-369877899	0.003822143	119910.637	1	-0.00382141	-0.0247903
-0.90023	-21608.304	-298626759	0.002815333	136640.414	1	-0.00281451	-0.0100515
-0.99203	-6501.1029	-93485860	0.000754143	156675.41	1	-0.00075322	-0.0018323
-0.99105	7491.34815	110767074	-0.000805014	160939.837	1	0.00080595	-0.012592
-0.90182	28524.706	437169645	-0.002791779	151798.02	1	0.002792651	0.00740635
-0.81001	40789.4238	637579484	-0.003800643	139057.486	1	0.00380144	-0.0009386
-0.71946	50179.729	796352299	-0.004506212	125400.771	1	0.004506933	0.00454839
-0.6284	58013.6029	932800722	-0.005049721	110972.985	1	0.005050364	0.00453274
-0.53675	64711.6519	1.053E+09	-0.005478983	95912.5185	1	0.005479546	-0.0021481
-0.44724	70317.5446	1.156E+09	-0.005809157	80773.3126	1	0.005809641	0.01482187
-0.35465	75271.2212	1.251E+09	-0.006073563	64717.1711	1	0.006073966	-0.0020964
-0.26528	79307.305	1.33E+09	-0.006263584	48871.3575	1	0.006263906	0.01631688
-0.17355	82727.3059	1.401E+09	-0.006398068	32276.4564	1	0.006398307	0.00892595
-0.08265	85410.5217	1.459E+09	-0.006474571	15511.8993	1	0.006474728	0.01056413
0.008943	87399.8386	1.507E+09	-0.006496521	-1693.58509	1	0.006496593	0.00457978
0.099283	88638.6671	1.542E+09	-0.006464504	-18968.9665	1	0.006464493	0.01237733
0.190548	89121.1969	1.564E+09	-0.006377221	-36732.4892	1	0.006377125	0.01001765
0.281887	88768.113	1.572E+09	-0.006232262	-54832.4491	1	0.00623208	0.00684405
0.374018	87471.4366	1.564E+09	-0.006023439	-73431.3595	1	0.00602317	-0.0050198
0.46467	85140.4192	1.536E+09	-0.005749966	-92091.7511	1	0.005749609	-0.0006513
0.55559	81553.6438	1.486E+09	-0.005397589	-111203.212	1	0.005397144	0.0007785
0.647047	76371.2163	1.407E+09	-0.004947365	-130887.353	1	0.00494683	-0.0036926
0.735588	69345.5022	1.292E+09	-0.004392333	-150478.14	1	0.004391708	0.02386905
0.829446	58650.8591	1.108E+09	-0.003615587	-172019.698	1	0.003614864	-0.0069668
0.920655	42139.9281	809465879	-0.002512523	-194231.977	1	0.002511699	-0.0087106
0.921644	-47476.497	-975404635	0.0024952	-207964.181	1	-0.00249611	-0.0195759
0.829238	-70815.188	-1.478E+09	0.003615629	-190127.894	1	-0.00361647	-0.0046883
0.736947	-87669.393	-1.852E+09	0.004380639	-171015.076	1	-0.00438141	0.00894283
0.647244	-100708.29	-2.149E+09	0.004944172	-151698.574	1	-0.00494487	-0.0058507
0.555331	-111698.84	-2.405E+09	0.005396536	-131345.795	1	-0.00539716	0.00362063
0.464415	-120722.33	-2.621E+09	0.005748574	-110755.506	1	-0.00574912	0.00214885
0.373226	-128217.18	-2.806E+09	0.006023161	-89705.1209	1	-0.00602363	0.00367945
0.281094	-134383.75	-2.963E+09	0.006231314	-68070.7353	1	-0.0062317	0.0155503
0.190347	-139180.35	-3.091E+09	0.006374893	-46425.3983	1	-0.0063752	0.01222352

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.098524	-142800.79	-3.194E+09	0.006462278	-24199.8166	1	-0.0064625	0.02071046
0.008217	-145172.03	-3.269E+09	0.006493692	-2032.13736	1	-0.00649383	0.01254946
-0.08274	-146357.57	-3.318E+09	0.006471473	20602.8875	1	-0.00647153	0.01157567
-0.17418	-146291.19	-3.339E+09	0.006394089	43668.2398	1	-0.00639406	0.01580676
-0.26641	-144868.71	-3.33E+09	0.006258042	67258.2555	1	-0.00625793	0.02872425
-0.35624	-142057.04	-3.288E+09	0.006065814	90563.7403	1	-0.00606561	0.01534613
-0.44763	-137573.75	-3.208E+09	0.005803684	114628.849	1	-0.00580339	0.01911554
-0.53902	-131182.82	-3.082E+09	0.005464809	139086.43	1	-0.00546443	0.0228815
-0.62949	-122560.22	-2.903E+09	0.005038645	163735.171	1	-0.00503817	0.01641462
-0.7207	-110840.99	-2.648E+09	0.004491395	189123.903	1	-0.00449082	0.01825085
-0.81054	-94998.392	-2.292E+09	0.003787775	214818.03	1	-0.0037871	0.00495878
-0.90194	-71111.065	-1.738E+09	0.002778308	242062.973	1	-0.00277753	0.00879674
-0.99396	-15264.368	-382112915	0.000573788	273276.221	1	-0.00057287	0.01943737
-0.99207	18938.0788	480345431	-0.000697239	276361.846	1	0.000698186	-0.001371
-0.8987	79356.0142	2.059E+09	-0.002820495	256048.384	1	0.002821411	-0.0267885
-0.80719	109309.533	2.869E+09	-0.003815973	232653.511	1	0.00381683	-0.0318465
-0.71675	131178.454	3.474E+09	-0.004516076	208458.376	1	0.00451687	-0.0252036
-0.62599	148605.846	3.966E+09	-0.005054948	183491.987	1	0.005055673	-0.02201
-0.53375	163010.704	4.382E+09	-0.005484479	157581.307	1	0.005485131	-0.0350198
-0.44259	174654.773	4.726E+09	-0.005817762	131526.418	1	0.005818339	-0.0362946
-0.35099	184151.926	5.014E+09	-0.006076374	104957.928	1	0.006076874	-0.0422807
-0.26102	191565.324	5.246E+09	-0.006265158	78512.6239	1	0.006265581	-0.0304171
-0.16871	197345.46	5.436E+09	-0.006397689	51042.8348	1	0.00639803	-0.0442653
-0.07842	201289.173	5.576E+09	-0.006471007	23859.2151	1	0.006471267	-0.0358891
0.013697	203601.373	5.672E+09	-0.006490396	-4190.56595	1	0.006490573	-0.0476329
0.103934	204177.118	5.719E+09	-0.006455603	-31973.3993	1	0.006455695	-0.038701
0.196793	202965.387	5.717E+09	-0.006363293	-60883.4492	1	0.006363298	-0.058565
0.286772	199939.779	5.663E+09	-0.006216784	-89215.7367	1	0.006216701	-0.0468074
0.377555	194872.78	5.552E+09	-0.006007838	-118134.383	1	0.006007665	-0.0438681
0.468995	187466.062	5.373E+09	-0.005728516	-147626.119	1	0.00572825	-0.0481477
0.559543	177463.92	5.118E+09	-0.005373264	-177228.102	1	0.005372904	-0.0426331
0.650546	164112.61	4.765E+09	-0.004920228	-207438.501	1	0.004919771	-0.0421229
0.740927	146511.251	4.285E+09	-0.004345465	-237999.554	1	0.004344907	-0.034773
0.831354	122432.754	3.611E+09	-0.003586419	-269328.836	1	0.003585754	-0.0279211
0.922118	85839.0521	2.56E+09	-0.002474352	-302055.234	1	0.002473567	-0.0247869
0.917097	-94249.028	-2.937E+09	0.002550252	-313927.558	1	-0.0025512	0.03036425
0.82645	-136199.5	-4.289E+09	0.003632124	-285812.249	1	-0.00363302	0.02593589
0.734209	-166541.06	-5.286E+09	0.004391449	-255944.685	1	-0.00439229	0.03900879
0.643416	-189796.73	-6.065E+09	0.004957667	-225799.46	1	-0.00495845	0.03618629
0.551781	-208690.82	-6.709E+09	0.005404967	-194816.73	1	-0.00540568	0.04261464
0.460746	-223926.4	-7.239E+09	0.005754528	-163580.901	1	-0.00575517	0.04244764
0.37003	-236180.71	-7.675E+09	0.006025274	-132060.432	1	-0.00602584	0.03877534
0.277881	-245995.93	-8.034E+09	0.006231398	-99676.8379	1	-0.00623189	0.05084386
0.187145	-253286.71	-8.312E+09	0.006373028	-67454.3179	1	-0.00637344	0.04739237

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.096537	-258343.47	-8.518E+09	0.006457752	-34960.0783	1	-0.00645808	0.04253468
0.005118	-261253.32	-8.655E+09	0.006488058	-1862.22821	1	-0.00648831	0.04658537
-0.08575	-261965.82	-8.719E+09	0.006464024	31345.6914	1	-0.00646419	0.04461963
-0.17707	-260438.83	-8.709E+09	0.00638486	65029.1097	1	-0.00638494	0.04756521
-0.26745	-256601.17	-8.621E+09	0.006250171	98679.8483	1	-0.00625016	0.04013584
-0.35826	-250226.38	-8.447E+09	0.006054571	132826.293	1	-0.00605447	0.03757625
-0.44895	-241054.57	-8.178E+09	0.005792899	167276.109	1	-0.0057927	0.03356071
-0.53966	-228634.01	-7.797E+09	0.005455247	202125.718	1	-0.00545495	0.02989125
-0.63042	-212265.38	-7.279E+09	0.005026228	237431.638	1	-0.00502583	0.02661855
-0.72139	-190712.79	-6.58E+09	0.004478296	273359.007	1	-0.0044778	0.02578867
-0.81094	-162241.9	-5.636E+09	0.003774144	309412.789	1	-0.00377353	0.00932667
-0.90271	-119817.7	-4.199E+09	0.002753947	347481.438	1	-0.00275321	0.01724263
-0.9922	-28167.534	-1.003E+09	0.000633289	388195.14	1	-0.00063239	0.00011594
-0.99269	26942.571	968423772	-0.000598458	391883.909	1	0.000599413	0.00542226
-0.89883	129132.819	4.719E+09	-0.002805341	360743.71	1	0.002806303	-0.0254419
-0.80703	177215.846	6.53E+09	-0.003807476	326596.969	1	0.003808397	-0.0336042
-0.71718	211473.756	7.842E+09	-0.004504788	292101.754	1	0.004505657	-0.0204419
-0.62433	239175.873	8.921E+09	-0.005056172	255744.967	1	0.005056979	-0.0402166
-0.53458	260678.729	9.771E+09	-0.005474215	220077.565	1	0.005474958	-0.0259614
-0.44304	278406.569	1.049E+10	-0.005809841	183259.567	1	0.005810514	-0.0313004
-0.35046	292697.933	1.107E+10	-0.006071505	145621.758	1	0.006072106	-0.0481344
-0.25999	303534.39	1.153E+10	-0.006260995	108490.223	1	0.006261521	-0.0417448
-0.16891	311571.615	1.189E+10	-0.006391619	70775.4991	1	0.006392066	-0.0421088
-0.07929	316807.425	1.214E+10	-0.006464778	33357.6018	1	0.006465146	-0.0263595
0.012746	319491.986	1.229E+10	-0.006484715	-5384.12037	1	0.006485	-0.0371906
0.103497	319463.385	1.234E+10	-0.006450105	-43893.633	1	0.006450304	-0.0339005
0.196279	316612.025	1.228E+10	-0.006358082	-83586.2025	1	0.006358192	-0.0529296
0.285623	311023.364	1.211E+10	-0.006213	-122122.683	1	0.006213021	-0.0341828
0.377449	302125.97	1.181E+10	-0.006001875	-162068.612	1	0.006001802	-0.0427089
0.466191	290129.279	1.139E+10	-0.005731652	-201022.437	1	0.005731485	-0.0173573
0.558275	273566.552	1.079E+10	-0.005371846	-241845.319	1	0.005371577	-0.0287158
0.649264	252153.7	9.995E+09	-0.004919742	-282644.95	1	0.004919368	-0.0280376
0.738855	224597.235	8.95E+09	-0.004351709	-323367.163	1	0.004351225	-0.012014
0.829447	187035.87	7.5E+09	-0.003594545	-365292.961	1	0.003593942	-0.0069788
0.921616	129747.962	5.246E+09	-0.002466911	-409255.512	1	0.002466168	-0.019271
0.920048	-136508.49	-5.7E+09	0.002489664	-421938.06	1	-0.00249065	-0.0020502
0.827176	-200224.03	-8.427E+09	0.003614664	-382371.871	1	-0.00361563	0.01795752
0.734129	-244693.7	-1.036E+10	0.00438342	-341415.473	1	-0.00438433	0.03989764
0.645291	-277490.87	-1.181E+10	0.004939729	-301581.57	1	-0.00494059	0.0155997
0.552975	-304728.26	-1.303E+10	0.005392834	-259627.709	1	-0.00539363	0.02949363
0.461606	-326377.99	-1.401E+10	0.005745083	-217641.559	1	-0.00574582	0.03299581
0.371578	-343418.61	-1.48E+10	0.006015018	-175879.711	1	-0.00601568	0.02177868
0.28014	-356906.94	-1.544E+10	0.006221105	-133103.889	1	-0.0062217	0.02603444
0.188961	-366864.12	-1.593E+10	0.00636489	-90111.624	1	-0.00636541	0.02745017

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.096696	-373597.69	-1.628E+10	0.006451862	-46280.2401	1	-0.0064523	0.04078921
0.005948	-377043.27	-1.649E+10	0.006482268	-2856.95255	1	-0.00648262	0.03746984
-0.08485	-377354.11	-1.656E+10	0.006458748	40896.0339	1	-0.00645902	0.03466819
-0.1761	-374441.76	-1.649E+10	0.006380147	85179.5374	1	-0.00638033	0.03685092
-0.26584	-368277.56	-1.628E+10	0.006247087	129044.897	1	-0.00624718	0.0224984
-0.35662	-358460.78	-1.59E+10	0.006052492	173746.529	1	-0.00605249	0.01953719
-0.44834	-344482.27	-1.534E+10	0.005788425	219268.377	1	-0.00578833	0.02685814
-0.53899	-326041.6	-1.458E+10	0.005451197	264649.875	1	-0.00545099	0.02245034
-0.62967	-302022.09	-1.356E+10	0.005022676	310492.968	1	-0.00502237	0.01847142
-0.72059	-270696.34	-1.221E+10	0.004475344	356984.739	1	-0.00447492	0.01697453
-0.81111	-229112.28	-1.039E+10	0.003762543	403978.735	1	-0.003762	0.01121198
-0.90131	-169455.83	-7.736E+09	0.002759671	451899.355	1	-0.00275898	0.0018008
-0.99291	-32178.447	-1.489E+09	0.000515428	504456.265	1	-0.00051453	0.0078389
-0.99213	36095.0779	1.68E+09	-0.000573758	507154.47	1	0.000574721	-0.0007301
-0.89936	177991.891	8.391E+09	-0.002784489	465661.765	1	0.002785497	-0.0195576
-0.80744	244473.161	1.16E+10	-0.003793862	420781.696	1	0.003794845	-0.029118
-0.71747	291425.909	1.39E+10	-0.004494481	375768.94	1	0.004495424	-0.0172984
-0.62573	328683.278	1.574E+10	-0.005041388	329172.425	1	0.00504228	-0.0248389
-0.53473	358168.109	1.722E+10	-0.00546674	282414.603	1	0.005467574	-0.0243309
-0.44384	381788.964	1.843E+10	-0.005800816	235274.248	1	0.005801587	-0.0225667
-0.35249	400599.297	1.94E+10	-0.006060351	187507.549	1	0.006061053	-0.0257982
-0.26161	414988.396	2.017E+10	-0.00625217	139626.744	1	0.0062528	-0.0239868
-0.17006	425483.726	2.074E+10	-0.006384477	91062.8037	1	0.006385032	-0.0294381
-0.08054	432074.576	2.113E+10	-0.006458349	43260.8329	1	0.006458826	-0.0126666
0.010825	435158.355	2.135E+10	-0.006479107	-5833.13269	1	0.006479501	-0.0160922
0.102093	434571.484	2.139E+10	-0.006445263	-55187.5652	1	0.006445571	-0.0184898
0.193657	430211.127	2.124E+10	-0.006355614	-105016.852	1	0.006355833	-0.0241251
0.284674	421941.284	2.09E+10	-0.006208826	-154871.182	1	0.006208951	-0.0237647
0.376444	409305.631	2.034E+10	-0.005998313	-205478.898	1	0.00599834	-0.031669
0.466181	392288.88	1.956E+10	-0.005725144	-255321.419	1	0.005725071	-0.0172492
0.557671	369423.825	1.849E+10	-0.005367515	-306537.67	1	0.005367336	-0.022075
0.647693	340277.508	1.71E+10	-0.004920866	-357386.217	1	0.004920575	-0.0107807
0.739693	301498.445	1.521E+10	-0.004336992	-409921.482	1	0.004336579	-0.0212152
0.830268	250107.145	1.268E+10	-0.003575947	-462405.254	1	0.003575401	-0.016002
0.921639	172808.744	8.82E+09	-0.002451123	-516653.475	1	0.002450419	-0.0195243
0.919245	-180879.3	-9.471E+09	0.002487003	-528637.994	1	-0.00248803	0.00677294
0.82778	-263704.19	-1.389E+10	0.003598332	-479029.357	1	-0.00359935	0.01133223
0.735896	-321549.43	-1.702E+10	0.004362232	-427901.73	1	-0.00436322	0.02048865
0.646997	-364603.51	-1.938E+10	0.004922645	-377701.737	1	-0.00492359	-0.003136
0.554483	-400215.85	-2.135E+10	0.005379312	-324893.864	1	-0.0053802	0.01294162
0.464368	-427962.76	-2.291E+10	0.005729124	-273000.029	1	-0.00572996	0.00266343
0.371834	-450558.53	-2.42E+10	0.006008127	-219305.993	1	-0.00600889	0.01896406
0.27993	-467840.57	-2.52E+10	0.006215499	-165608.887	1	-0.0062162	0.02833802
0.189411	-480296.82	-2.595E+10	0.006358459	-112385.445	1	-0.00635908	0.02251112

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.098995	-488489.87	-2.647E+10	0.006444603	-58906.7346	1	-0.00644515	0.0155371
0.007748	-492596.86	-2.676E+10	0.006476433	-4623.72922	1	-0.0064769	0.01770022
-0.08356	-492554.79	-2.684E+10	0.006453682	50007.6066	1	-0.00645406	0.02055448
-0.17358	-488376.21	-2.669E+10	0.006377214	104175.384	1	-0.00637751	0.00927112
-0.26387	-479879.09	-2.63E+10	0.006244651	158812.729	1	-0.00624485	0.00084098
-0.35683	-466291.93	-2.563E+10	0.006045799	215411.62	1	-0.0060459	0.02179604
-0.44844	-447575.47	-2.468E+10	0.005781618	271555.095	1	-0.00578161	0.0279852
-0.53848	-423244.27	-2.341E+10	0.005446417	327122.9	1	-0.00544631	0.01693827
-0.62909	-391582.12	-2.173E+10	0.005018272	383480.199	1	-0.00501805	0.01210263
-0.71952	-350704.91	-1.954E+10	0.004474189	440254.804	1	-0.00447384	0.00527383
-0.81003	-296438.53	-1.659E+10	0.00376235	497780.718	1	-0.00376187	-0.0006345
-0.89948	-219739.63	-1.236E+10	0.002771092	555713.6	1	-0.00277045	-0.0181999
-0.99183	42616.3884	2.435E+09	-0.000527501	622394.714	1	0.00052847	-0.0039868
-0.99101	47572.792	2.719E+09	-0.000588654	622057.789	1	0.000589627	-0.0129317
-0.8998	226203.725	1.306E+10	-0.002764865	570641.639	1	0.002765919	-0.0147631
-0.80944	310016.978	1.8E+10	-0.003765767	516032.509	1	0.003766814	-0.00714
-0.71883	370377.838	2.159E+10	-0.004476849	460169.189	1	0.004477867	-0.002316
-0.62696	417728.548	2.444E+10	-0.005027445	402817.778	1	0.005028421	-0.0113098
-0.53569	455085.236	2.671E+10	-0.005455873	345301.623	1	0.005456798	-0.0137184
-0.44497	484742.297	2.854E+10	-0.005790683	287688.377	1	0.00579155	-0.0101107
-0.35377	508261.154	3.001E+10	-0.006051018	229383.719	1	0.006051822	-0.0117127
-0.26302	526151.901	3.115E+10	-0.006243689	171006.887	1	0.006244424	-0.0084741
-0.17159	539089.24	3.2E+10	-0.006376905	111860.423	1	0.006377567	-0.0126652
-0.0804	547191.399	3.257E+10	-0.006452649	52551.2196	1	0.006453233	-0.0141807
0.011473	550647.499	3.286E+10	-0.006473306	-7518.69096	1	0.006473806	-0.0232082
0.101486	549467.645	3.287E+10	-0.006439886	-66678.3764	1	0.006440301	-0.0118145
0.192976	543534.326	3.26E+10	-0.006350623	-127122.526	1	0.006350947	-0.0166521
0.282796	532839.641	3.204E+10	-0.006206452	-186778.556	1	0.006206681	-0.0031373
0.373995	516653.256	3.115E+10	-0.005998575	-247686.203	1	0.005998704	-0.0047671
0.465834	494303.458	2.989E+10	-0.005719805	-309388.795	1	0.005719827	-0.0134337
0.556262	465411.74	2.823E+10	-0.005366724	-370541.315	1	0.005366635	-0.0066031
0.647176	427943.423	2.604E+10	-0.004916111	-432480.605	1	0.004915902	-0.0051086
0.737903	379475.113	2.317E+10	-0.004341222	-494852.152	1	0.004340883	-0.0015604
0.829248	314061.943	1.926E+10	-0.003575424	-558413.599	1	0.003574939	-0.0047904
0.920623	216742.234	1.336E+10	-0.002451852	-623302.631	1	0.00245119	-0.0083643
0.91949	-223640.66	-1.408E+10	0.002467881	-635825.694	1	-0.00246895	0.00407556
0.829239	-325808.81	-2.062E+10	0.003573545	-576395.088	1	-0.00357463	-0.0046956
0.737906	-397466.33	-2.526E+10	0.004339179	-514960.664	1	-0.00434024	-0.0015855
0.646746	-452132.7	-2.883E+10	0.00491641	-452877.942	1	-0.00491744	-0.0003893
0.5563	-495078.08	-3.166E+10	0.00536439	-390722.602	1	-0.00536537	-0.0070155
0.464853	-529730.79	-3.397E+10	0.00572093	-327418.051	1	-0.00572186	-0.0026583
0.374082	-556921.07	-3.581E+10	0.005995998	-264182.087	1	-0.00599687	-0.0057237
0.282913	-577937.06	-3.726E+10	0.006203775	-200306.671	1	-0.00620458	-0.0044174
0.191969	-593202.36	-3.834E+10	0.006349348	-136252.068	1	-0.00635008	-0.0055833

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.100497	-603175.78	-3.907E+10	0.00643784	-71502.3061	1	-0.00643849	-0.0009558
0.010516	-607908.57	-3.947E+10	0.00647052	-7499.47784	1	-0.00647109	-0.0126954
-0.08131	-607597.06	-3.955E+10	0.006449131	58127.9711	1	-0.00644962	-0.0041616
-0.17244	-602072.72	-3.928E+10	0.006372686	123570.928	1	-0.00637308	-0.0032781
-0.26267	-591268.65	-3.867E+10	0.006240826	188678.087	1	-0.00624113	-0.0123478
-0.35447	-574435.28	-3.766E+10	0.006045488	255263.23	1	-0.00604569	-0.0040353
-0.44557	-551291.49	-3.624E+10	0.005784553	321690.007	1	-0.00578465	-0.0035358
-0.53568	-521146.84	-3.435E+10	0.005451263	387781.457	1	-0.00545124	-0.0138805
-0.62685	-481794.85	-3.185E+10	0.0050227	455095.229	1	-0.00502256	-0.0125635
-0.7186	-430575.17	-2.855E+10	0.004472054	523382.063	1	-0.00447178	-0.0048384
-0.80839	-364239.23	-2.424E+10	0.003767291	590901.691	1	-0.00376687	-0.0186714
-0.89952	-267983.69	-1.792E+10	0.002757002	660532.244	1	-0.00275641	-0.0178679
-0.98991	-59369.331	-4.004E+09	0.000604647	733271.262	1	-0.00060377	-0.0250648
-0.99077	53290.3939	3.61E+09	-0.000539961	737159.989	1	0.000540939	-0.0156476
-0.90163	271729.298	1.857E+10	-0.002725636	676713.085	1	0.002726733	0.00538544
-0.81028	375618.914	2.578E+10	-0.003748114	610897.662	1	0.003749223	0.00204479
-0.71964	449080.329	3.094E+10	-0.00446296	544472.383	1	0.004464052	0.00653183
-0.6294	505597.866	3.494E+10	-0.005007079	477641.462	1	0.005008139	0.01542076
-0.53846	550807.698	3.816E+10	-0.005437446	409762.687	1	0.005438463	0.01670402
-0.44749	586836.659	4.077E+10	-0.005775986	341412.007	1	0.005776951	0.01760583
-0.35596	615334.426	4.285E+10	-0.006039423	272241.137	1	0.006040328	0.01228693
-0.26423	637044.033	4.447E+10	-0.00623556	202559.873	1	0.006236399	0.0048024
-0.17523	652125.259	4.562E+10	-0.006366876	134628.627	1	0.006367646	0.02730176
-0.08358	661852.746	4.64E+10	-0.006445174	64356.9642	1	0.006445867	0.02071502
0.007027	665860.473	4.679E+10	-0.006467831	-5422.83949	1	0.006468443	0.02562262
0.099328	664240.837	4.678E+10	-0.006435535	-76825.1823	1	0.006436059	0.0118881
0.190767	656845.646	4.636E+10	-0.006347591	-147878.157	1	0.006348022	0.00761348
0.280554	643709.703	4.553E+10	-0.00620476	-217963.039	1	0.006205095	0.02148264
0.370647	624227.955	4.426E+10	-0.006001158	-288615.717	1	0.006001391	0.03199753
0.462024	597296.588	4.245E+10	-0.00572636	-360637.082	1	0.005726481	0.02840588
0.554538	561503.056	4.001E+10	-0.00536731	-433964.271	1	0.005367311	0.01232832
0.645028	516356.757	3.689E+10	-0.004920444	-506145.578	1	0.004920319	0.01848897
0.737044	456909.225	3.274E+10	-0.004338747	-580114.619	1	0.004338481	0.00787468
0.828342	377876.604	2.718E+10	-0.003573774	-654275.551	1	0.003573348	0.00515049
0.918342	262801.978	1.899E+10	-0.002472588	-728650.754	1	0.002471971	0.01668975
0.921865	-261960.6	-1.927E+10	0.002414522	-744688.979	1	-0.00241563	-0.021999
0.830244	-387500.91	-2.863E+10	0.003553071	-673741.347	1	-0.00355421	-0.0157386
0.73858	-473620.24	-3.512E+10	0.004325647	-601424.665	1	-0.00432678	-0.008996
0.649909	-537271.11	-3.995E+10	0.004891072	-530726.147	1	-0.00489218	-0.0351266
0.557937	-589484.49	-4.395E+10	0.005350205	-456827.288	1	-0.00535128	-0.0249988
0.466192	-630856.55	-4.715E+10	0.005709783	-382640.618	1	-0.00571081	-0.0173725
0.37722	-662557.46	-4.962E+10	0.005981445	-310305.776	1	-0.00598241	-0.0401911
0.285124	-687779.97	-5.163E+10	0.006193484	-235065.984	1	-0.00619439	-0.0287007
0.194317	-705803.26	-5.309E+10	0.006340463	-160541.098	1	-0.0063413	-0.0313754

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.102375	-717575.47	-5.409E+10	0.006430812	-84757.9462	1	-0.00643157	-0.0215806
0.013071	-723001.83	-5.461E+10	0.00646458	-10843.3832	1	-0.00646526	-0.0407578
-0.08044	-722410.04	-5.468E+10	0.006443828	66876.5508	1	-0.00644442	-0.0136779
-0.17035	-715703.53	-5.428E+10	0.00636924	141902.132	1	-0.00636975	-0.0263018
-0.26053	-702682.91	-5.341E+10	0.006238655	217478.272	1	-0.00623906	-0.0357959
-0.35177	-682649.26	-5.2E+10	0.006046023	294266.934	1	-0.00604633	-0.0337576
-0.44289	-655034.88	-5E+10	0.005786832	371313.46	1	-0.00578703	-0.033003
-0.53404	-618682.58	-4.734E+10	0.005451215	448780.364	1	-0.00545129	-0.0318649
-0.62522	-571796.59	-4.386E+10	0.005023833	526715.929	1	-0.00502378	-0.0304271
-0.7166	-511179.59	-3.932E+10	0.004477241	605361.108	1	-0.00447704	-0.026783
-0.80756	-431385.94	-3.328E+10	0.003764889	684342.15	1	-0.00376453	-0.0278406
-0.89963	-315582.14	-2.445E+10	0.002741701	765429.183	1	-0.00274115	-0.0166162
-0.99261	-29036.072	-2.269E+09	0.00024979	851890.169	1	-0.00024889	0.00462041
-0.99322	11930.1727	933345133	-0.000102495	853416.701	1	0.000103437	0.0113127
-0.90334	315989.773	2.494E+10	-0.002687591	783102.525	1	0.002688731	0.02415417
-0.81234	439213.8	3.48E+10	-0.003719047	706991.835	1	0.003720216	0.02473174
-0.72191	526051.455	4.181E+10	-0.004438931	630215.76	1	0.004440096	0.03153673
-0.6312	593130.126	4.727E+10	-0.004989961	552488.399	1	0.004991103	0.03524678
-0.54006	646463.884	5.164E+10	-0.005423868	473849.66	1	0.005424974	0.03422323
-0.45035	688252.867	5.51E+10	-0.005760125	396008.504	1	0.005761186	0.04894786
-0.35849	721868.547	5.792E+10	-0.006026904	315910.731	1	0.006027911	0.04011675
-0.2675	747149.28	6.007E+10	-0.00622368	236206.076	1	0.006224624	0.04075809
-0.17634	765189.946	6.164E+10	-0.006359741	156014.109	1	0.006360617	0.03949063
-0.08478	776401.52	6.267E+10	-0.006438745	75157.1729	1	0.006439545	0.0339526
0.006327	780910.643	6.315E+10	-0.00646212	-5619.11635	1	0.00646284	0.0333143
0.096794	778869.975	6.311E+10	-0.006431401	-86133.8473	1	0.006432034	0.03971952
0.188764	770010.272	6.251E+10	-0.006344275	-168303.452	1	0.006344813	0.02961076
0.279079	754345.698	6.136E+10	-0.006201586	-249313.182	1	0.006202025	0.03768603
0.369679	731169.06	5.959E+10	-0.005997484	-330912.227	1	0.005997816	0.0426254
0.45892	700225.946	5.719E+10	-0.005730422	-411636.194	1	0.005730642	0.06249931
0.552988	657518.016	5.382E+10	-0.005367127	-497141.929	1	0.005367219	0.0293524
0.643486	604514.54	4.96E+10	-0.00492139	-579864.071	1	0.004921348	0.03542243
0.734744	535497.429	4.405E+10	-0.004346618	-663842.835	1	0.004346427	0.03313454
0.826902	442267.64	3.65E+10	-0.003577378	-749423.388	1	0.003577014	0.02097021
0.919497	302908.407	2.51E+10	-0.00243892	-836745.762	1	0.00243834	0.00400344
0.923875	-298973.64	-2.516E+10	0.002365895	-853778.98	1	-0.00236704	-0.0440839
0.831447	-448188.49	-3.787E+10	0.003530518	-771495.721	1	-0.00353171	-0.0289494
0.740289	-548304.08	-4.646E+10	0.004304566	-688983.88	1	-0.00430577	-0.0277642
0.650675	-623249.24	-5.295E+10	0.004879104	-607109.705	1	-0.00488029	-0.0435361
0.559882	-683190.27	-5.817E+10	0.005334608	-523594.722	1	-0.00533577	-0.0463603
0.468901	-730936.79	-6.237E+10	0.00569385	-439442.589	1	-0.00569497	-0.0471205
0.377437	-768717.46	-6.573E+10	0.005974644	-354437.485	1	-0.00597571	-0.0425722
0.28713	-797297.66	-6.83E+10	0.006183555	-270147.739	1	-0.00618456	-0.0507384
0.19588	-818221.23	-7.022E+10	0.006332569	-184638.743	1	-0.00633351	-0.0485414

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.104628	-831672.97	-7.151E+10	0.006423506	-98803.8621	1	-0.00642438	-0.0463241
0.014826	-837894.96	-7.217E+10	0.006458681	-14025.5775	1	-0.00645947	-0.060034
-0.07803	-837101.59	-7.224E+10	0.006439331	73950.8258	1	-0.00644003	-0.0402279
-0.16962	-829025.91	-7.167E+10	0.006364238	161046.194	1	-0.00636485	-0.034319
-0.25973	-813726.68	-7.047E+10	0.006234109	247059.869	1	-0.00623462	-0.0445468
-0.35091	-790311.53	-6.858E+10	0.006041987	334412.431	1	-0.00604239	-0.0431917
-0.44355	-757489.73	-6.586E+10	0.005778239	423537.056	1	-0.00577853	-0.0257118
-0.53407	-715333.25	-6.232E+10	0.005444258	511007.996	1	-0.00544442	-0.0315695
-0.62467	-661105.13	-5.772E+10	0.00501929	599004.215	1	-0.00501932	-0.036463
-0.71719	-589692.27	-5.161E+10	0.004464861	689409.393	1	-0.00446474	-0.0203075
-0.80787	-497081.04	-4.363E+10	0.003752052	778720.96	1	-0.00375176	-0.0244053
-0.90032	-361555.81	-3.185E+10	0.002718246	870931.39	1	-0.00271774	-0.0090184
-0.99158	-36508.815	-3.24E+09	0.000272194	966416.254	1	-0.0002713	-0.0067772
-0.99237	10712.3225	951597036	-7.97741E-05	968197.143	1	8.0714E-05	0.00199183
-0.90444	360003.851	3.223E+10	-0.002657717	889314.125	1	0.002658899	0.03627247
-0.81258	504075.22	4.529E+10	-0.003706725	801809.264	1	0.003707955	0.02735871
-0.72282	603452.338	5.436E+10	-0.00442421	715160.32	1	0.004425448	0.04152657
-0.63102	681527.856	6.154E+10	-0.004983425	625816.858	1	0.00498465	0.03329448
-0.54049	742319.938	6.717E+10	-0.005415182	537160.951	1	0.005416378	0.03893454
-0.44882	791163.2	7.173E+10	-0.005758729	446941.229	1	0.005759885	0.03211612
-0.35754	829217.712	7.533E+10	-0.006023143	356712.181	1	0.006024248	0.02961686
-0.2666	857940.114	7.808E+10	-0.006219346	266466.211	1	0.006220392	0.03090208
-0.17608	878238.003	8.006E+10	-0.006354206	176296.854	1	0.006355186	0.03670503
-0.08343	890955.757	8.136E+10	-0.00643373	83682.2539	1	0.006434636	0.01916132
0.007598	895773.445	8.194E+10	-0.006456324	-7633.90273	1	0.006457149	0.01934505
0.09798	893079.623	8.183E+10	-0.006424873	-98603.3272	1	0.006425611	0.02668364
0.189276	882652.564	8.102E+10	-0.00633778	-190806.084	1	0.006338422	0.02398381
0.280625	864111.27	7.945E+10	-0.00619264	-283386.247	1	0.006193179	0.02070414
0.371087	837147.208	7.711E+10	-0.005987598	-375402.68	1	0.005988028	0.02716332
0.463294	799859.582	7.381E+10	-0.005708981	-469562.199	1	0.00570929	0.0144584
0.553077	752546.172	6.958E+10	-0.005359791	-561640.742	1	0.005359972	0.02837991
0.645681	689849.256	6.392E+10	-0.004901586	-657082.63	1	0.004901621	0.01131551
0.737321	609507.19	5.661E+10	-0.00431944	-752106.559	1	0.004319315	0.0048339
0.827851	503508.577	4.689E+10	-0.00355753	-846743.592	1	0.003557222	0.01054275
0.919603	344128.525	3.216E+10	-0.002421731	-943982.464	1	0.002421189	0.0028406
0.920151	-347871.37	-3.296E+10	0.00241099	-957632.675	1	-0.00241218	-0.0031827
0.829313	-512929.33	-4.878E+10	0.003541123	-866136.594	1	-0.00354238	-0.0055123
0.73911	-625328.24	-5.962E+10	0.004304456	-773973.264	1	-0.00430573	-0.0148088
0.648616	-711164.15	-6.796E+10	0.004883056	-680750.181	1	-0.00488432	-0.0209254
0.557279	-779356.93	-7.463E+10	0.005339161	-586087.626	1	-0.00534041	-0.0177737
0.465199	-833814.01	-8E+10	0.005700172	-490182.125	1	-0.00570138	-0.0064586
0.375286	-875597.26	-8.415E+10	0.005974165	-396137.216	1	-0.00597533	-0.0189509
0.283869	-908063.33	-8.742E+10	0.006183931	-300155.267	1	-0.00618504	-0.0149204
0.193779	-931119.74	-8.979E+10	0.006329452	-205233.177	1	-0.0063305	-0.0254699

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.10141	-946151.3	-9.139E+10	0.006419922	-107581.31	1	-0.0064209	-0.0109834
0.010494	-952758.83	-9.218E+10	0.006453302	-11150.9611	1	-0.0064542	-0.0124617
-0.08109	-951302.62	-9.218E+10	0.00643197	86304.8164	1	-0.00643278	-0.0065613
-0.17199	-941697.8	-9.14E+10	0.006355716	183333.756	1	-0.00635643	-0.0082922
-0.26197	-923858.48	-8.981E+10	0.006224185	279711.711	1	-0.0062248	-0.019953
-0.35299	-896797.59	-8.733E+10	0.006030695	377523.716	1	-0.0060312	-0.0203123
-0.44492	-859298.1	-8.382E+10	0.00576733	476679.017	1	-0.00576771	-0.0106418
-0.53625	-810428.08	-7.92E+10	0.005428325	575575.674	1	-0.00542857	-0.0076338
-0.62747	-747652.97	-7.321E+10	0.004997006	674817.094	1	-0.00499711	-0.0056871
-0.71914	-666389.05	-6.539E+10	0.004443197	775077.784	1	-0.00444314	0.00103643
-0.80898	-561224.89	-5.521E+10	0.003731887	874050.758	1	-0.00373165	-0.0122242
-0.89961	-409619.87	-4.042E+10	0.002714402	975013.812	1	-0.00271394	-0.0168402
-0.99155	1329.07614	132132763	-8.7355E-06	1082668.54	1	9.66593E-06	-0.0070264
-0.99122	25187.7845	2.505E+09	-0.000165474	1082755.98	1	0.000166427	-0.0106365
-0.90206	410678.987	4.112E+10	-0.002677848	992082.732	1	0.002679075	0.01005287
-0.81101	571104.34	5.737E+10	-0.003711167	894731.81	1	0.003712457	0.0101227
-0.72077	683534.809	6.883E+10	-0.004429961	797092.725	1	0.004431271	0.01894599
-0.62887	771357.899	7.783E+10	-0.004987465	696942.59	1	0.004988771	0.00964832
-0.53828	839633.849	8.488E+10	-0.005417634	597670.595	1	0.005418919	0.01465141
-0.4471	894112.793	9.055E+10	-0.00575793	497313.159	1	0.00575918	0.01327242
-0.35584	936727.61	9.503E+10	-0.006021238	396470.071	1	0.006022442	0.01098679
-0.26494	968798.139	9.844E+10	-0.006216394	295667.282	1	0.006217542	0.01265177
-0.17331	991584.816	1.009E+11	-0.006351603	193715.356	1	0.006352686	0.00627947
-0.08247	1005282.72	1.025E+11	-0.006428504	92319.4519	1	0.006429515	0.00855715
0.009075	1010422.14	1.031E+11	-0.00645051	-10174.6109	1	0.00645144	0.00312655
0.099955	1006980.37	1.029E+11	-0.006417821	-112235.364	1	0.006418663	0.0049918
0.190567	994913.86	1.019E+11	-0.006330296	-214305.673	1	0.006331041	0.00980614
0.28124	973812.13	9.987E+10	-0.006185473	-316763.93	1	0.006186114	0.01395557
0.373227	942469.817	9.681E+10	-0.005975759	-421051.02	1	0.005976285	0.00366182
0.4642	900520.511	9.265E+10	-0.00569935	-524553.14	1	0.005699751	0.00450578
0.555284	845872.831	8.718E+10	-0.005343158	-628582.691	1	0.005343424	0.00413942
0.646685	775489.88	8.008E+10	-0.004888354	-733441	1	0.004888468	0.00028553
0.736093	686824.809	7.107E+10	-0.00431966	-836565.485	1	0.004319608	0.01832174
0.83019	561840.258	5.829E+10	-0.003523756	-945903.939	1	0.0035235	-0.0151389
0.921086	381150.37	3.967E+10	-0.002382007	-1052878.82	1	0.002381498	-0.0134492
0.920927	-386399.07	-4.071E+10	0.002382719	-1065612.81	1	-0.00238394	-0.0116984
0.829964	-573490.91	-6.061E+10	0.003524057	-963430.593	1	-0.00352537	-0.0126558
0.738185	-702037.07	-7.438E+10	0.004302497	-858976.359	1	-0.00430384	-0.0046528
0.647291	-798524.39	-8.478E+10	0.00488287	-754752.205	1	-0.00488422	-0.0063721
0.556451	-874359.55	-9.3E+10	0.005335855	-650022.364	1	-0.00533719	-0.0086744
0.464947	-934882.4	-9.961E+10	0.005694527	-544061.148	1	-0.00569583	-0.0036951
0.374033	-982041.61	-1.048E+11	0.005971286	-438380.308	1	-0.00597255	-0.005194
0.282664	-1018133.8	-1.088E+11	0.00618028	-331804.201	1	-0.00618149	-0.001686
0.19263	-1043676.3	-1.117E+11	0.006325092	-226452.955	1	-0.00632624	-0.0128527

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	$\boldsymbol{B_{ii}}$	b
0.099744	-1060287.9	-1.137E+11	0.006415267	-117432.159	1	-0.00641634	0.00731859
0.009486	-1067313.7	-1.146E+11	0.006447638	-11184.7005	1	-0.00644864	-0.0013896
-0.08202	-1065377.6	-1.145E+11	0.006425723	96842.1993	1	-0.00642664	0.0035975
-0.1734	-1054217.6	-1.135E+11	0.006348268	205037.156	1	-0.00634908	0.00721553
-0.26441	-1033566.7	-1.114E+11	0.006213873	313114.574	1	-0.00621458	0.00676678
-0.35471	-1003009.1	-1.083E+11	0.006020263	420688.101	1	-0.00602086	-0.0014002
-0.447	-960331.92	-1.039E+11	0.005754082	530982.155	1	-0.00575455	0.01216316
-0.5376	-905497.18	-9.81E+10	0.005415778	639661.511	1	-0.00541611	0.00727544
-0.62815	-835236.86	-9.064E+10	0.004985939	748715.733	1	-0.00498613	0.00175111
-0.71879	-744889.92	-8.1E+10	0.004437182	858414.513	1	-0.0044372	-0.002791
-0.81053	-623938.43	-6.8E+10	0.003707489	970181.719	1	-0.00370731	0.00485493
-0.90337	-446860.25	-4.884E+10	0.002646785	1084481.3	1	-0.00264635	0.02449665
-0.99065	-5147.7573	-566304782	3.02769E-05	1196933.73	1	-2.9353E-05	-0.0169749
-0.90104	458675.471	5.079E+10	-0.002678298	1095905.85	1	0.002679568	-0.0010849
-0.81	637390.941	7.078E+10	-0.003710446	987951.074	1	0.003711797	-0.0009891
-0.71978	762528.34	8.487E+10	-0.004428338	879828.786	1	0.00442972	0.00810314
-0.62792	860193.916	9.592E+10	-0.00498507	769019.899	1	0.004986457	-0.0007806
-0.53688	936419.549	1.046E+11	-0.005416648	658652.177	1	0.005418021	-0.0006896
-0.44572	996825.364	1.115E+11	-0.005756004	547699.728	1	0.005757347	-0.001824
-0.3545	1043999.68	1.17E+11	-0.006018434	436268.084	1	0.006019736	-0.0037237
-0.26478	1079045	1.211E+11	-0.006210736	326321.087	1	0.006211986	0.01087504
-0.17265	1104364.23	1.241E+11	-0.006346543	213087.497	1	0.00634773	-0.0009498
-0.08013	1119567.88	1.26E+11	-0.006424009	99034.6749	1	0.006425124	-0.0171436
0.010756	1124854.75	1.268E+11	-0.006444673	-13312.1022	1	0.006445707	-0.0153349
0.100965	1120712.43	1.265E+11	-0.006411394	-125128.342	1	0.00641234	-0.0060923
0.19321	1106617.82	1.25E+11	-0.006321058	-239787.452	1	0.006321905	-0.0192144
0.284862	1082279.44	1.225E+11	-0.006172457	-354039.135	1	0.006173196	-0.0258336
0.375552	1047270.55	1.187E+11	-0.005963382	-467427.814	1	0.005964003	-0.0218679
0.466322	1000109.05	1.135E+11	-0.005685492	-581280.359	1	0.005685984	-0.0187949
0.558137	938091.153	1.066E+11	-0.005323583	-696851.004	1	0.005323931	-0.0271904
0.649614	858764.031	9.779E+10	-0.004864232	-812469.107	1	0.004864421	-0.031884
0.740485	757024.79	8.637E+10	-0.004279042	-927893.288	1	0.004279051	-0.0299104
0.830698	622126.522	7.114E+10	-0.003508037	-1043255.6	1	0.003507837	-0.0207157
0.924299	411482.008	4.72E+10	-0.002312513	-1164381.49	1	0.002312029	-0.0487402
0.921607	-424140.03	-4.918E+10	0.002355702	-1173652.24	1	-0.00235696	-0.0191689
0.829214	-635979.95	-7.396E+10	0.003520923	-1059123.92	1	-0.00352229	-0.0044195
0.737515	-778332.53	-9.072E+10	0.004298697	-944081.839	1	-0.00430011	0.00270721
0.6467	-885112.25	-1.034E+11	0.00487857	-829372.168	1	-0.00488	0.00012425
0.55448	-970173.86	-1.135E+11	0.005337584	-712309.202	1	-0.005339	0.01296801
0.463995	-1036196.8	-1.214E+11	0.00569131	-596985.08	1	-0.00569271	0.00676462
0.372047	-1088742.6	-1.277E+11	0.005970344	-479390.356	1	-0.0059717	0.01662088
0.281262	-1128138.9	-1.325E+11	0.006177008	-362918.735	1	-0.00617832	0.01370776
0.190129	-1156410.1	-1.36E+11	0.006322457	-245661.665	1	-0.00632371	0.01462004
0.098458	-1174195.9	-1.383E+11	0.006410353	-127386.181	1	-0.00641154	0.02144009

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.007687	-1181680.3	-1.394E+11	0.006442014	-9958.11773	1	-0.00644312	0.01837689
-0.08373	-1179155.9	-1.393E+11	0.00641904	108619.229	1	-0.00642006	0.02241978
-0.17559	-1166308	-1.379E+11	0.006339898	228081.767	1	-0.00634082	0.03126588
-0.2659	-1143186.5	-1.354E+11	0.006205219	345859.231	1	-0.00620603	0.02319156
-0.35717	-1108507.3	-1.315E+11	0.006007955	465214.953	1	-0.00600865	0.02558443
-0.44821	-1061390.6	-1.261E+11	0.005743659	584625.821	1	-0.00574422	0.02542439
-0.53864	-1000352.2	-1.19E+11	0.005404568	703634.398	1	-0.00540499	0.01862564
-0.62944	-921821.98	-1.098E+11	0.004971593	823576.051	1	-0.00497186	0.01587926
-0.721	-819943.09	-9.786E+10	0.004413541	945070.169	1	-0.00441363	0.02147402
-0.81118	-687296.73	-8.219E+10	0.003691359	1065459.52	1	-0.00369124	0.01199329
-0.90138	-496927.71	-5.958E+10	0.002661416	1187003.06	1	-0.00266103	0.00257547
-0.98976	-3431.2502	-413863670	1.82591E-05	1311151.56	1	-1.7335E-05	-0.0267402
-0.90116	503610.466	6.111E+10	-0.002662604	1200951.38	1	0.002663916	0.00020562
-0.80921	703259.618	8.556E+10	-0.003707702	1081219.41	1	0.003709114	-0.0096874
-0.71825	842180.567	1.027E+11	-0.00443046	961613.833	1	0.004431915	-0.0087196
-0.62727	948634.487	1.158E+11	-0.004981108	841264.642	1	0.004982577	-0.0079707
-0.53432	1034108.89	1.265E+11	-0.005420505	717762.727	1	0.005421966	-0.028764
-0.44417	1099579.41	1.347E+11	-0.005754673	597519.735	1	0.00575611	-0.0189319
-0.35352	1150987.58	1.412E+11	-0.006014742	476230.349	1	0.006016142	-0.0145196
-0.26159	1190179.23	1.462E+11	-0.00621051	352870.097	1	0.006211861	-0.0241345
-0.17064	1217278.66	1.497E+11	-0.006343038	230476.27	1	0.006344328	-0.0230907
-0.07934	1233496.04	1.519E+11	-0.006418683	107300.116	1	0.006419903	-0.0258063
0.012054	1239067.09	1.528E+11	-0.006438853	-16322.8321	1	0.006439992	-0.0295932
0.101594	1234248.58	1.524E+11	-0.00640522	-137740.494	1	0.006406271	-0.0130061
0.194324	1218331.93	1.506E+11	-0.006313773	-263803.61	1	0.006314722	-0.0314486
0.284743	1191608.25	1.475E+11	-0.006166708	-387049.917	1	0.006167548	-0.0245238
0.376976	1152062.32	1.428E+11	-0.005953371	-513112.223	1	0.005954088	-0.0375072
0.467584	1099749.9	1.365E+11	-0.005674581	-637320.651	1	0.005675164	-0.0326599
0.557759	1032315.15	1.283E+11	-0.00531829	-761332.591	1	0.005318726	-0.0230491
0.649589	944346.593	1.176E+11	-0.004856733	-888090.339	1	0.004857002	-0.0316033
0.739923	832692.599	1.038E+11	-0.004274452	-1013355.21	1	0.004274532	-0.023742
0.831946	680387.79	8.502E+10	-0.003484779	-1141761	1	0.003484631	-0.0344327
0.923736	451554.984	5.658E+10	-0.002305838	-1271253.36	1	0.002305392	-0.0425549
0.921129	-464699.32	-5.881E+10	0.002347712	-1280291.29	1	-0.00234901	-0.0139226
0.828031	-699218.69	-8.873E+10	0.003522111	-1154048.34	1	-0.00352354	0.00857125
0.737099	-854116.75	-1.086E+11	0.00429306	-1029377.76	1	-0.00429454	0.00727841
0.645952	-971719.08	-1.238E+11	0.004875152	-903636.302	1	-0.00487666	0.00833265
0.553798	-1064887.7	-1.358E+11	0.005333642	-775923.463	1	-0.00533515	0.02046358
0.463379	-1137151.6	-1.453E+11	0.005686935	-650154.393	1	-0.00568842	0.01352731
0.371502	-1194613.8	-1.528E+11	0.005965595	-521949.915	1	-0.00596705	0.02261028
0.280227	-1237871	-1.586E+11	0.006173028	-394218.492	1	-0.00617444	0.02508245
0.188003	-1268933.8	-1.627E+11	0.006319318	-264814.93	1	-0.00632067	0.03796472
0.098135	-1287777	-1.654E+11	0.006404815	-138396.78	1	-0.0064061	0.02498392
0.008028	-1295775.8	-1.666E+11	0.006436279	-11335.1729	1	-0.00643749	0.01462751

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.0839	-1292830.5	-1.664E+11	0.00641321	118602.189	1	-0.00641433	0.02422387
-0.17509	-1278675	-1.648E+11	0.006334665	247825.948	1	-0.00633569	0.02582308
-0.26534	-1253206.7	-1.617E+11	0.006200303	376017.554	1	-0.00620122	0.01696295
-0.35599	-1215340.1	-1.57E+11	0.006004785	505121.771	1	-0.00600558	0.01259129
-0.44803	-1162971.6	-1.504E+11	0.005737813	636565.674	1	-0.00573847	0.02345518
-0.53886	-1095479.8	-1.419E+11	0.005396783	766678.855	1	-0.0053973	0.02104199
-0.62907	-1009657.2	-1.31E+11	0.004966101	896351.078	1	-0.00496645	0.01183745
-0.72012	-898414.97	-1.167E+11	0.004411146	1027779.34	1	-0.00441131	0.01189623
-0.80993	-753643.43	-9.812E+10	0.003692895	1158117.15	1	-0.00369284	-0.001746
-0.90116	-542315.17	-7.078E+10	0.002650471	1291677.71	1	-0.00265013	0.00018106
-0.98877	18613.3833	2.443E+09	-9.04179E-05	1425286.89	1	9.13617E-05	-0.0376246
-0.90191	546065.717	7.205E+10	-0.002637716	1306910.08	1	0.002639068	0.00843517
-0.8083	769256.313	1.017E+11	-0.003706059	1174131.68	1	0.003707531	-0.0196833
-0.71778	920357.772	1.22E+11	-0.004425245	1044574.12	1	0.004426773	-0.0138017
-0.62645	1037108.33	1.376E+11	-0.004978	913125.063	1	0.00497955	-0.0169078
-0.53603	1128127.02	1.499E+11	-0.005406516	782452.169	1	0.005408066	-0.0099859
-0.44452	1200937.63	1.598E+11	-0.005747093	649752.045	1	0.005748625	-0.0150414
-0.35399	1257054.61	1.675E+11	-0.006007443	518074.59	1	0.006008942	-0.0093872
-0.2616	1300039.62	1.735E+11	-0.006204562	383342.021	1	0.006206014	-0.024047
-0.17188	1329217.83	1.776E+11	-0.006335807	252170.577	1	0.006337202	-0.0094005
-0.08068	1346951.88	1.801E+11	-0.00641224	118509.219	1	0.006413566	-0.0110585
0.010042	1353055.17	1.812E+11	-0.006433285	-14766.2192	1	0.006434532	-0.0074869
0.101262	1347675.26	1.807E+11	-0.006399696	-149081.581	1	0.006400852	-0.0093605
0.192191	1330607.71	1.786E+11	-0.006310708	-283280.472	1	0.006311764	-0.0080214
0.282015	1301749.4	1.749E+11	-0.006166044	-416167.778	1	0.006166989	0.0054363
0.374252	1258757.19	1.693E+11	-0.005954454	-552967.718	1	0.005955272	-0.0075994
0.464376	1202204.45	1.619E+11	-0.005679229	-686995.288	1	0.00567991	0.00257288
0.556101	1127526.2	1.521E+11	-0.005318689	-823809.751	1	0.005319215	-0.0048345
0.647065	1032608.88	1.395E+11	-0.004863364	-959956.609	1	0.004863717	-0.0038851
0.737552	910974.164	1.233E+11	-0.004283105	-1095957.18	1	0.00428326	0.00229825
0.828533	747728.145	1.014E+11	-0.003508449	-1233479.56	1	0.003508367	0.00305333
0.920926	499438.81	6.787E+10	-0.002337037	-1374528.47	1	0.002336638	-0.0116887
0.922061	-499848.32	-6.855E+10	0.002316112	-1388870.81	1	-0.00231744	-0.0241524
0.828976	-757861.88	-1.042E+11	0.003502068	-1251847.14	1	-0.00350355	-0.0018092
0.738495	-926795.48	-1.277E+11	0.004274209	-1117278.09	1	-0.00427576	-0.0080544
0.646823	-1056194.7	-1.457E+11	0.00486265	-980149.565	1	-0.00486424	-0.0012275
0.556339	-1156301.8	-1.598E+11	0.005315484	-844227.581	1	-0.00531708	-0.0074427
0.464651	-1236488.2	-1.711E+11	0.005676041	-706027.921	1	-0.00567763	-0.0004399
0.374021	-1298482.8	-1.799E+11	0.00595274	-569020.34	1	-0.0059543	-0.0050584
0.282925	-1345810.8	-1.866E+11	0.006161845	-430943.228	1	-0.00616336	-0.0045544
0.192002	-1379523.3	-1.915E+11	0.006308401	-292788.939	1	-0.00630986	-0.0059482
0.101686	-1400522.3	-1.947E+11	0.006396727	-155238.716	1	-0.00639812	-0.0140188
0.009919	-1409610.4	-1.962E+11	0.006430455	-15160.3102	1	-0.00643177	-0.0061447
-0.08135	-1406565.5	-1.96E+11	0.006408869	124468.278	1	-0.0064101	-0.0037645

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.17307	-1391210.1	-1.94E+11	0.006331213	265112.002	1	-0.00633234	0.00362447
-0.26384	-1363396.9	-1.904E+11	0.006197072	404613.378	1	-0.00619809	0.00053164
-0.35554	-1321657	-1.848E+11	0.005999758	545893.889	1	-0.00600065	0.00773059
-0.44594	-1265689.6	-1.772E+11	0.005738262	685515.59	1	-0.00573902	0.00056286
-0.53679	-1192411.5	-1.671E+11	0.005398648	826229.525	1	-0.00539925	-0.0016261
-0.62705	-1099214.9	-1.543E+11	0.004969449	966471.182	1	-0.00496988	-0.0103153
-0.71899	-977189.25	-1.373E+11	0.004410548	1109877.51	1	-0.00441078	-0.0005277
-0.80914	-819110.73	-1.153E+11	0.003690203	1251196.79	1	-0.00369021	-0.010482
-0.90198	-583962.48	-8.241E+10	0.002624405	1397954.05	1	-0.0026241	0.00921628
-0.90016	595768.068	8.492E+10	-0.002648817	1409234.22	1	0.002650214	-0.0107525
-0.80996	829765.209	1.185E+11	-0.003680566	1270798.21	1	0.003682096	-0.0014183
-0.72079	992864.222	1.421E+11	-0.004396058	1132812.83	1	0.004397655	0.01925198
-0.62865	1121772.85	1.608E+11	-0.004958818	989482.728	1	0.004960449	0.00719399
-0.5356	1223714.43	1.756E+11	-0.005401516	844180.107	1	0.005403154	-0.0147822
-0.44939	1298553.34	1.866E+11	-0.005724609	709144.948	1	0.005726236	0.0384185
-0.34824	1366462.23	1.966E+11	-0.006015524	550253.529	1	0.00601712	-0.0724905
-0.27268	1404874.78	2.023E+11	-0.006178385	431272.004	1	0.006179945	0.09765976
-0.17004	1441752.58	2.079E+11	-0.006332109	269274.037	1	0.006333607	-0.0296293
-0.07832	1460683.69	2.108E+11	-0.006407751	124156.604	1	0.006409181	-0.0370457
0.011165	1466843.75	2.12E+11	-0.006427495	-17718.9545	1	0.006428846	-0.0198283
0.194282	1441591.56	2.088E+11	-0.006302165	-308993.757	1	0.006303322	-0.0309889
0.274431	1413874.63	2.049E+11	-0.006174574	-436891.851	1	0.00617563	0.08873269
0.382542	1358888.62	1.972E+11	-0.005925884	-609826.118	1	0.005926788	-0.0986395
0.461351	1304661.89	1.896E+11	-0.00568318	-736214.986	1	0.005683959	0.03579743
0.55122	1225940.86	1.783E+11	-0.00533325	-880708.934	1	0.005333872	0.04877591
0.642477	1123709.2	1.637E+11	-0.004881485	-1027897.95	1	0.004881926	0.04650071
0.731004	996414.014	1.454E+11	-0.00432182	-1171226.09	1	0.00432206	0.07421313
0.822637	821448.787	1.2E+11	-0.003556293	-1320346.32	1	0.003556285	0.06780855
0.912008	570888.477	8.361E+10	-0.00246569	-1467042.95	1	0.002465366	0.08625531
0.912973	-572405.66	-8.459E+10	0.002448617	-1481871.5	1	-0.00245	0.07565448
0.824685	-828103.95	-1.226E+11	0.003534194	-1341494.69	1	-0.00353574	0.0453159
0.736634	-1004433.5	-1.49E+11	0.004279125	-1200262.14	1	-0.00428075	0.01237987
0.648416	-1139235.7	-1.693E+11	0.004846029	-1058031.23	1	-0.00484769	-0.0187299
0.558697	-1247231.4	-1.855E+11	0.005298039	-912824.919	1	-0.00529972	-0.0333446

$A^T A$					
193.1976094	-1350569.283	-1.93121E+11	0.000939573	-162467736.4	-8.345412087
-1350569.283	2.85101E+14	3.2171E+19	-1835209.003	2.12104E+12	6219336.727
-1.93121E+11	3.2171E+19	3.83652E+24	-1.85938E+11	3.12203E+17	9.56383E+11
0.000939573	-1835209.003	-1.85938E+11	0.016352738	-8884.696749	-0.062798418
-162467736.4	2.12104E+12	3.12203E+17	-8884.696749	1.75353E+14	4510855.812
-8.345412087	6219336.727	9.56383E+11	-0.062798418	4510855.812	587

$(A^TA)^{-1}$					
0.023465399	1.46136E-09	-9.79556E-15	0.0637533	2.17396E-08	0.000173846
1.46136E-09	2.02407E-13	-1.32859E-18	7.61268E-06	1.63519E-15	8.42745E-10
-9.79556E-15	-1.32859E-18	9.30158E-24	-4.33676E-11	-1.1615E-20	-5.76777E-15
0.0637533	7.61268E-06	-4.33676E-11	422.5535602	6.46931E-08	0.0356152
2.17396E-08	1.63519E-15	-1.1615E-20	6.46931E-08	2.5846E-14	1.18977E-10
0.000173846	8.42745E-10	-5.76777E-15	0.0356152	1.18977E-10	0.001709413

$A^T b$
-1.02876E-13
4.00761E-06
0.449004173
-2.60539E-14
1.2479E-07
3.93408E-13

$\hat{x} = -(A^TA)^{-1}A^Tb$
1.64525E-16
1.66689E-20
-2.08484E-26
4.27873E-14
6.88134E-22
5.29187E-16

A	10.98296	σ_A	0.00417215
θ	0.000592	σ_θ	1.2253E-08
θ'	9.29E-13	$\sigma_{\theta'}$	8.3066E-14
B	9273.193	σ_B	0.55986975
λ	8.53E-08	σ_λ	4.3787E-09
Q	-0.1028	σ_Q	0.00112608

Maximum residual 0.098639492

$\hat{\sigma}_0$ 0.027236175

Least squares adjustment computations using the second set of time observation data are summarised below.

$\frac{\partial F}{\partial A}$ (A)	$\frac{\partial F}{\partial \theta}$ (A)	$\frac{\partial F}{\partial \theta'}$ (A)	$\frac{\partial F}{\partial B}$ (A)	$\frac{\partial F}{\partial \lambda}$ (A)	$\frac{\partial F}{\partial Q}$ (A)	B_{ii}	b
0.503339	-18613.6	-1.4E+08	-0.00636	-45385.3	1	0.006355	-0.2737
0.584307	-15858.8	-1.2E+08	-0.00597	-53853.9	1	0.005967	-0.27257
0.662499	-13084.4	-1E+08	-0.00551	-62433.6	1	0.005507	-0.2372
0.742075	-10152.6	-7.9E+07	-0.00493	-71653.9	1	0.004926	-0.21889
0.819545	-7199.12	-5.8E+07	-0.00421	-81247.4	1	0.004208	-0.17461
0.897764	-4122.7	-3.4E+07	-0.00323	-91870.4	1	0.003229	-0.13958
0.973651	-1051.57	-9124473	-0.00165	-104224	1	0.001651	-0.07577
0.962946	-1486.42	-1.4E+07	0.001961	-112962	1	-0.00196	0.0563
0.873491	-5080.93	-5E+07	0.00357	-106746	1	-0.00357	0.15987
0.790596	-8305.87	-8.4E+07	0.004497	-99083.8	1	-0.0045	0.182513
0.707196	-11441.3	-1.2E+08	0.005195	-90481	1	-0.0052	0.211391
0.625133	-14413.3	-1.5E+08	0.005737	-81408.3	1	-0.00574	0.223772
0.541989	-17302.6	-1.9E+08	0.006178	-71730.8	1	-0.00618	0.249496
0.461093	-19988.4	-2.2E+08	0.006524	-61917.6	1	-0.00652	0.247471
0.380019	-22547.4	-2.5E+08	0.006802	-51733.8	1	-0.0068	0.247661
0.29818	-24985.8	-2.8E+08	0.007019	-41129.7	1	-0.00702	0.257276
0.216992	-27250	-3.1E+08	0.007179	-30308.4	1	-0.00718	0.258866
0.135451	-29355.5	-3.4E+08	0.007286	-19151.5	1	-0.00729	0.264799
0.055363	-31244.4	-3.6E+08	0.007343	-7919.92	1	-0.00734	0.252826
-0.02687	-32980.7	-3.9E+08	0.007351	3889.84	1	-0.00735	0.267315
-0.10833	-34475.9	-4.1E+08	0.007311	15864.51	1	-0.00731	0.272222
-0.1873	-35686.9	-4.3E+08	0.007223	27739.93	1	-0.00722	0.246512
-0.2668	-36637.7	-4.4E+08	0.007087	39964.92	1	-0.00709	0.227274
-0.34843	-37293.4	-4.6E+08	0.006892	52811.24	1	-0.00689	0.234303
-0.42857	-37568.6	-4.7E+08	0.006642	65728.68	1	-0.00664	0.222869
-0.50853	-37409.2	-4.7E+08	0.006329	78945.71	1	-0.00633	0.209312
-0.58864	-36716	-4.7E+08	0.005941	92552.87	1	-0.00594	0.19771
-0.66733	-35372.6	-4.6E+08	0.005471	106324.5	1	-0.00547	0.168446
-0.74672	-33118.7	-4.3E+08	0.004883	120714.6	1	-0.00488	0.147852
-0.82645	-29509.6	-3.9E+08	0.004129	135825.4	1	-0.00413	0.131387
-0.80429	46196.73	7.11E+08	-0.00436	152643.5	1	0.004358	-0.14196
-0.72028	55852.17	8.71E+08	-0.00509	138646.2	1	0.005093	-0.17832
-0.63785	63834.79	1.01E+09	-0.00566	124266.3	1	0.005657	-0.19525
-0.55626	70650.28	1.13E+09	-0.0061	109543.4	1	0.006106	-0.20185
-0.47465	76579.95	1.23E+09	-0.00647	94410.34	1	0.006468	-0.20856
-0.39257	81765.66	1.33E+09	-0.00676	78824.1	1	0.00676	-0.22122
-0.31175	86177.15	1.42E+09	-0.00698	63153.4	1	0.006984	-0.21829
-0.22917	90015.34	1.49E+09	-0.00715	46832.87	1	0.007155	-0.23697
-0.14904	93115.75	1.56E+09	-0.00727	30708.3	1	0.007269	-0.22559

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.06791	95635.08	1.61E+09	-0.00733	14106.38	1	0.007334	-0.22647
0.013081	97522.58	1.66E+09	-0.00735	-2739.44	1	0.00735	-0.22557
0.09398	98764.36	1.69E+09	-0.00732	-19838.4	1	0.007318	-0.22358
0.173671	99328.62	1.71E+09	-0.00724	-36949.9	1	0.007239	-0.2067
0.255691	99179.84	1.72E+09	-0.00711	-54845	1	0.007105	-0.21855
0.334777	98273.04	1.72E+09	-0.00692	-72382.7	1	0.006925	-0.19421
0.415896	96474.93	1.71E+09	-0.00668	-90675.9	1	0.006682	-0.19495
0.495884	93717.62	1.67E+09	-0.00638	-109039	1	0.006379	-0.18173
0.575259	89840.27	1.62E+09	-0.00601	-127614	1	0.006007	-0.16094
0.655831	84472.4	1.53E+09	-0.00554	-146879	1	0.005541	-0.15493
0.73476	77388.85	1.42E+09	-0.00498	-166233	1	0.004975	-0.12866
0.81484	67562.65	1.25E+09	-0.00425	-186502	1	0.004245	-0.11657
0.892831	53712.94	1.01E+09	-0.00329	-207161	1	0.003286	-0.07873
0.971983	28448.3	5.46E+08	-0.00167	-230227	1	0.001673	-0.0552
0.964225	-34901.5	-7E+08	0.001899	-238180	1	-0.0019	0.040517
0.879246	-66167.4	-1.4E+09	0.003478	-221397	1	-0.00348	0.088874
0.797089	-86036.1	-1.8E+09	0.004422	-203227	1	-0.00442	0.102409
0.714486	-101646	-2.1E+09	0.005131	-184044	1	-0.00513	0.12146
0.633794	-114204	-2.4E+09	0.005676	-164697	1	-0.00568	0.116924
0.551938	-124927	-2.7E+09	0.006122	-144590	1	-0.00612	0.126753
0.470131	-134000	-2.9E+09	0.006482	-124087	1	-0.00648	0.135982
0.388436	-141654	-3.1E+09	0.006768	-103253	1	-0.00677	0.143824
0.306951	-148034	-3.2E+09	0.006992	-82145.8	1	-0.00699	0.149073
0.227201	-153154	-3.3E+09	0.007155	-61192.8	1	-0.00716	0.132919
0.144711	-157338	-3.5E+09	0.00727	-39227.3	1	-0.00727	0.150562
0.064766	-160340	-3.5E+09	0.007332	-17664.2	1	-0.00733	0.136815
-0.01679	-162340	-3.6E+09	0.007346	4607.183	1	-0.00735	0.142924
-0.09704	-163246	-3.7E+09	0.007312	26793.04	1	-0.00731	0.133
-0.17784	-163055	-3.7E+09	0.00723	49400.78	1	-0.00723	0.129746
-0.25802	-161711	-3.7E+09	0.007097	72112.96	1	-0.0071	0.118906
-0.33981	-159064	-3.6E+09	0.006908	95576.18	1	-0.00691	0.127935
-0.41909	-155138	-3.6E+09	0.006668	118619.4	1	-0.00667	0.105991
-0.49941	-149613	-3.5E+09	0.00636	142288.2	1	-0.00636	0.096809
-0.57998	-142235	-3.3E+09	0.005978	166397.1	1	-0.00598	0.090833
-0.65968	-132721	-3.1E+09	0.005512	190655	1	-0.00551	0.074073
-0.74095	-120058	-2.8E+09	0.00492	215896.7	1	-0.00492	0.076618
-0.82002	-103623	-2.5E+09	0.004185	241102.7	1	-0.00418	0.052153
-0.8096	121073.6	3.14E+09	-0.00429	258621.4	1	0.004293	-0.07649
-0.72751	143604.4	3.75E+09	-0.00503	234348.2	1	0.005027	-0.08911
-0.6465	161600.3	4.25E+09	-0.00559	209749.7	1	0.005595	-0.08861
-0.56571	176501.4	4.67E+09	-0.00605	184734.3	1	0.006051	-0.08518
-0.48478	189003.6	5.03E+09	-0.00642	159262.7	1	0.00642	-0.08361
-0.4027	199583.8	5.35E+09	-0.00672	133063.5	1	0.006721	-0.09616
-0.32174	208189.5	5.61E+09	-0.00695	106891	1	0.006953	-0.09497

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.2395	215228.4	5.83E+09	-0.00713	79992.24	1	0.007131	-0.10961
-0.1596	220518.8	6E+09	-0.00725	53576.27	1	0.007251	-0.09527
-0.07923	224356.5	6.14E+09	-0.00732	26730.28	1	0.007322	-0.08674
0.002252	226750	6.23E+09	-0.00734	-763.566	1	0.007345	-0.09197
0.083125	227626.6	6.29E+09	-0.00732	-28323.9	1	0.007319	-0.08967
0.162861	226991.7	6.3E+09	-0.00725	-55764.3	1	0.007246	-0.07335
0.245003	224709.1	6.27E+09	-0.00712	-84316.1	1	0.007119	-0.0867
0.324283	220831.7	6.19E+09	-0.00694	-112156	1	0.006945	-0.06475
0.405141	215024.1	6.06E+09	-0.00671	-140851	1	0.006711	-0.06226
0.485004	207225.9	5.87E+09	-0.00642	-169513	1	0.006417	-0.0475
0.565891	196902.5	5.61E+09	-0.00605	-198901	1	0.006047	-0.04538
0.646617	183652.3	5.26E+09	-0.00559	-228639	1	0.005591	-0.04127
0.726754	166758.1	4.81E+09	-0.00503	-258643	1	0.005029	-0.02989
0.807125	144559.5	4.2E+09	-0.00431	-289357	1	0.004313	-0.0214
0.887952	113488	3.33E+09	-0.00334	-321181	1	0.003342	-0.01853
0.967988	61077.53	1.81E+09	-0.00177	-354729	1	0.001765	-0.00591
0.968423	-62923.1	-1.9E+09	0.00175	-364565	1	-0.00175	-0.01128
0.8847	-124035	-3.8E+09	0.003387	-337413	1	-0.00339	0.021584
0.802946	-161313	-5E+09	0.004353	-308809	1	-0.00435	0.030156
0.722308	-189344	-5.9E+09	0.005061	-279676	1	-0.00506	0.024959
0.641691	-211971	-6.7E+09	0.005619	-249934	1	-0.00562	0.019506
0.56058	-230766	-7.3E+09	0.006071	-219524	1	-0.00607	0.020141
0.479909	-246306	-7.9E+09	0.006435	-188875	1	-0.00644	0.015351
0.399288	-259180	-8.3E+09	0.006727	-157889	1	-0.00673	0.009946
0.317684	-269820	-8.7E+09	0.006958	-126197	1	-0.00696	0.016656
0.237687	-278118	-9E+09	0.007129	-94829.1	1	-0.00713	0.003559
0.15662	-284504	-9.2E+09	0.00725	-62754.6	1	-0.00725	0.003654
0.075664	-288927	-9.4E+09	0.00732	-30445	1	-0.00732	0.002377
-0.0052	-291427	-9.5E+09	0.007341	2101.377	1	-0.00734	-2.7E-05
-0.08603	-292007	-9.6E+09	0.007313	34904.39	1	-0.00731	-0.00288
-0.1657	-290657	-9.6E+09	0.007238	67505.89	1	-0.00724	-0.01998
-0.2466	-287249	-9.5E+09	0.007112	100887.9	1	-0.00711	-0.02191
-0.32806	-281608	-9.4E+09	0.006931	134789.5	1	-0.00693	-0.01706
-0.40875	-273631	-9.2E+09	0.006695	168679.1	1	-0.0067	-0.02161
-0.48945	-262984	-8.8E+09	0.006395	202899.2	1	-0.00639	-0.02606
-0.57052	-249168	-8.4E+09	0.006019	237639.6	1	-0.00602	-0.02593
-0.65174	-231520	-7.9E+09	0.005554	272861.9	1	-0.00555	-0.02394
-0.7329	-208972	-7.1E+09	0.004975	308562	1	-0.00497	-0.02261
-0.81368	-179622	-6.2E+09	0.00424	344739.3	1	-0.00424	-0.02606
-0.81636	192823.5	7.02E+09	-0.00421	366571.2	1	0.004211	0.007005
-0.73526	228731.4	8.38E+09	-0.00495	332130.4	1	0.004955	0.00645
-0.65483	257168.9	9.47E+09	-0.00553	297325.1	1	0.005533	0.014202
-0.57583	280129.4	1.04E+10	-0.00599	262654.6	1	0.005991	0.039579
-0.49356	299919.9	1.11E+10	-0.00637	226127.3	1	0.006376	0.024652

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.41199	316069.7	1.18E+10	-0.00668	189538.4	1	0.006683	0.018361
-0.33256	328888.3	1.23E+10	-0.00692	153585.6	1	0.006919	0.03843
-0.25067	339379.1	1.28E+10	-0.0071	116212.8	1	0.007103	0.028225
-0.16986	347201.5	1.31E+10	-0.00723	79042.47	1	0.007232	0.031374
-0.09086	352526.9	1.33E+10	-0.00731	42427.93	1	0.007308	0.056674
-0.01008	355650.6	1.35E+10	-0.00734	4723.447	1	0.007338	0.06015
0.071357	356431	1.36E+10	-0.00732	-33562.4	1	0.00732	0.055512
0.152901	354788.1	1.36E+10	-0.00725	-72176.8	1	0.007251	0.04953
0.233406	350695.9	1.35E+10	-0.00713	-110577	1	0.007134	0.056367
0.314585	343933.9	1.33E+10	-0.00696	-149586	1	0.006963	0.054897
0.395141	334384.3	1.29E+10	-0.00674	-188598	1	0.006736	0.061107
0.475874	321649.3	1.25E+10	-0.00645	-228017	1	0.006447	0.065128
0.557187	305133.3	1.19E+10	-0.00608	-268079	1	0.006083	0.062003
0.63891	284046.7	1.11E+10	-0.00563	-308754	1	0.00563	0.053808
0.719273	257722.9	1.01E+10	-0.00508	-349231	1	0.005077	0.062407
0.800884	223098.4	8.83E+09	-0.00436	-390960	1	0.004364	0.055594
0.882582	175480.5	6.99E+09	-0.0034	-433662	1	0.003403	0.047715
0.964648	95712.32	3.85E+09	-0.00183	-478567	1	0.001833	0.035295
0.96988	-89697.2	-3.7E+09	0.001675	-490843	1	-0.00168	-0.02925
0.888792	-179621	-7.4E+09	0.003313	-454180	1	-0.00331	-0.02889
0.808886	-233984	-9.8E+09	0.004281	-415932	1	-0.00428	-0.04312
0.729827	-274607	-1.2E+10	0.004992	-377170	1	-0.00499	-0.0678
0.649745	-307569	-1.3E+10	0.00556	-337283	1	-0.00556	-0.07985
0.56892	-334811	-1.4E+10	0.006021	-296534	1	-0.00602	-0.08275
0.488338	-357212	-1.5E+10	0.006393	-255497	1	-0.00639	-0.08863
0.408193	-375539	-1.6E+10	0.00669	-214326	1	-0.00669	-0.09992
0.327545	-390490	-1.7E+10	0.006926	-172571	1	-0.00693	-0.10499
0.246717	-402286	-1.7E+10	0.007106	-130418	1	-0.00711	-0.10784
0.165878	-411106	-1.8E+10	0.007232	-87969.8	1	-0.00723	-0.11057
0.085085	-417076	-1.8E+10	0.007307	-45266.6	1	-0.00731	-0.11385
0.005508	-420241	-1.8E+10	0.007334	-2939.6	1	-0.00734	-0.13214
-0.07054	-420756	-1.8E+10	0.007315	37758.35	1	-0.00732	-0.19392
-0.15676	-418324	-1.8E+10	0.007242	84188	1	-0.00724	-0.1303
-0.23717	-413059	-1.8E+10	0.007123	127774.4	1	-0.00712	-0.13835
-0.32047	-404368	-1.8E+10	0.006944	173231.5	1	-0.00694	-0.11066
-0.40079	-392597	-1.7E+10	0.006713	217367.7	1	-0.00671	-0.11974
-0.4828	-376743	-1.7E+10	0.006414	262760.8	1	-0.00642	-0.10805
-0.56318	-356860	-1.6E+10	0.006048	307612	1	-0.00605	-0.11643
-0.6444	-331503	-1.5E+10	0.005591	353341.5	1	-0.00559	-0.11446
-0.7266	-298937	-1.3E+10	0.005015	400124.9	1	-0.00502	-0.10041
-0.80778	-257190	-1.2E+10	0.004289	446973.6	1	-0.00429	-0.09889
-0.82216	262197.7	1.23E+10	-0.00414	475723.4	1	0.004137	0.078521
-0.7427	311363.1	1.47E+10	-0.00488	431734.3	1	0.004885	0.098256
-0.66153	351161.6	1.66E+10	-0.00548	386122.2	1	0.005481	0.096831

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.58224	382850.6	1.82E+10	-0.00595	341071.8	1	0.005949	0.118693
-0.50101	409582.4	1.95E+10	-0.00634	294493.5	1	0.006337	0.116549
-0.42097	431198.8	2.06E+10	-0.00664	248236.6	1	0.006646	0.129155
-0.34138	448647.6	2.15E+10	-0.00689	201915.7	1	0.006889	0.147303
-0.26038	462668.2	2.22E+10	-0.00708	154468.7	1	0.007078	0.148089
-0.17925	473205.7	2.28E+10	-0.00721	106645.7	1	0.007213	0.147141
-0.09864	480368.9	2.32E+10	-0.00729	58851.93	1	0.007297	0.152667
-0.01737	484347	2.35E+10	-0.00733	10392.18	1	0.007331	0.150081
0.062235	485109.6	2.36E+10	-0.00732	-37340.8	1	0.007318	0.168038
0.143792	482633.7	2.35E+10	-0.00725	-86518.9	1	0.007255	0.161903
0.225529	476734.8	2.33E+10	-0.00714	-136089	1	0.007141	0.153543
0.305689	467442	2.29E+10	-0.00698	-184988	1	0.006978	0.164637
0.387001	454181.2	2.23E+10	-0.00675	-234890	1	0.006755	0.161528
0.469044	436455.3	2.15E+10	-0.00647	-285572	1	0.006467	0.149383
0.550645	413855.3	2.05E+10	-0.00611	-336340	1	0.006107	0.142709
0.632742	385126.3	1.91E+10	-0.00566	-387829	1	0.005659	0.129899
0.714406	348986.2	1.74E+10	-0.0051	-439531	1	0.005104	0.12245
0.796529	302106.9	1.51E+10	-0.0044	-492150	1	0.004396	0.109321
0.878381	238415.1	1.2E+10	-0.00345	-545508	1	0.003447	0.099549
0.96503	125271.2	6.36E+09	-0.00179	-604155	1	0.001793	0.030581
0.970912	-114558	-5.9E+09	0.00161	-617239	1	-0.00161	-0.04197
0.892653	-232687	-1.2E+10	0.00324	-571849	1	-0.00324	-0.07653
0.814215	-304451	-1.6E+10	0.004214	-524201	1	-0.00422	-0.10886
0.735049	-358697	-1.9E+10	0.00494	-475165	1	-0.00494	-0.13222
0.655268	-402197	-2.1E+10	0.005515	-425111	1	-0.00552	-0.14799
0.575511	-437675	-2.3E+10	0.005978	-374582	1	-0.00598	-0.16406
0.494411	-467315	-2.5E+10	0.006359	-322785	1	-0.00636	-0.16355
0.415181	-491039	-2.6E+10	0.006659	-271826	1	-0.00666	-0.18613
0.334864	-510483	-2.7E+10	0.0069	-219845	1	-0.0069	-0.19529
0.255453	-525586	-2.8E+10	0.007082	-168151	1	-0.00708	-0.21562
0.173091	-537231	-2.9E+10	0.007216	-114239	1	-0.00722	-0.19954
0.093036	-544814	-2.9E+10	0.007295	-61559.5	1	-0.0073	-0.21194
0.012964	-548815	-3E+10	0.007327	-8599.5	1	-0.00733	-0.22412
-0.06897	-549224	-3E+10	0.00731	45867.04	1	-0.00731	-0.21335
-0.14984	-545925	-3E+10	0.007244	99905.78	1	-0.00725	-0.21562
-0.23029	-538868	-2.9E+10	0.007128	153935.9	1	-0.00713	-0.22315
-0.31256	-527539	-2.9E+10	0.006956	209483.7	1	-0.00696	-0.20821
-0.39416	-511855	-2.8E+10	0.006727	264885.1	1	-0.00673	-0.20159
-0.47533	-491364	-2.7E+10	0.006437	320325.1	1	-0.00644	-0.20022
-0.55653	-465254	-2.5E+10	0.006073	376145.1	1	-0.00607	-0.19847
-0.63814	-432227	-2.4E+10	0.005621	432655.9	1	-0.00562	-0.1917
-0.72084	-389905	-2.2E+10	0.00505	490428	1	-0.00505	-0.1714
-0.80478	-334170	-1.9E+10	0.004307	549730.9	1	-0.00431	-0.13587
-0.82637	330250.5	1.9E+10	-0.00408	585283.8	1	0.004079	0.130441

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.74672	393752.8	2.27E+10	-0.00484	530894.2	1	0.004841	0.147768
-0.66707	443848.8	2.57E+10	-0.00543	475842.4	1	0.005436	0.165249
-0.58791	484349.3	2.81E+10	-0.00591	420618.1	1	0.005911	0.188599
-0.50817	517873.2	3.01E+10	-0.0063	364582.4	1	0.006299	0.204963
-0.42804	545482.6	3.18E+10	-0.00661	307902.1	1	0.006614	0.216428
-0.34769	567864.6	3.32E+10	-0.00686	250729.7	1	0.006865	0.225091
-0.26694	585548.8	3.43E+10	-0.00706	192973.7	1	0.007058	0.228992
-0.18717	598646.1	3.52E+10	-0.00719	135627.1	1	0.007196	0.244911
-0.10614	607732.7	3.58E+10	-0.00728	77087.73	1	0.007285	0.245181
-0.02557	612681.6	3.62E+10	-0.00732	18613.19	1	0.007324	0.251238
0.055167	613598.8	3.63E+10	-0.00731	-40253.3	1	0.007315	0.255238
0.136125	610418.2	3.62E+10	-0.00725	-99555.3	1	0.007257	0.256491
0.217909	602911.2	3.58E+10	-0.00715	-159745	1	0.007148	0.247552
0.298169	591130.4	3.52E+10	-0.00699	-219096	1	0.006989	0.257414
0.379639	574356.4	3.43E+10	-0.00677	-279646	1	0.006771	0.25234
0.460331	552456.8	3.31E+10	-0.00649	-339935	1	0.006493	0.256877
0.543311	523664.8	3.14E+10	-0.00613	-402299	1	0.006135	0.233179
0.626295	487287.3	2.93E+10	-0.00569	-465082	1	0.00569	0.209433
0.709289	441306.8	2.66E+10	-0.00513	-528366	1	0.005134	0.185567
0.791234	382929.8	2.32E+10	-0.00444	-591468	1	0.004436	0.174641
0.873983	303023.7	1.84E+10	-0.00349	-656096	1	0.003493	0.1538
0.96179	162440.3	9.95E+09	-0.00186	-726795	1	0.001857	0.070555
0.972924	-133855	-8.3E+09	0.001508	-744534	1	-0.00151	-0.0668
0.895262	-284746	-1.8E+10	0.003184	-689587	1	-0.00319	-0.10871
0.816592	-375285	-2.4E+10	0.004176	-631640	1	-0.00418	-0.13819
0.738839	-441945	-2.8E+10	0.004899	-573420	1	-0.0049	-0.17897
0.659207	-496144	-3.1E+10	0.00548	-513155	1	-0.00548	-0.19659
0.579422	-540298	-3.4E+10	0.005949	-452276	1	-0.00595	-0.21231
0.498677	-576915	-3.7E+10	0.006333	-390246	1	-0.00634	-0.21618
0.419201	-606366	-3.9E+10	0.006638	-328832	1	-0.00664	-0.23572
0.337438	-630736	-4E+10	0.006886	-265315	1	-0.00689	-0.22703
0.257597	-649306	-4.1E+10	0.007071	-202987	1	-0.00707	-0.24207
0.176517	-663267	-4.2E+10	0.007205	-139401	1	-0.00721	-0.24182
0.095388	-672543	-4.3E+10	0.007287	-75493.2	1	-0.00729	-0.24096
0.014207	-677264	-4.4E+10	0.00732	-11268	1	-0.00732	-0.23945
-0.06766	-677442	-4.4E+10	0.007304	53777.3	1	-0.00731	-0.2295
-0.14906	-673002	-4.3E+10	0.007238	118733.9	1	-0.00724	-0.22523
-0.23003	-663871	-4.3E+10	0.007122	183622.4	1	-0.00712	-0.22635
-0.31224	-649522	-4.2E+10	0.00695	249794.1	1	-0.00695	-0.21225
-0.39431	-629674	-4.1E+10	0.00672	316173.8	1	-0.00672	-0.19975
-0.47591	-603857	-3.9E+10	0.006427	382502.6	1	-0.00643	-0.19308
-0.55846	-570637	-3.7E+10	0.006055	449971.5	1	-0.00606	-0.17472
-0.64117	-528723	-3.5E+10	0.005593	518003.4	1	-0.00559	-0.15429
-0.72384	-475788	-3.1E+10	0.005016	586505.5	1	-0.00502	-0.13444

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.80716	-406683	-2.7E+10	0.004271	656218.8	1	-0.00427	-0.10654
-0.82378	403680.7	2.74E+10	-0.00409	690431.1	1	0.004095	0.098477
-0.74346	480706.2	3.28E+10	-0.00486	625144.4	1	0.004858	0.107668
-0.6627	541612.4	3.7E+10	-0.00545	558808	1	0.005456	0.111258
-0.58279	590428.4	4.05E+10	-0.00593	492673.4	1	0.00593	0.125501
-0.5018	630903.5	4.33E+10	-0.00632	425218.8	1	0.006319	0.126397
-0.42091	663841.2	4.57E+10	-0.00663	357472.6	1	0.006632	0.128466
-0.33926	690510.3	4.76E+10	-0.00688	288746.7	1	0.006881	0.121094
-0.25836	711073.3	4.92E+10	-0.00707	220347.3	1	0.007069	0.123063
-0.17791	726168.8	5.03E+10	-0.0072	152042.2	1	0.007202	0.13064
-0.09742	736202.8	5.11E+10	-0.00728	83422.65	1	0.007285	0.137696
-0.01451	741395	5.16E+10	-0.00732	12446.82	1	0.007319	0.114775
0.066734	741465.6	5.17E+10	-0.0073	-57371.5	1	0.007304	0.112539
0.146939	736610.7	5.14E+10	-0.00724	-126571	1	0.007239	0.123074
0.230216	726225.3	5.08E+10	-0.00712	-198710	1	0.007121	0.095716
0.310123	710862.3	4.98E+10	-0.00695	-268217	1	0.006954	0.10994
0.392771	688960.5	4.84E+10	-0.00672	-340419	1	0.006723	0.090342
0.47283	661276.1	4.66E+10	-0.00643	-410682	1	0.006437	0.102676
0.555998	624768.8	4.41E+10	-0.00606	-484043	1	0.006065	0.076663
0.637026	580097.9	4.1E+10	-0.00561	-555929	1	0.005616	0.077049
0.719235	523075	3.71E+10	-0.00505	-629358	1	0.005048	0.062872
0.800347	450735.4	3.21E+10	-0.00433	-702437	1	0.004335	0.062225
0.882527	350765	2.5E+10	-0.00336	-777426	1	0.003359	0.048402
0.96658	177328.8	1.27E+10	-0.00169	-856347	1	0.001686	0.011465
0.969842	-168057	-1.2E+10	0.00158	-868258	1	-0.00158	-0.02877
0.889142	-349071	-2.5E+10	0.00326	-800497	1	-0.00326	-0.03321
0.80893	-456315	-3.3E+10	0.004244	-730906	1	-0.00425	-0.04366
0.728457	-536362	-3.9E+10	0.004972	-660128	1	-0.00498	-0.0509
0.647764	-599883	-4.4E+10	0.005545	-588514	1	-0.00555	-0.05541
0.567622	-651088	-4.8E+10	0.006002	-516900	1	-0.00601	-0.06674
0.486748	-693328	-5.1E+10	0.006376	-444219	1	-0.00638	-0.06902
0.406797	-727298	-5.4E+10	0.006673	-372012	1	-0.00668	-0.08269
0.325223	-754990	-5.6E+10	0.006911	-298007	1	-0.00691	-0.07635
0.245138	-775988	-5.8E+10	0.007088	-225051	1	-0.00709	-0.08837
0.164488	-791376	-5.9E+10	0.007213	-151291	1	-0.00722	-0.09342
0.083298	-801306	-6E+10	0.007288	-76756.4	1	-0.00729	-0.0918
0.001544	-805819	-6E+10	0.007314	-1424.93	1	-0.00732	-0.08323
-0.07963	-804888	-6E+10	0.00729	73645.07	1	-0.00729	-0.08185
-0.16026	-798550	-6E+10	0.007218	148494	1	-0.00722	-0.08706
-0.24098	-786612	-5.9E+10	0.007095	223692.4	1	-0.0071	-0.09133
-0.32283	-768460	-5.8E+10	0.006916	300251.3	1	-0.00692	-0.08149
-0.40393	-743970	-5.6E+10	0.006681	376405.4	1	-0.00668	-0.08106
-0.4845	-712488	-5.4E+10	0.006384	452394.8	1	-0.00639	-0.08709
-0.56694	-671763	-5.1E+10	0.006004	530514.5	1	-0.00601	-0.07011

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.64844	-621280	-4.7E+10	0.005539	608170.9	1	-0.00554	-0.06467
-0.73021	-557483	-4.2E+10	0.004956	686597.9	1	-0.00496	-0.05584
-0.81308	-473758	-3.6E+10	0.004197	766754.2	1	-0.0042	-0.03352
-0.8197	479535.9	3.76E+10	-0.00412	793494.9	1	0.004127	0.048151
-0.73845	569931.8	4.48E+10	-0.00489	716864.8	1	0.00489	0.045799
-0.65856	639831.8	5.05E+10	-0.00547	640850.9	1	0.005474	0.060191
-0.5789	696334.3	5.5E+10	-0.00594	564562.6	1	0.005943	0.077475
-0.49776	743343.9	5.89E+10	-0.00633	486436.8	1	0.006329	0.076529
-0.41623	781701.6	6.2E+10	-0.00664	407550.9	1	0.006641	0.070636
-0.33671	811661.6	6.45E+10	-0.00688	330294.3	1	0.006881	0.089675
-0.2541	835784.8	6.66E+10	-0.00707	249714.7	1	0.007071	0.07053
-0.17365	852925.1	6.81E+10	-0.0072	170947.9	1	0.007202	0.078017
-0.09377	864056.6	6.91E+10	-0.00728	92470.24	1	0.007282	0.092617
-0.01209	869552.4	6.96E+10	-0.00731	11944.62	1	0.007314	0.084983
0.068484	869194.9	6.97E+10	-0.00729	-67766.2	1	0.007296	0.090953
0.149192	863020.1	6.93E+10	-0.00723	-147881	1	0.007231	0.095287
0.231206	850591.4	6.85E+10	-0.00711	-229577	1	0.007112	0.083505
0.312142	831912.8	6.71E+10	-0.00694	-310489	1	0.006942	0.085024
0.392467	806573.4	6.52E+10	-0.00671	-391090	1	0.006717	0.094095
0.47506	772671.6	6.25E+10	-0.00642	-474302	1	0.006421	0.075173
0.555057	731132.1	5.93E+10	-0.00606	-555253	1	0.006062	0.08828
0.636501	678296.9	5.51E+10	-0.00561	-638076	1	0.005611	0.083535
0.71863	611320.9	4.98E+10	-0.00504	-722093	1	0.005043	0.07034
0.799659	526478.7	4.3E+10	-0.00433	-805614	1	0.00433	0.070715
0.882561	407999.3	3.34E+10	-0.00334	-892030	1	0.003343	0.047976
0.966092	205107.3	1.69E+10	-0.00167	-981332	1	0.001671	0.017483
0.970348	-189569	-1.6E+10	0.001529	-994501	1	-0.00153	-0.03502
0.889979	-402925	-3.4E+10	0.003232	-916676	1	-0.00323	-0.04353
0.808979	-529479	-4.4E+10	0.004232	-835911	1	-0.00423	-0.04426
0.728718	-622374	-5.2E+10	0.00496	-754911	1	-0.00496	-0.05411
0.648652	-695657	-5.9E+10	0.00553	-673472	1	-0.00553	-0.06637
0.568703	-755016	-6.4E+10	0.005988	-591664	1	-0.00599	-0.08007
0.486966	-804506	-6.8E+10	0.006367	-507605	1	-0.00637	-0.0717
0.406573	-843970	-7.1E+10	0.006666	-424568	1	-0.00667	-0.07993
0.326194	-875462	-7.4E+10	0.006901	-341219	1	-0.00691	-0.08833
0.245635	-899807	-7.6E+10	0.00708	-257380	1	-0.00708	-0.0945
0.163897	-917651	-7.8E+10	0.007207	-172018	1	-0.00721	-0.08612
0.083951	-928744	-7.9E+10	0.007281	-88251.9	1	-0.00728	-0.09986
0.002268	-933762	-8E+10	0.007307	-2388.4	1	-0.00731	-0.09217
-0.08002	-932406	-8E+10	0.007283	84391.32	1	-0.00729	-0.07703
-0.16117	-924681	-7.9E+10	0.00721	170249.1	1	-0.00721	-0.07592
-0.24237	-910382	-7.8E+10	0.007086	256445.2	1	-0.00709	-0.07416
-0.323	-889308	-7.6E+10	0.006909	342320.1	1	-0.00691	-0.0795
-0.40401	-860658	-7.4E+10	0.006674	428909.2	1	-0.00668	-0.08012

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.48604	-823130	-7.1E+10	0.00637	516927.6	1	-0.00637	-0.06811
-0.56731	-776195	-6.7E+10	0.005994	604500.6	1	-0.006	-0.06547
-0.64912	-717102	-6.2E+10	0.005525	693071.7	1	-0.00553	-0.05624
-0.73189	-641721	-5.6E+10	0.004932	783204.5	1	-0.00493	-0.03516
-0.81228	-547181	-4.8E+10	0.004194	871412.5	1	-0.00419	-0.04334
-0.81847	552288.7	4.91E+10	-0.00413	898529.7	1	0.004129	0.033014
-0.738	655074.5	5.84E+10	-0.00488	812192.7	1	0.004883	0.040296
-0.65688	736429	6.58E+10	-0.00547	724477.7	1	0.005476	0.0395
-0.57624	801728.3	7.18E+10	-0.00595	636771.5	1	0.005949	0.044622
-0.49452	855597.4	7.68E+10	-0.00633	547469.6	1	0.006335	0.036501
-0.41291	899154.6	8.08E+10	-0.00664	457917.1	1	0.006645	0.029781
-0.33224	933485.9	8.41E+10	-0.00688	369057.2	1	0.006886	0.034582
-0.25131	960082.8	8.66E+10	-0.00707	279597.1	1	0.00707	0.03611
-0.17088	979298.4	8.85E+10	-0.00719	190404.5	1	0.007199	0.043853
-0.08927	991813.8	8.97E+10	-0.00727	99624.1	1	0.007278	0.037094
-0.00764	997503.1	9.04E+10	-0.0073	8540.423	1	0.007308	0.030078
0.072855	996483.2	9.04E+10	-0.00728	-81552.8	1	0.007288	0.037022
0.153455	988789.4	8.99E+10	-0.00722	-172033	1	0.00722	0.042697
0.236478	973654	8.86E+10	-0.00709	-265525	1	0.007097	0.018472
0.317213	951527.4	8.67E+10	-0.00692	-356732	1	0.006923	0.022471
0.398921	921086	8.41E+10	-0.00669	-449348	1	0.00669	0.014469
0.479585	882201.4	8.07E+10	-0.00639	-541109	1	0.006395	0.01934
0.561189	832586.4	7.63E+10	-0.00602	-634302	1	0.006024	0.012627
0.642063	771156.1	7.08E+10	-0.00557	-727075	1	0.005567	0.014917
0.723896	693112	6.38E+10	-0.00499	-821450	1	0.004992	0.005364
0.805044	593583.1	5.47E+10	-0.00426	-915678	1	0.004264	0.004278
0.885924	457964.7	4.23E+10	-0.00328	-1010558	1	0.003279	0.006494
0.968903	217188.3	2.02E+10	-0.00155	-1110254	1	0.001547	-0.01719
0.969905	-213811	-2E+10	0.00151	-1119850	1	-0.00151	-0.02955
0.888331	-461076	-4.3E+10	0.003241	-1030280	1	-0.00324	-0.02321
0.806814	-605495	-5.7E+10	0.004242	-938404	1	-0.00425	-0.01756
0.725174	-712333	-6.7E+10	0.004978	-845399	1	-0.00498	-0.01039
0.644851	-795247	-7.5E+10	0.005545	-753255	1	-0.00555	-0.01948
0.562759	-863742	-8.2E+10	0.006011	-658578	1	-0.00601	-0.00674
0.482305	-918349	-8.7E+10	0.006379	-565377	1	-0.00638	-0.0142
0.401243	-962901	-9.2E+10	0.006676	-471110	1	-0.00668	-0.01418
0.319638	-998480	-9.5E+10	0.006911	-375879	1	-0.00692	-0.00744
0.238987	-1025384	-9.8E+10	0.007086	-281456	1	-0.00709	-0.01248
0.157204	-1044871	-1E+11	0.007208	-185413	1	-0.00721	-0.00355
0.075489	-1056874	-1E+11	0.00728	-89164.5	1	-0.00728	0.004533
-0.00438	-1061580	-1E+11	0.007301	5175.778	1	-0.0073	-0.0102
-0.08657	-1059212	-1E+11	0.007273	102545.4	1	-0.00728	0.003794
-0.16817	-1049520	-1E+11	0.007195	199486.8	1	-0.0072	0.010408
-0.2486	-1032547	-9.9E+10	0.007067	295328.9	1	-0.00707	0.0027

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.33011	-1007386	-9.7E+10	0.006883	392749	1	-0.00689	0.008283
-0.41134	-973658	-9.4E+10	0.006642	490135.9	1	-0.00665	0.010315
-0.49243	-930352	-9E+10	0.006335	587700.8	1	-0.00634	0.010742
-0.57419	-875403	-8.5E+10	0.00595	686443	1	-0.00595	0.01941
-0.65445	-808103	-7.8E+10	0.005482	783792.7	1	-0.00548	0.009575
-0.73651	-721601	-7E+10	0.004884	883831.8	1	-0.00489	0.021816
-0.81737	-611312	-6E+10	0.004127	983099.8	1	-0.00413	0.019405
-0.81576	627499.1	6.24E+10	-0.00414	1001448	1	0.004146	-0.00049
-0.73421	744179.3	7.42E+10	-0.0049	903351	1	0.004904	-0.00649
-0.65189	836385	8.36E+10	-0.0055	803636.7	1	0.0055	-0.02203
-0.57191	908642.1	9.1E+10	-0.00596	706249.9	1	0.005964	-0.00874
-0.4895	969190.1	9.72E+10	-0.00634	605481.9	1	0.006349	-0.02537
-0.40936	1016825	1.02E+11	-0.00665	507124.1	1	0.00665	-0.014
-0.32865	1055056	1.06E+11	-0.00688	407728.3	1	0.006889	-0.00977
-0.24597	1085080	1.09E+11	-0.00707	305604.1	1	0.007074	-0.02969
-0.16552	1106072	1.12E+11	-0.0072	205927.7	1	0.0072	-0.02225
-0.08451	1119417	1.13E+11	-0.00727	105290.1	1	0.007275	-0.02158
-0.00294	1125180	1.14E+11	-0.0073	3662.697	1	0.007302	-0.02797
0.078659	1123299	1.14E+11	-0.00727	-98264.9	1	0.007279	-0.03458
0.159718	1113764	1.13E+11	-0.0072	-199799	1	0.007206	-0.03458
0.240839	1096314	1.11E+11	-0.00708	-301692	1	0.007082	-0.03533
0.321952	1070505	1.09E+11	-0.0069	-403870	1	0.006905	-0.036
0.40342	1035470	1.05E+11	-0.00666	-506802	1	0.006668	-0.04103
0.482767	991576.2	1.01E+11	-0.00637	-607376	1	0.006375	-0.0199
0.565547	933931.4	9.54E+10	-0.00599	-712674	1	0.005994	-0.04113
0.64508	864825.1	8.85E+10	-0.00554	-814251	1	0.00554	-0.02231
0.726466	776260.4	7.96E+10	-0.00496	-918692	1	0.004962	-0.02634
0.80775	662433.2	6.8E+10	-0.00422	-1023656	1	0.004224	-0.0291
0.889317	505347	5.2E+10	-0.00321	-1129988	1	0.003213	-0.03537
0.970775	228579.3	2.36E+10	-0.00145	-1238628	1	0.001446	-0.04028
0.969281	-238367	-2.5E+10	0.001497	-1244853	1	-0.0015	-0.02185
0.886303	-520419	-5.4E+10	0.003255	-1142964	1	-0.00326	0.001811
0.805185	-680954	-7.1E+10	0.004247	-1040998	1	-0.00425	0.002532
0.723467	-800515	-8.4E+10	0.004981	-937293	1	-0.00499	0.01066
0.642197	-894134	-9.4E+10	0.005553	-833508	1	-0.00556	0.013261
0.560493	-970096	-1E+11	0.006014	-728667	1	-0.00602	0.021218
0.479483	-1031290	-1.1E+11	0.006382	-624303	1	-0.00639	0.020606
0.398922	-1080487	-1.1E+11	0.006676	-520157	1	-0.00668	0.014462
0.316756	-1120201	-1.2E+11	0.006911	-413604	1	-0.00692	0.028103
0.235534	-1150072	-1.2E+11	0.007085	-307964	1	-0.00709	0.030113
0.153764	-1171380	-1.2E+11	0.007206	-201316	1	-0.00721	0.038876
0.073266	-1184168	-1.3E+11	0.007274	-96047.8	1	-0.00728	0.031953
-0.0089	-1189057	-1.3E+11	0.007294	11687.58	1	-0.0073	0.045656
-0.08924	-1185905	-1.3E+11	0.007264	117295.1	1	-0.00727	0.036772

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.17016	-1174682	-1.3E+11	0.007185	223932.9	1	-0.00719	0.034981
-0.25163	-1154922	-1.2E+11	0.007054	331591.8	1	-0.00706	0.040082
-0.33299	-1126211	-1.2E+11	0.006869	439394.5	1	-0.00687	0.043763
-0.41349	-1088205	-1.2E+11	0.006627	546375.2	1	-0.00663	0.036927
-0.49491	-1038909	-1.1E+11	0.006317	654906.3	1	-0.00632	0.041369
-0.57498	-978098	-1.1E+11	0.005937	762000.7	1	-0.00594	0.029141
-0.65595	-901513	-9.7E+10	0.005463	870729.2	1	-0.00547	0.028092
-0.7377	-804270	-8.7E+10	0.004864	981017.9	1	-0.00487	0.036587
-0.8172	-682225	-7.4E+10	0.004117	1088933	1	-0.00412	0.017311
-0.81457	700475.9	7.71E+10	-0.00414	1105678	1	0.004146	-0.01513
-0.73302	830331.1	9.15E+10	-0.0049	996992.3	1	0.004904	-0.02117
-0.65116	932368.2	1.03E+11	-0.00549	887209.1	1	0.005496	-0.03107
-0.56979	1014018	1.12E+11	-0.00596	777566.8	1	0.005967	-0.03495
-0.48889	1079974	1.2E+11	-0.00634	668148.2	1	0.006344	-0.03293
-0.40773	1133480	1.26E+11	-0.00664	558011.2	1	0.006649	-0.03411
-0.32593	1176258	1.31E+11	-0.00688	446649	1	0.006889	-0.04337
-0.2444	1208849	1.34E+11	-0.00707	335364.5	1	0.00707	-0.04908
-0.16399	1231897	1.37E+11	-0.00719	225306	1	0.007195	-0.04109
-0.08363	1246348	1.39E+11	-0.00726	115038.3	1	0.00727	-0.03249
-0.00212	1252488	1.4E+11	-0.00729	2915.615	1	0.007296	-0.03807
0.079407	1250116	1.4E+11	-0.00727	-109502	1	0.007272	-0.04381
0.160973	1239116	1.39E+11	-0.00719	-222257	1	0.007198	-0.05006
0.242001	1219355	1.37E+11	-0.00707	-334550	1	0.007074	-0.04967
0.321893	1190763	1.34E+11	-0.00689	-445555	1	0.006898	-0.03527
0.402743	1151863	1.29E+11	-0.00666	-558196	1	0.006663	-0.03268
0.483584	1101812	1.24E+11	-0.00636	-671152	1	0.006364	-0.02999
0.565246	1038193	1.17E+11	-0.00598	-785626	1	0.005987	-0.03743
0.645591	960165.4	1.08E+11	-0.00552	-898665	1	0.005528	-0.0286
0.725584	862999.6	9.75E+10	-0.00496	-1011699	1	0.004959	-0.01545
0.806481	736936.6	8.35E+10	-0.00422	-1126655	1	0.004226	-0.01345
0.887832	562969.1	6.39E+10	-0.00322	-1243248	1	0.00322	-0.01705
0.967993	265228.7	3.02E+10	-0.00151	-1360467	1	0.00151	-0.00597
0.967748	-268099	-3.1E+10	0.001517	-1368480	1	-0.00152	-0.00295
0.884949	-578457	-6.7E+10	0.003259	-1256035	1	-0.00326	0.018514
0.80306	-757476	-8.7E+10	0.004257	-1142462	1	-0.00426	0.028758
0.721178	-889616	-1E+11	0.004989	-1027914	1	-0.00499	0.038899
0.640696	-991944	-1.1E+11	0.005553	-914687	1	-0.00556	0.031777
0.559475	-1075378	-1.2E+11	0.006011	-799926	1	-0.00602	0.033768
0.478505	-1142945	-1.3E+11	0.006379	-685106	1	-0.00638	0.032676
0.396904	-1197860	-1.4E+11	0.006676	-569028	1	-0.00668	0.039347
0.316432	-1240695	-1.4E+11	0.006905	-454228	1	-0.00691	0.032107
0.235852	-1273398	-1.5E+11	0.007078	-338970	1	-0.00708	0.026198
0.153576	-1297007	-1.5E+11	0.007199	-220991	1	-0.0072	0.041203
0.073145	-1310983	-1.5E+11	0.007268	-105377	1	-0.00727	0.03345

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.00837	-1316202	-1.5E+11	0.007287	12065.61	1	-0.00729	0.039011
-0.08982	-1312472	-1.5E+11	0.007257	129700.9	1	-0.00726	0.043847
-0.17124	-1299672	-1.5E+11	0.007177	247571.9	1	-0.00718	0.048309
-0.25262	-1277482	-1.5E+11	0.007046	365671	1	-0.00705	0.052283
-0.33388	-1245389	-1.5E+11	0.006859	483884.7	1	-0.00686	0.054717
-0.41373	-1203340	-1.4E+11	0.006619	600367.5	1	-0.00662	0.039894
-0.49505	-1148547	-1.4E+11	0.006309	719318.2	1	-0.00631	0.043116
-0.57646	-1079672	-1.3E+11	0.005921	838766.2	1	-0.00592	0.047376
-0.65587	-996074	-1.2E+11	0.005454	955711.2	1	-0.00546	0.027106
-0.73789	-887703	-1.1E+11	0.004852	1076998	1	-0.00485	0.038842
-0.81817	-750502	-8.9E+10	0.004094	1196414	1	-0.0041	0.029319
-0.81391	772392.5	9.31E+10	-0.00414	1210350	1	0.004141	-0.02329
-0.73285	914872.9	1.1E+11	-0.00489	1091804	1	0.004895	-0.0233
-0.65023	1028413	1.24E+11	-0.00549	970291.2	1	0.005493	-0.04253
-0.56938	1117749	1.35E+11	-0.00596	850870.4	1	0.005961	-0.03992
-0.48908	1189899	1.44E+11	-0.00633	731836.6	1	0.006336	-0.03061
-0.40695	1249491	1.52E+11	-0.00664	609720.8	1	0.006644	-0.04384
-0.32631	1295854	1.58E+11	-0.00688	489492.4	1	0.006882	-0.03866
-0.24373	1332111	1.62E+11	-0.00706	366051.9	1	0.007065	-0.05743
-0.16337	1357286	1.65E+11	-0.00718	245651	1	0.00719	-0.0487
-0.0819	1373145	1.68E+11	-0.00726	123281.4	1	0.007265	-0.05386
-0.00222	1379506	1.69E+11	-0.00728	3345.932	1	0.00729	-0.0368
0.080414	1376623	1.68E+11	-0.00726	-121322	1	0.007265	-0.05622
0.161317	1364333	1.67E+11	-0.00719	-243657	1	0.007191	-0.0543
0.242841	1342151	1.65E+11	-0.00706	-367215	1	0.007066	-0.06003
0.32208	1310632	1.61E+11	-0.00689	-487595	1	0.006891	-0.03758
0.403924	1266918	1.56E+11	-0.00665	-612239	1	0.006653	-0.04725
0.484111	1211848	1.49E+11	-0.00635	-734688	1	0.006355	-0.03648
0.565645	1141519	1.41E+11	-0.00597	-859562	1	0.005977	-0.04234
0.646295	1054812	1.3E+11	-0.00551	-983499	1	0.005515	-0.0373
0.726083	947526.3	1.17E+11	-0.00494	-1106600	1	0.004946	-0.02161
0.807074	807786.9	1E+11	-0.00421	-1232206	1	0.004208	-0.02077
0.887506	617479.6	7.66E+10	-0.00321	-1357929	1	0.003209	-0.01302
0.96944	274997.4	3.42E+10	-0.00142	-1488484	1	0.001423	-0.02382
0.967385	-290552	-3.6E+10	0.001495	-1493385	1	-0.0015	0.001529
0.884273	-634671	-8E+10	0.003253	-1369761	1	-0.00326	0.026861
0.802852	-830440	-1E+11	0.004247	-1246283	1	-0.00425	0.031312
0.721138	-975464	-1.2E+11	0.004979	-1121377	1	-0.00498	0.039389
0.639899	-1088864	-1.4E+11	0.005549	-996548	1	-0.00555	0.041614
0.558732	-1180246	-1.5E+11	0.006006	-871335	1	-0.00601	0.042944
0.477816	-1254202	-1.6E+11	0.006374	-746097	1	-0.00638	0.041165
0.396274	-1314262	-1.7E+11	0.00667	-619524	1	-0.00668	0.047121
0.314741	-1361633	-1.7E+11	0.006902	-492633	1	-0.00691	0.052963
0.233046	-1397604	-1.8E+11	0.007076	-365178	1	-0.00708	0.060807

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.15313	-1422437	-1.8E+11	0.007193	-240212	1	-0.0072	0.046702
0.072175	-1437641	-1.8E+11	0.007261	-113342	1	-0.00727	0.045418
-0.00868	-1443087	-1.8E+11	0.007281	13643.34	1	-0.00729	0.042878
-0.09006	-1438794	-1.8E+11	0.00725	141734.1	1	-0.00726	0.046852
-0.17141	-1424556	-1.8E+11	0.00717	270052.9	1	-0.00718	0.050425
-0.25214	-1400234	-1.8E+11	0.00704	397686.5	1	-0.00704	0.046433
-0.33445	-1364360	-1.7E+11	0.006851	528103.1	1	-0.00686	0.061812
-0.41314	-1318704	-1.7E+11	0.006614	653083.8	1	-0.00662	0.032535
-0.49542	-1257672	-1.6E+11	0.006299	784116.1	1	-0.0063	0.047634
-0.57718	-1181424	-1.5E+11	0.005909	914699.8	1	-0.00591	0.056316
-0.65687	-1088917	-1.4E+11	0.005439	1042389	1	-0.00544	0.039405
-0.73745	-971700	-1.3E+11	0.004845	1172021	1	-0.00485	0.033517
-0.81796	-820505	-1.1E+11	0.004084	1302206	1	-0.00409	0.02668
-0.81377	842956	1.1E+11	-0.00413	1315666	1	0.004131	-0.025
-0.73171	1000711	1.31E+11	-0.00489	1185013	1	0.004895	-0.03738
-0.65	1123393	1.48E+11	-0.00548	1054235	1	0.005486	-0.04542
-0.56925	1220941	1.61E+11	-0.00595	924494.3	1	0.005954	-0.04156
-0.48699	1301503	1.72E+11	-0.00633	791883.6	1	0.006338	-0.05642
-0.40646	1365092	1.8E+11	-0.00663	661707.4	1	0.006639	-0.04985
-0.32644	1415270	1.87E+11	-0.00687	532028.9	1	0.006875	-0.03699
-0.24451	1454546	1.92E+11	-0.00705	398942.4	1	0.007057	-0.0477
-0.16367	1482180	1.96E+11	-0.00718	267314.4	1	0.007183	-0.0451
-0.08226	1499402	1.99E+11	-0.00725	134498.6	1	0.007258	-0.04936
-0.00147	1506278	2E+11	-0.00728	2410.574	1	0.007283	-0.04602
0.081089	1502875	2E+11	-0.00725	-132859	1	0.007259	-0.06455
0.160752	1489477	1.98E+11	-0.00718	-263651	1	0.007186	-0.04733
0.241069	1465573	1.95E+11	-0.00706	-395793	1	0.007063	-0.03817
0.322513	1430078	1.91E+11	-0.00688	-530082	1	0.006883	-0.04291
0.403723	1382465	1.84E+11	-0.00664	-664295	1	0.006646	-0.04477
0.483833	1322181	1.77E+11	-0.00634	-797018	1	0.006349	-0.03306
0.565771	1244740	1.66E+11	-0.00596	-933139	1	0.005969	-0.0439
0.645408	1151041	1.54E+11	-0.00551	-1065847	1	0.005512	-0.02635
0.726755	1031241	1.38E+11	-0.00493	-1201907	1	0.00493	-0.0299
0.80713	879128.8	1.18E+11	-0.00419	-1336993	1	0.004196	-0.02146
0.887551	670413	9.02E+10	-0.00319	-1473143	1	0.003192	-0.01358
0.96798	303719.8	4.1E+10	-0.00144	-1611725	1	0.001441	-0.00581
0.965624	-322587	-4.4E+10	0.001522	-1615974	1	-0.00152	0.023259
0.883744	-690202	-9.4E+10	0.003245	-1483537	1	-0.00325	0.033385
0.801115	-906414	-1.2E+11	0.004253	-1347506	1	-0.00426	0.052748
0.719709	-1063306	-1.5E+11	0.004981	-1212505	1	-0.00499	0.057025
0.639354	-1185276	-1.6E+11	0.005544	-1078613	1	-0.00555	0.048335
0.557287	-1285701	-1.8E+11	0.006005	-941366	1	-0.00601	0.060765
0.476385	-1365921	-1.9E+11	0.006372	-805652	1	-0.00638	0.058829
0.395947	-1430232	-2E+11	0.006664	-670361	1	-0.00667	0.051152

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	$\boldsymbol{B_{ii}}$	b
0.313366	-1482263	-2E+11	0.006898	-531126	1	-0.0069	0.069933
0.233433	-1520415	-2.1E+11	0.007068	-396053	1	-0.00707	0.056034
0.151845	-1547840	-2.1E+11	0.007188	-257891	1	-0.00719	0.062556
0.070947	-1564085	-2.2E+11	0.007255	-120616	1	-0.00726	0.060566
-0.01043	-1569712	-2.2E+11	0.007274	17752.41	1	-0.00728	0.064504
-0.09115	-1564736	-2.2E+11	0.007243	155271.8	1	-0.00725	0.060295
-0.17184	-1549096	-2.1E+11	0.007163	293012	1	-0.00717	0.055694
-0.25306	-1522186	-2.1E+11	0.007031	431959.8	1	-0.00704	0.057767
-0.3336	-1483783	-2.1E+11	0.006846	570023.4	1	-0.00685	0.051357
-0.41331	-1433295	-2E+11	0.006606	706967.2	1	-0.00661	0.03471
-0.49601	-1366235	-1.9E+11	0.006289	849385.3	1	-0.00629	0.054907
-0.57716	-1283565	-1.8E+11	0.005901	989515.4	1	-0.00591	0.056012
-0.65674	-1182772	-1.6E+11	0.00543	1127351	1	-0.00543	0.037739
-0.73759	-1054427	-1.5E+11	0.004834	1267904	1	-0.00484	0.035164
-0.81921	-886668	-1.2E+11	0.004058	1410493	1	-0.00406	0.042094
-0.81248	915879.5	1.3E+11	-0.00413	1418953	1	0.004133	-0.04092
-0.73079	1086086	1.54E+11	-0.00489	1278297	1	0.004892	-0.0487
-0.64957	1218367	1.73E+11	-0.00547	1137767	1	0.00548	-0.05072
-0.56842	1324656	1.88E+11	-0.00594	996865.6	1	0.00595	-0.05176
-0.48621	1411856	2.01E+11	-0.00633	853674.9	1	0.006334	-0.06599
-0.4052	1481050	2.11E+11	-0.00663	712207.4	1	0.006636	-0.0654
-0.32466	1535536	2.19E+11	-0.00687	571236.6	1	0.006873	-0.05897
-0.24219	1578024	2.25E+11	-0.00705	426568.9	1	0.007055	-0.07636
-0.16254	1607215	2.3E+11	-0.00717	286551.4	1	0.007178	-0.05903
-0.08002	1625794	2.33E+11	-0.00725	141205.6	1	0.007253	-0.07707
0.000713	1632812	2.34E+11	-0.00727	-1259.12	1	0.007277	-0.07298
0.082022	1628815	2.33E+11	-0.00725	-145025	1	0.007252	-0.07606
0.162772	1613716	2.32E+11	-0.00717	-288078	1	0.007177	-0.07224
0.243555	1587138	2.28E+11	-0.00705	-431472	1	0.007052	-0.06884
0.32319	1549070	2.23E+11	-0.00687	-573110	1	0.006875	-0.05126
0.404835	1496793	2.15E+11	-0.00663	-718635	1	0.006636	-0.05849
0.48481	1431104	2.06E+11	-0.00633	-861511	1	0.006337	-0.04512
0.566592	1346821	1.94E+11	-0.00595	-1007984	1	0.005957	-0.05403
0.647392	1243033	1.79E+11	-0.00549	-1153119	1	0.00549	-0.05083
0.727597	1114070	1.61E+11	-0.00491	-1297685	1	0.004914	-0.04029
0.808687	946447.5	1.37E+11	-0.00416	-1444505	1	0.004167	-0.04067
0.888988	717928.8	1.04E+11	-0.00315	-1590902	1	0.003155	-0.03131
0.967731	322910.3	4.7E+10	-0.00141	-1736867	1	0.001414	-0.00273
0.966599	-333477	-4.9E+10	0.001453	-1742754	1	-0.00146	0.011233
0.883099	-745838	-1.1E+11	0.003239	-1596956	1	-0.00324	0.041336
0.799571	-982112	-1.4E+11	0.004257	-1448610	1	-0.00426	0.071793
0.719314	-1149269	-1.7E+11	0.004974	-1305105	1	-0.00498	0.061891
0.638612	-1281759	-1.9E+11	0.005539	-1160170	1	-0.00555	0.057489
0.556604	-1390191	-2E+11	0.006001	-1012387	1	-0.00601	0.069194

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.47576	-1476766	-2.2E+11	0.006367	-866289	1	-0.00637	0.066528
0.394843	-1546549	-2.3E+11	0.00666	-719696	1	-0.00667	0.064777
0.312305	-1602545	-2.4E+11	0.006894	-569829	1	-0.0069	0.083015
0.231848	-1643790	-2.4E+11	0.007064	-423433	1	-0.00707	0.075589
0.151471	-1672763	-2.5E+11	0.007182	-276897	1	-0.00719	0.067171
0.077107	-1689184	-2.5E+11	0.007245	-141076	1	-0.00725	-0.01542
-0.01244	-1696061	-2.5E+11	0.007267	22785.67	1	-0.00727	0.089293
-0.09132	-1690511	-2.5E+11	0.007236	167406	1	-0.00724	0.062413
-0.17484	-1672604	-2.5E+11	0.007152	320815.2	1	-0.00716	0.092742
-0.25366	-1643939	-2.4E+11	0.007023	465875.1	1	-0.00703	0.065138
-0.33522	-1601522	-2.4E+11	0.006835	616266.3	1	-0.00684	0.071301
-0.41533	-1546141	-2.3E+11	0.006591	764291.7	1	-0.0066	0.059579
-0.49682	-1474184	-2.2E+11	0.006278	915210.3	1	-0.00628	0.064919
-0.57734	-1385132	-2.1E+11	0.005892	1064704	1	-0.0059	0.058295
-0.65767	-1274671	-1.9E+11	0.005415	1214264	1	-0.00542	0.049299
-0.7375	-1137154	-1.7E+11	0.004824	1363394	1	-0.00483	0.034085
-0.81796	-958281	-1.4E+11	0.004059	1514395	1	-0.00406	0.026732
-0.81372	982305.3	1.49E+11	-0.0041	1526556	1	0.004107	-0.02568
-0.73168	1167781	1.78E+11	-0.00487	1374683	1	0.004875	-0.03774
-0.64894	1313401	2E+11	-0.00547	1220814	1	0.005475	-0.05843
-0.56979	1425411	2.18E+11	-0.00593	1073118	1	0.005935	-0.03486
-0.48675	1520676	2.32E+11	-0.00632	917718.5	1	0.006324	-0.05935
-0.40585	1595296	2.44E+11	-0.00662	765969.7	1	0.006627	-0.05733
-0.32431	1654769	2.53E+11	-0.00686	612663.3	1	0.006867	-0.06333
-0.24305	1699880	2.61E+11	-0.00704	459585.7	1	0.007047	-0.06579
-0.1629	1731554	2.66E+11	-0.00716	308308.3	1	0.007172	-0.05456
-0.08163	1751322	2.69E+11	-0.00724	154629.7	1	0.007246	-0.05719
0.000207	1759024	2.7E+11	-0.00726	-392.256	1	0.007271	-0.06674
0.080862	1754735	2.7E+11	-0.00724	-153457	1	0.007246	-0.06176
0.162136	1738335	2.68E+11	-0.00717	-307974	1	0.007172	-0.0644
0.243432	1709395	2.63E+11	-0.00704	-462815	1	0.007045	-0.06732
0.323553	1667908	2.57E+11	-0.00686	-615709	1	0.006868	-0.05574
0.404031	1612208	2.49E+11	-0.00663	-769589	1	0.006632	-0.04857
0.48447	1540888	2.38E+11	-0.00633	-923723	1	0.006331	-0.04092
0.56521	1451163	2.25E+11	-0.00595	-1078795	1	0.005956	-0.03698
0.647785	1336677	2.07E+11	-0.00547	-1237826	1	0.005479	-0.05567
0.726621	1199815	1.86E+11	-0.00491	-1390147	1	0.004912	-0.02825
0.806329	1022739	1.59E+11	-0.00418	-1544791	1	0.00418	-0.01158
0.887198	776085.7	1.21E+11	-0.00316	-1702687	1	0.003166	-0.00923
0.968514	331268.9	5.17E+10	-0.00135	-1864029	1	0.001347	-0.0124
0.966753	-348727	-5.5E+10	0.001411	-1868261	1	-0.00141	0.009325
0.88417	-794881	-1.2E+11	0.003208	-1713444	1	-0.00321	0.028129
0.802006	-1047724	-1.6E+11	0.00422	-1556910	1	-0.00423	0.041749
0.719951	-1232926	-1.9E+11	0.004959	-1399573	1	-0.00496	0.054033

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.638572	-1376958	-2.2E+11	0.005531	-1242877	1	-0.00554	0.057987
0.558138	-1491651	-2.4E+11	0.005985	-1087510	1	-0.00599	0.050264
0.476458	-1586036	-2.5E+11	0.006356	-929319	1	-0.00636	0.057918
0.395662	-1661117	-2.6E+11	0.00665	-772475	1	-0.00666	0.054669
0.314353	-1720660	-2.7E+11	0.006882	-614304	1	-0.00689	0.057752
0.233449	-1765500	-2.8E+11	0.007055	-456615	1	-0.00706	0.055835
0.152006	-1797153	-2.9E+11	0.007174	-297580	1	-0.00718	0.060571
0.070073	-1815995	-2.9E+11	0.007243	-137301	1	-0.00725	0.071352
-0.00881	-1822138	-2.9E+11	0.007261	17286.24	1	-0.00727	0.044557
-0.09116	-1816066	-2.9E+11	0.00723	178936.1	1	-0.00724	0.060469
-0.16765	-1798841	-2.9E+11	0.007155	329334.4	1	-0.00716	0.004055
-0.2528	-1766066	-2.8E+11	0.007018	497064	1	-0.00702	0.054509
-0.3332	-1721191	-2.7E+11	0.006833	655733.8	1	-0.00684	0.046353
-0.41437	-1660954	-2.7E+11	0.006587	816240.5	1	-0.00659	0.047732
-0.4948	-1584717	-2.5E+11	0.006278	975617.6	1	-0.00628	0.039985
-0.57583	-1488596	-2.4E+11	0.005891	1136547	1	-0.0059	0.039584
-0.65617	-1370030	-2.2E+11	0.005416	1296534	1	-0.00542	0.0307
-0.73758	-1219163	-2E+11	0.004813	1459190	1	-0.00482	0.03511
-0.81818	-1025889	-1.6E+11	0.004044	1620897	1	-0.00405	0.029386
-0.81213	1055643	1.72E+11	-0.00411	1628902	1	0.004112	-0.04517
-0.7312	1251721	2.04E+11	-0.00486	1468581	1	0.004869	-0.04358
-0.64945	1406322	2.29E+11	-0.00546	1305950	1	0.005463	-0.05213
-0.56854	1529169	2.5E+11	-0.00593	1144482	1	0.005934	-0.05027
-0.48757	1628798	2.66E+11	-0.00631	982464.5	1	0.006313	-0.0492
-0.40573	1709857	2.8E+11	-0.00661	818340.6	1	0.006621	-0.0588
-0.32482	1773159	2.9E+11	-0.00685	655739.5	1	0.006859	-0.05698
-0.24365	1821533	2.98E+11	-0.00703	492311.7	1	0.00704	-0.05833
-0.16242	1855894	3.04E+11	-0.00716	328467.2	1	0.007166	-0.06042
-0.0818	1876804	3.08E+11	-0.00723	165569.4	1	0.00724	-0.05501
-0.00063	1884964	3.1E+11	-0.00726	1273.303	1	0.007265	-0.05643
0.079375	1880455	3.09E+11	-0.00723	-160922	1	0.007241	-0.04341
0.162338	1862462	3.06E+11	-0.00716	-329403	1	0.007165	-0.06689
0.242417	1831804	3.02E+11	-0.00703	-492309	1	0.007041	-0.0548
0.323047	1786997	2.94E+11	-0.00686	-656624	1	0.006862	-0.0495
0.404001	1726775	2.85E+11	-0.00662	-821908	1	0.006625	-0.0482
0.483843	1650728	2.72E+11	-0.00632	-985249	1	0.006327	-0.03319
0.56597	1552590	2.57E+11	-0.00594	-1153636	1	0.005944	-0.04636
0.646596	1432517	2.37E+11	-0.00547	-1319365	1	0.005478	-0.041
0.726218	1284261	2.13E+11	-0.0049	-1483528	1	0.004905	-0.02328
0.807501	1089788	1.81E+11	-0.00415	-1651775	1	0.004156	-0.02604
0.887569	825991.3	1.37E+11	-0.00314	-1818512	1	0.003145	-0.0138
0.967991	350236.2	5.83E+10	-0.00133	-1988554	1	0.001329	-0.00595
0.967599	-355099	-5.9E+10	0.001342	-1995137	1	-0.00134	-0.00111
0.884333	-846367	-1.4E+11	0.003189	-1828355	1	-0.00319	0.026122

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.801879	-1119052	-1.9E+11	0.004209	-1660593	1	-0.00421	0.043325
0.720007	-1317259	-2.2E+11	0.004948	-1492999	1	-0.00495	0.053353
0.63923	-1470641	-2.5E+11	0.005518	-1327000	1	-0.00552	0.049865
0.558484	-1594145	-2.7E+11	0.005975	-1160569	1	-0.00598	0.045996
0.476928	-1695156	-2.9E+11	0.006347	-992049	1	-0.00635	0.052124
0.396243	-1775513	-3E+11	0.006641	-824965	1	-0.00665	0.047507
0.31392	-1840009	-3.1E+11	0.006876	-654152	1	-0.00688	0.063098
0.234224	-1887248	-3.2E+11	0.007046	-488488	1	-0.00705	0.046274
0.152285	-1921337	-3.3E+11	0.007167	-317866	1	-0.00717	0.057131
0.072186	-1941148	-3.3E+11	0.007235	-150796	1	-0.00724	0.045285
-0.00898	-1947958	-3.3E+11	0.007254	18783.75	1	-0.00726	0.046651
-0.09126	-1941316	-3.3E+11	0.007223	190962.8	1	-0.00723	0.0617
-0.17116	-1921625	-3.3E+11	0.007144	358435.7	1	-0.00715	0.047389
-0.25162	-1888140	-3.2E+11	0.007013	527356.4	1	-0.00702	0.039955
-0.33419	-1838673	-3.1E+11	0.006823	701017.7	1	-0.00683	0.058598
-0.41418	-1774901	-3E+11	0.006581	869555.8	1	-0.00659	0.045377
-0.49555	-1692104	-2.9E+11	0.006267	1041346	1	-0.00627	0.049222
-0.57643	-1589019	-2.7E+11	0.00588	1212478	1	-0.00589	0.047038
-0.65617	-1462730	-2.5E+11	0.005407	1381610	1	-0.00541	0.030743
-0.73745	-1301251	-2.2E+11	0.004804	1554525	1	-0.00481	0.033415
-0.81755	-1095445	-1.9E+11	0.004039	1725637	1	-0.00404	0.021655
-0.8134	1120651	1.94E+11	-0.00408	1736817	1	0.004087	-0.02954
-0.73174	1333124	2.31E+11	-0.00485	1564479	1	0.004855	-0.03698
-0.64932	1499931	2.6E+11	-0.00545	1389851	1	0.005455	-0.0537
-0.56949	1629642	2.83E+11	-0.00591	1220178	1	0.005921	-0.03863
-0.48766	1737366	3.02E+11	-0.0063	1045842	1	0.006306	-0.04814
-0.40645	1823357	3.17E+11	-0.0066	872462.1	1	0.006612	-0.04995
-0.32454	1891809	3.29E+11	-0.00685	697229.8	1	0.006853	-0.0605
-0.24401	1943008	3.39E+11	-0.00702	524651.2	1	0.007033	-0.05397
-0.16286	1979658	3.45E+11	-0.00715	350448.7	1	0.007159	-0.05509
-0.08172	2002066	3.5E+11	-0.00723	175999.6	1	0.007234	-0.056
-0.00062	2010656	3.51E+11	-0.00725	1328.24	1	0.007259	-0.05658
0.079906	2005647	3.51E+11	-0.00723	-172358	1	0.007235	-0.04996
0.160472	1986974	3.48E+11	-0.00715	-346411	1	0.007161	-0.04387
0.242224	1953597	3.42E+11	-0.00703	-523315	1	0.007035	-0.05242
0.322229	1906068	3.34E+11	-0.00685	-696726	1	0.006858	-0.03941
0.404205	1840845	3.23E+11	-0.00661	-874723	1	0.006617	-0.05072
0.484466	1758936	3.09E+11	-0.00631	-1049326	1	0.006317	-0.04087
0.565495	1655318	2.91E+11	-0.00593	-1225967	1	0.005939	-0.04049
0.646489	1526363	2.68E+11	-0.00547	-1402956	1	0.00547	-0.03969
0.726387	1367166	2.41E+11	-0.00489	-1578044	1	0.004894	-0.02536
0.80713	1160472	2.05E+11	-0.00414	-1755647	1	0.004148	-0.02146
0.887788	875731.1	1.55E+11	-0.00312	-1934082	1	0.003125	-0.01651
0.968758	353677.3	6.26E+10	-0.00126	-2115892	1	0.001258	-0.01541

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.966934	-375224	-6.7E+10	0.00133	-2119064	1	-0.00133	0.0071
0.884377	-897683	-1.6E+11	0.003173	-1943037	1	-0.00318	0.025577
0.801941	-1189423	-2.1E+11	0.004197	-1764641	1	-0.0042	0.042555
0.721086	-1398871	-2.5E+11	0.00493	-1588661	1	-0.00494	0.040035
0.639242	-1565004	-2.8E+11	0.005509	-1409869	1	-0.00552	0.049714
0.558617	-1696573	-3E+11	0.005966	-1233239	1	-0.00597	0.044363
0.476656	-1804787	-3.2E+11	0.00634	-1053262	1	-0.00635	0.055476
0.396586	-1889779	-3.4E+11	0.006633	-877076	1	-0.00664	0.043274
0.314358	-1958485	-3.5E+11	0.006868	-695804	1	-0.00688	0.057696
0.234176	-2009115	-3.6E+11	0.00704	-518739	1	-0.00705	0.046868
0.152308	-2045332	-3.7E+11	0.007161	-337655	1	-0.00717	0.056839
0.071692	-2066447	-3.7E+11	0.007229	-159056	1	-0.00724	0.051381
-0.00882	-2073512	-3.7E+11	0.007248	19590.66	1	-0.00726	0.044664
-0.09045	-2066458	-3.7E+11	0.007217	200979	1	-0.00722	0.051644
-0.17087	-2045310	-3.7E+11	0.007137	379963.7	1	-0.00715	0.043748
-0.2524	-2008966	-3.6E+11	0.007005	561703	1	-0.00701	0.049536
-0.33321	-1957246	-3.5E+11	0.006819	742144.8	1	-0.00683	0.046508
-0.41422	-1888357	-3.4E+11	0.006573	923337.6	1	-0.00658	0.045906
-0.49449	-1801274	-3.3E+11	0.006264	1103195	1	-0.00627	0.036105
-0.57581	-1690858	-3.1E+11	0.005875	1285807	1	-0.00588	0.039403
-0.6568	-1553898	-2.8E+11	0.005393	1468093	1	-0.0054	0.03854
-0.73748	-1382588	-2.5E+11	0.004793	1650195	1	-0.0048	0.033813
-0.81837	-1160198	-2.1E+11	0.004017	1833493	1	-0.00402	0.031809
-0.81352	1188351	2.18E+11	-0.00407	1842483	1	0.004073	-0.02806
-0.7321	1414407	2.6E+11	-0.00484	1660107	1	0.004842	-0.03254
-0.65033	1591055	2.93E+11	-0.00543	1476253	1	0.00544	-0.04133
-0.56878	1732185	3.19E+11	-0.00591	1292381	1	0.005917	-0.04736
-0.48803	1845260	3.4E+11	-0.00629	1109880	1	0.006297	-0.04355
-0.40693	1936719	3.57E+11	-0.0066	926226.4	1	0.006603	-0.04402
-0.32623	2008666	3.71E+11	-0.00683	743129.1	1	0.006842	-0.03964
-0.24411	2064332	3.81E+11	-0.00702	556508.9	1	0.007026	-0.05269
-0.16303	2103234	3.89E+11	-0.00714	371959	1	0.007153	-0.0529
-0.08256	2126878	3.94E+11	-0.00722	188495.8	1	0.007227	-0.0457
-0.00093	2136077	3.96E+11	-0.00724	2135.216	1	0.007253	-0.05266
0.078936	2130796	3.95E+11	-0.00722	-180493	1	0.007229	-0.03799
0.160602	2110618	3.91E+11	-0.00715	-367508	1	0.007155	-0.04548
0.242281	2074997	3.85E+11	-0.00702	-554839	1	0.007029	-0.05312
0.322211	2024346	3.76E+11	-0.00684	-738449	1	0.006851	-0.03919
0.403036	1955927	3.64E+11	-0.00661	-924420	1	0.006614	-0.0363
0.483776	1868346	3.48E+11	-0.00631	-1110527	1	0.006312	-0.03235
0.56523	1757497	3.27E+11	-0.00593	-1298652	1	0.005932	-0.03723
0.646134	1620363	3.02E+11	-0.00546	-1485928	1	0.005464	-0.03531
0.725934	1451119	2.71E+11	-0.00488	-1671149	1	0.004888	-0.01978
0.805898	1233544	2.3E+11	-0.00415	-1857398	1	0.00415	-0.00626

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.886596	931225.6	1.74E+11	-0.00312	-2046374	1	0.003128	-0.00181
0.967688	376983.9	7.07E+10	-0.00126	-2238951	1	0.001262	-0.0022
0.966962	-386522	-7.3E+10	0.001289	-2244333	1	-0.00129	0.006755
0.884048	-950095	-1.8E+11	0.003162	-2056862	1	-0.00317	0.029635
0.802532	-1257661	-2.4E+11	0.004179	-1869906	1	-0.00418	0.035271
0.721984	-1480227	-2.8E+11	0.004912	-1684170	1	-0.00492	0.028959
0.640397	-1656823	-3.1E+11	0.005492	-1495377	1	-0.0055	0.035472
0.55902	-1798239	-3.4E+11	0.005956	-1306562	1	-0.00596	0.039387
0.477188	-1913150	-3.6E+11	0.00633	-1116267	1	-0.00634	0.048921
0.39723	-2003416	-3.8E+11	0.006623	-929971	1	-0.00663	0.035324
0.315664	-2075970	-3.9E+11	0.006858	-739593	1	-0.00687	0.041578
0.234449	-2130532	-4E+11	0.007032	-549724	1	-0.00704	0.043502
0.153239	-2168721	-4.1E+11	0.007153	-359574	1	-0.00716	0.045359
0.0727	-2191213	-4.2E+11	0.007221	-170714	1	-0.00723	0.038939
-0.00833	-2198802	-4.2E+11	0.007241	19579.49	1	-0.00725	0.038603
-0.09048	-2191198	-4.2E+11	0.00721	212765.1	1	-0.00722	0.051982
-0.17083	-2168646	-4.1E+11	0.007131	402009.9	1	-0.00714	0.043236
-0.25228	-2129977	-4.1E+11	0.006998	594148	1	-0.00701	0.048149
-0.33247	-2075440	-4E+11	0.006814	783587	1	-0.00682	0.037418
-0.41396	-2001827	-3.8E+11	0.006567	976412.4	1	-0.00657	0.042733
-0.49415	-1909362	-3.7E+11	0.006258	1166495	1	-0.00626	0.032012
-0.57493	-1792923	-3.4E+11	0.005871	1358329	1	-0.00588	0.028471
-0.655	-1649444	-3.2E+11	0.005396	1548923	1	-0.0054	0.016294
-0.73807	-1462110	-2.8E+11	0.004778	1747190	1	-0.00478	0.041072
-0.81771	-1229188	-2.4E+11	0.004012	1937968	1	-0.00402	0.023567
-0.81284	1258278	2.44E+11	-0.00406	1946260	1	0.004069	-0.03652
-0.73229	1495646	2.91E+11	-0.00482	1755402	1	0.00483	-0.03025
-0.65025	1683797	3.28E+11	-0.00542	1560336	1	0.005432	-0.04226
-0.57026	1830939	3.56E+11	-0.00589	1369606	1	0.005901	-0.02913
-0.48818	1953097	3.81E+11	-0.00628	1173469	1	0.006289	-0.04173
-0.40718	2049986	4E+11	-0.00659	979544.7	1	0.006595	-0.04098
-0.32656	2126206	4.15E+11	-0.00683	786209.6	1	0.006835	-0.03548
-0.2451	2184833	4.27E+11	-0.00701	590525.7	1	0.007018	-0.04046
-0.16354	2226401	4.35E+11	-0.00714	394296.4	1	0.007146	-0.0467
-0.08372	2251335	4.4E+11	-0.00721	201997.4	1	0.007221	-0.03135
-0.00217	2261212	4.43E+11	-0.00724	5239.827	1	0.007247	-0.03741
0.078222	2255657	4.42E+11	-0.00721	-188998	1	0.007224	-0.02919
0.159247	2234525	4.38E+11	-0.00714	-385040	1	0.007151	-0.02877
0.240879	2196948	4.31E+11	-0.00702	-582836	1	0.007025	-0.03582
0.321326	2143025	4.21E+11	-0.00684	-778055	1	0.006847	-0.02827
0.402638	2070072	4.07E+11	-0.0066	-975683	1	0.006609	-0.03139
0.482287	1978586	3.89E+11	-0.0063	-1169594	1	0.006311	-0.01399
0.563253	1862158	3.66E+11	-0.00593	-1367078	1	0.005935	-0.01284
0.644201	1717256	3.38E+11	-0.00546	-1564937	1	0.005468	-0.01146

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.724467	1537380	3.03E+11	-0.00488	-1761633	1	0.00489	-0.00168
0.806157	1301691	2.57E+11	-0.00413	-1962489	1	0.004135	-0.00945
0.886023	983788.8	1.94E+11	-0.00312	-2159875	1	0.00312	0.005272
0.967868	383671	7.6E+10	-0.00121	-2364920	1	0.001213	-0.00443
0.966403	-403777	-8E+10	0.001273	-2368220	1	-0.00128	0.013644
0.884881	-996427	-2E+11	0.003133	-2173391	1	-0.00314	0.019351
0.803306	-1324604	-2.6E+11	0.004159	-1975756	1	-0.00416	0.025717
0.721906	-1563449	-3.1E+11	0.004903	-1777519	1	-0.00491	0.029914
0.641351	-1748440	-3.5E+11	0.005477	-1580682	1	-0.00549	0.023699
0.56114	-1896671	-3.8E+11	0.005936	-1384190	1	-0.00594	0.013238
0.478519	-2020069	-4E+11	0.006317	-1181366	1	-0.00633	0.032498
0.398712	-2115797	-4.2E+11	0.006611	-985086	1	-0.00662	0.017047
0.316724	-2193260	-4.4E+11	0.006848	-783108	1	-0.00686	0.028498
0.236752	-2250432	-4.5E+11	0.007021	-585789	1	-0.00703	0.015088
0.154494	-2291674	-4.6E+11	0.007145	-382534	1	-0.00715	0.029873
0.073451	-2315743	-4.7E+11	0.007214	-181993	1	-0.00722	0.029676
-0.00692	-2323827	-4.7E+11	0.007235	17167.19	1	-0.00724	0.02123
-0.08783	-2316194	-4.7E+11	0.007206	217931.3	1	-0.00721	0.019401
-0.16757	-2292986	-4.6E+11	0.007128	416042.1	1	-0.00714	0.003031
-0.24958	-2252296	-4.5E+11	0.006997	620109.9	1	-0.00701	0.01481
-0.33032	-2194600	-4.4E+11	0.006812	821302	1	-0.00682	0.010864
-0.41021	-2118814	-4.3E+11	0.006572	1020678	1	-0.00658	-0.00358
-0.49252	-2018971	-4.1E+11	0.006257	1226446	1	-0.00626	0.011872
-0.5733	-1896096	-3.8E+11	0.005871	1428748	1	-0.00588	0.008367
-0.65339	-1744651	-3.5E+11	0.005398	1629758	1	-0.0054	-0.00358
-0.73455	-1552092	-3.1E+11	0.004797	1833987	1	-0.0048	-0.00227
-0.81555	-1303272	-2.6E+11	0.004023	2038495	1	-0.00403	-0.00301
-0.81529	1316723	2.69E+11	-0.00402	2057669	1	0.00403	-0.00627
-0.73464	1570510	3.22E+11	-0.00479	1856157	1	0.004801	-0.00123
-0.65306	1769971	3.63E+11	-0.0054	1651620	1	0.005405	-0.00766
-0.572	1928625	3.96E+11	-0.00588	1447865	1	0.005884	-0.00766
-0.49115	2056609	4.22E+11	-0.00626	1244222	1	0.006269	-0.00501
-0.40986	2160185	4.44E+11	-0.00657	1039068	1	0.006579	-0.00792
-0.32887	2241703	4.61E+11	-0.00681	834356.2	1	0.006822	-0.00702
-0.24698	2304418	4.74E+11	-0.007	627039.8	1	0.007008	-0.01729
-0.16726	2347874	4.84E+11	-0.00713	424924.2	1	0.007135	-0.0008
-0.08638	2375150	4.89E+11	-0.0072	219601	1	0.007213	0.001455
-0.0055	2386013	4.92E+11	-0.00723	13979.62	1	0.007241	0.003606
0.07484	2380720	4.91E+11	-0.00721	-190517	1	0.007219	0.012538
0.156409	2358776	4.87E+11	-0.00714	-398434	1	0.007148	0.006246
0.238019	2319578	4.79E+11	-0.00702	-606742	1	0.007024	-0.00054
0.31902	2262673	4.68E+11	-0.00684	-813790	1	0.006846	0.000182
0.400333	2186027	4.52E+11	-0.0066	-1021948	1	0.006609	-0.00295
0.480512	2089131	4.33E+11	-0.0063	-1227532	1	0.006312	0.00791

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.561968	1965686	4.07E+11	-0.00593	-1436761	1	0.005934	0.003011
0.642906	1812896	3.76E+11	-0.00546	-1645077	1	0.005467	0.00452
0.723175	1623213	3.37E+11	-0.00488	-1852175	1	0.00489	0.01427
0.804549	1375837	2.86E+11	-0.00414	-2062793	1	0.00414	0.010378
0.886063	1032924	2.15E+11	-0.0031	-2274815	1	0.003104	0.004774
0.966988	403667.4	8.42E+10	-0.00121	-2488063	1	0.001209	0.006436
0.967289	-399620	-8.4E+10	0.001194	-2495498	1	-0.0012	0.002713
0.887096	-1034405	-2.2E+11	0.003082	-2293632	1	-0.00309	-0.00798
0.805268	-1386548	-2.9E+11	0.004126	-2084832	1	-0.00413	0.001512
0.724825	-1638138	-3.4E+11	0.004869	-1878534	1	-0.00488	-0.00609
0.643436	-1837033	-3.9E+11	0.005455	-1669138	1	-0.00546	-0.00203
0.562559	-1995610	-4.2E+11	0.005921	-1460548	1	-0.00593	-0.00427
0.481668	-2123908	-4.5E+11	0.006297	-1251503	1	-0.00631	-0.00635
0.401026	-2226767	-4.7E+11	0.006596	-1042737	1	-0.00661	-0.01151
0.319205	-2308931	-4.9E+11	0.006835	-830577	1	-0.00684	-0.0021
0.23937	-2369694	-5E+11	0.00701	-623265	1	-0.00702	-0.01721
0.157811	-2413430	-5.1E+11	0.007134	-411180	1	-0.00714	-0.01104
0.07687	-2439426	-5.2E+11	0.007206	-200418	1	-0.00722	-0.0125
-0.0046	-2448579	-5.2E+11	0.007228	12002.58	1	-0.00724	-0.00743
-0.08545	-2440921	-5.2E+11	0.007201	223087.8	1	-0.00721	-0.00999
-0.1663	-2416371	-5.1E+11	0.007123	434434.8	1	-0.00713	-0.01261
-0.24713	-2374355	-5E+11	0.006995	646023.3	1	-0.007	-0.01544
-0.32896	-2312979	-4.9E+11	0.006809	860541.6	1	-0.00682	-0.00587
-0.40989	-2232055	-4.7E+11	0.006566	1072984	1	-0.00657	-0.00755
-0.49111	-2128203	-4.5E+11	0.006255	1286553	1	-0.00626	-0.00552
-0.57188	-1998878	-4.2E+11	0.005871	1499286	1	-0.00588	-0.00915
-0.65241	-1838486	-3.9E+11	0.005395	1711829	1	-0.0054	-0.01568
-0.73471	-1632346	-3.5E+11	0.004785	1929581	1	-0.00479	-0.00036
-0.81517	-1371088	-2.9E+11	0.004015	2143168	1	-0.00402	-0.0077
-0.81659	1378039	2.96E+11	-0.004	2166691	1	0.004003	0.009837
-0.73525	1649290	3.55E+11	-0.00478	1952930	1	0.004786	0.00634
-0.65391	1859597	4.01E+11	-0.00538	1738455	1	0.005391	0.002813
-0.57352	2026048	4.37E+11	-0.00586	1525993	1	0.005868	0.011144
-0.49288	2161211	4.66E+11	-0.00625	1312428	1	0.006254	0.016318
-0.41176	2270679	4.9E+11	-0.00656	1097224	1	0.006566	0.0156
-0.33038	2357438	5.1E+11	-0.0068	880968.5	1	0.006812	0.011578
-0.24974	2422981	5.24E+11	-0.00699	666386.1	1	0.006996	0.016769
-0.16783	2470353	5.35E+11	-0.00712	448115.6	1	0.007128	0.006253
-0.08761	2498980	5.41E+11	-0.0072	234079.3	1	0.007206	0.016646
-0.00563	2510644	5.44E+11	-0.00722	15043.31	1	0.007235	0.005229
0.075811	2504806	5.43E+11	-0.0072	-202811	1	0.007213	0.000562
0.156144	2481857	5.39E+11	-0.00713	-417985	1	0.007142	0.009527
0.237118	2440897	5.3E+11	-0.00701	-635159	1	0.007019	0.01058
0.318067	2381075	5.17E+11	-0.00683	-852561	1	0.006842	0.011934

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.398798	2301086	5E+11	-0.0066	-1069684	1	0.006608	0.015984
0.479473	2198584	4.78E+11	-0.0063	-1286986	1	0.006309	0.020729
0.561376	2067900	4.5E+11	-0.00592	-1507969	1	0.005929	0.010326
0.641805	1908056	4.16E+11	-0.00546	-1725396	1	0.005466	0.0181
0.722848	1706335	3.72E+11	-0.00488	-1944990	1	0.004883	0.018299
0.80443	1444653	3.15E+11	-0.00413	-2166719	1	0.00413	0.011848
0.885707	1083591	2.37E+11	-0.00309	-2388663	1	0.003093	0.009167
0.967851	395919.5	8.67E+10	-0.00113	-2615893	1	0.001127	-0.00421
0.968995	-376759	-8.3E+10	0.001069	-2625082	1	-0.00107	-0.01833
0.887172	-1082280	-2.4E+11	0.003065	-2408714	1	-0.00307	-0.00891
0.805396	-1454179	-3.2E+11	0.004112	-2189472	1	-0.00412	-7.2E-05
0.725185	-1718944	-3.8E+11	0.004856	-1973386	1	-0.00486	-0.01054
0.644434	-1927280	-4.3E+11	0.00544	-1755179	1	-0.00545	-0.01434
0.563284	-2095250	-4.6E+11	0.005909	-1535382	1	-0.00592	-0.01322
0.482031	-2231131	-4.9E+11	0.006287	-1314880	1	-0.0063	-0.01083
0.401497	-2339372	-5.2E+11	0.006588	-1095962	1	-0.0066	-0.01731
0.320329	-2425342	-5.4E+11	0.006825	-874987	1	-0.00683	-0.01597
0.239461	-2490222	-5.5E+11	0.007003	-654521	1	-0.00701	-0.01833
0.157397	-2536436	-5.6E+11	0.007128	-430493	1	-0.00714	-0.00594
0.07769	-2563348	-5.7E+11	0.007199	-212620	1	-0.00721	-0.02263
-0.00371	-2573076	-5.7E+11	0.007222	10156.81	1	-0.00723	-0.01843
-0.08566	-2564888	-5.7E+11	0.007194	234738.4	1	-0.0072	-0.00737
-0.16644	-2538935	-5.6E+11	0.007116	456364.1	1	-0.00713	-0.01088
-0.24776	-2494247	-5.5E+11	0.006986	679781.9	1	-0.007	-0.00761
-0.32785	-2430982	-5.4E+11	0.006805	900091	1	-0.00681	-0.01962
-0.4098	-2344772	-5.2E+11	0.006559	1125839	1	-0.00657	-0.00864
-0.49094	-2235482	-5E+11	0.006249	1349702	1	-0.00626	-0.00764
-0.57257	-2097641	-4.7E+11	0.005859	1575290	1	-0.00587	-0.00061
-0.6538	-1926642	-4.3E+11	0.005377	1800223	1	-0.00538	0.001544
-0.73464	-1712823	-3.8E+11	0.004776	2024588	1	-0.00478	-0.00117
-0.81623	-1432898	-3.2E+11	0.003991	2251742	1	-0.004	0.005358
-0.8171	1441584	3.25E+11	-0.00398	2273833	1	0.003985	0.016065
-0.73685	1724464	3.89E+11	-0.00475	2052572	1	0.004762	0.026077
-0.65456	1949141	4.41E+11	-0.00537	1824960	1	0.005378	0.010893
-0.57436	2124198	4.8E+11	-0.00585	1602597	1	0.005856	0.021465
-0.49236	2268801	5.14E+11	-0.00624	1374814	1	0.006249	0.009868
-0.41184	2382960	5.4E+11	-0.00655	1150765	1	0.006559	0.016521
-0.33109	2473531	5.61E+11	-0.00679	925746.9	1	0.006803	0.020363
-0.24941	2543321	5.77E+11	-0.00698	697825.5	1	0.006991	0.012754
-0.16815	2592663	5.88E+11	-0.00711	470746	1	0.007122	0.010187
-0.08742	2622889	5.96E+11	-0.00719	244889.8	1	0.0072	0.014276
-0.00609	2635000	5.99E+11	-0.00722	17067.52	1	0.007228	0.01093
0.075866	2628781	5.98E+11	-0.0072	-212786	1	0.007207	-0.00012
0.154973	2605057	5.93E+11	-0.00713	-434922	1	0.007137	0.023968

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.237602	2561099	5.83E+11	-0.007	-667237	1	0.007012	0.004605
0.317911	2498564	5.69E+11	-0.00683	-893323	1	0.006836	0.013857
0.399639	2413224	5.5E+11	-0.00659	-1123716	1	0.006598	0.005611
0.48019	2305262	5.26E+11	-0.00629	-1351124	1	0.006298	0.011882
0.561952	2167719	4.95E+11	-0.00591	-1582324	1	0.005918	0.003213
0.643551	1996487	4.56E+11	-0.00544	-1813493	1	0.005446	-0.00344
0.723088	1787504	4.09E+11	-0.00487	-2039320	1	0.004872	0.015334
0.804789	1511209	3.46E+11	-0.00411	-2271963	1	0.004114	0.007426
0.886455	1127510	2.58E+11	-0.00306	-2505573	1	0.003065	-6E-05
0.968196	392447.4	9.01E+10	-0.00106	-2742407	1	0.001064	-0.00847
0.968767	-381865	-8.8E+10	0.001032	-2749846	1	-0.00104	-0.01552
0.886639	-1133073	-2.6E+11	0.003057	-2522095	1	-0.00306	-0.00233
0.806042	-1519246	-3.5E+11	0.004093	-2295589	1	-0.0041	-0.00804
0.725371	-1799791	-4.2E+11	0.004845	-2067825	1	-0.00485	-0.01283
0.644795	-2018492	-4.7E+11	0.005429	-1839661	1	-0.00544	-0.01879
0.563319	-2195785	-5.1E+11	0.005901	-1608431	1	-0.00591	-0.01365
0.481661	-2339155	-5.4E+11	0.006281	-1376257	1	-0.00629	-0.00627
0.400665	-2453271	-5.7E+11	0.006583	-1145590	1	-0.00659	-0.00704
0.320102	-2542686	-5.9E+11	0.006818	-915829	1	-0.00683	-0.01317
0.238169	-2611461	-6.1E+11	0.006998	-681842	1	-0.00701	-0.00239
0.157889	-2658770	-6.2E+11	0.007121	-452286	1	-0.00713	-0.01201
0.077088	-2687330	-6.2E+11	0.007193	-220956	1	-0.0072	-0.01519
-0.00483	-2697328	-6.3E+11	0.007215	13854.72	1	-0.00723	-0.00459
-0.08613	-2688530	-6.3E+11	0.007187	247170.1	1	-0.0072	-0.00163
-0.16683	-2661113	-6.2E+11	0.007109	479048	1	-0.00712	-0.00605
-0.24864	-2613652	-6.1E+11	0.006978	714405.9	1	-0.00699	0.003236
-0.32918	-2546452	-5.9E+11	0.006794	946403.7	1	-0.0068	-0.00315
-0.40941	-2457591	-5.7E+11	0.006553	1177797	1	-0.00656	-0.01345
-0.492	-2340531	-5.5E+11	0.006236	1416362	1	-0.00624	0.005451
-0.57346	-2195546	-5.1E+11	0.005846	1652052	1	-0.00585	0.010437
-0.65407	-2016887	-4.7E+11	0.005366	1885705	1	-0.00537	0.004898
-0.73552	-1789985	-4.2E+11	0.004758	2122300	1	-0.00476	0.00962
-0.81645	-1497301	-3.5E+11	0.003976	2358128	1	-0.00398	0.008075
-0.81585	1511951	3.57E+11	-0.00398	2376025	1	0.003987	0.000603
-0.7348	1810507	4.28E+11	-0.00476	2142067	1	0.004769	0.000805
-0.65372	2041993	4.83E+11	-0.00537	1907279	1	0.005374	0.000485
-0.57356	2225221	5.27E+11	-0.00584	1674668	1	0.005852	0.011657
-0.49162	2376536	5.63E+11	-0.00624	1436416	1	0.006245	0.000709
-0.41009	2497360	5.92E+11	-0.00655	1198998	1	0.006558	-0.00508
-0.32936	2591798	6.15E+11	-0.00679	963566.9	1	0.006801	-0.00102
-0.24828	2664030	6.32E+11	-0.00698	726803.3	1	0.006986	-0.00128
-0.16648	2715702	6.45E+11	-0.00711	487657.6	1	0.007117	-0.01033
-0.08581	2746923	6.53E+11	-0.00718	251490.7	1	0.007195	-0.00561
-0.00512	2759146	6.56E+11	-0.00721	15028.96	1	0.007222	-0.00096

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
0.075591	2752557	6.55E+11	-0.00719	-221804	1	0.007201	0.003274
0.157523	2726420	6.49E+11	-0.00712	-462489	1	0.007128	-0.00749
0.238332	2680830	6.38E+11	-0.00699	-700160	1	0.007004	-0.0044
0.31855	2615035	6.23E+11	-0.00682	-936381	1	0.006828	0.005981
0.400176	2525355	6.02E+11	-0.00658	-1177062	1	0.006589	-0.00101
0.481636	2410381	5.75E+11	-0.00628	-1417593	1	0.006285	-0.00596
0.562256	2267651	5.41E+11	-0.0059	-1656011	1	0.005909	-0.00053
0.643277	2089124	4.99E+11	-0.00543	-1896039	1	0.005439	-5.9E-05
0.723891	1866476	4.46E+11	-0.00485	-2135371	1	0.004855	0.005428
0.804622	1579321	3.78E+11	-0.0041	-2375720	1	0.004104	0.009478
0.88606	1177296	2.82E+11	-0.00305	-2619233	1	0.003055	0.004806
0.968421	387694.9	9.31E+10	-0.001	-2868631	1	0.001003	-0.01124
0.967736	-401317	-9.7E+10	0.001036	-2872262	1	-0.00104	-0.00279
0.887479	-1175091	-2.8E+11	0.003027	-2639344	1	-0.00303	-0.01269
0.804879	-1591175	-3.8E+11	0.004094	-2396526	1	-0.0041	0.006308
0.723419	-1887011	-4.6E+11	0.00485	-2155976	1	-0.00486	0.01126
0.644082	-2112001	-5.1E+11	0.005424	-1921041	1	-0.00543	-0.00999
0.563145	-2296425	-5.6E+11	0.005893	-1680861	1	-0.0059	-0.0115
0.482089	-2445623	-5.9E+11	0.006272	-1439903	1	-0.00628	-0.01154
0.401203	-2565183	-6.2E+11	0.006574	-1199080	1	-0.00658	-0.01369
0.319636	-2660019	-6.4E+11	0.006813	-955890	1	-0.00682	-0.00742
0.239469	-2730583	-6.6E+11	0.006989	-716570	1	-0.007	-0.01843
0.157551	-2781225	-6.8E+11	0.007114	-471723	1	-0.00713	-0.00784
0.076816	-2810979	-6.8E+11	0.007186	-230125	1	-0.0072	-0.01183
-0.00445	-2821306	-6.9E+11	0.007208	13337.22	1	-0.00722	-0.0093
-0.08452	-2812384	-6.8E+11	0.007181	253480.7	1	-0.00719	-0.02155
-0.16574	-2783716	-6.8E+11	0.007104	497383.4	1	-0.00711	-0.01948
-0.24694	-2734634	-6.7E+11	0.006974	741474.6	1	-0.00698	-0.01782
-0.328	-2664069	-6.5E+11	0.00679	985469.8	1	-0.0068	-0.01776
-0.40872	-2570565	-6.3E+11	0.006548	1228741	1	-0.00656	-0.02196
-0.48974	-2450646	-6E+11	0.006238	1473247	1	-0.00625	-0.02247
-0.57029	-2301388	-5.6E+11	0.005854	1716734	1	-0.00586	-0.02868
-0.65192	-2112988	-5.2E+11	0.005371	1963897	1	-0.00538	-0.02166
-0.73308	-1877373	-4.6E+11	0.004768	2210163	1	-0.00477	-0.02045
-0.81487	-1569025	-3.8E+11	0.003981	2459077	1	-0.00399	-0.01138
-0.81671	1572914	3.88E+11	-0.00396	2484254	1	0.003965	0.011265
-0.73645	1884342	4.65E+11	-0.00474	2242170	1	0.004746	0.021074
-0.65529	2128253	5.26E+11	-0.00535	1996698	1	0.005355	0.019939
-0.57493	2321424	5.74E+11	-0.00583	1753087	1	0.005837	0.028535
-0.49419	2478325	6.13E+11	-0.00622	1507904	1	0.006227	0.032492
-0.41342	2604652	6.45E+11	-0.00653	1262230	1	0.00654	0.035981
-0.33234	2704865	6.7E+11	-0.00678	1015313	1	0.006787	0.035802
-0.25086	2781587	6.89E+11	-0.00696	766820.3	1	0.006975	0.030526
-0.17091	2835292	7.03E+11	-0.00709	522744.2	1	0.007106	0.044299

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.09035	2868973	7.12E+11	-0.00717	276507.9	1	0.007186	0.050481
-0.00859	2882854	7.16E+11	-0.0072	26296.08	1	0.007216	0.041759
0.071487	2876851	7.14E+11	-0.00719	-219013	1	0.007197	0.053907
0.15339	2850461	7.08E+11	-0.00712	-470205	1	0.007127	0.043494
0.234199	2803731	6.97E+11	-0.00699	-718327	1	0.007006	0.046579
0.316108	2734241	6.8E+11	-0.00682	-970123	1	0.006828	0.036103
0.396671	2642400	6.58E+11	-0.00658	-1218091	1	0.006594	0.04223
0.477185	2524710	6.29E+11	-0.00629	-1466241	1	0.006296	0.048951
0.559414	2373685	5.92E+11	-0.00591	-1720049	1	0.005916	0.034528
0.640106	2188905	5.46E+11	-0.00544	-1969539	1	0.005451	0.039052
0.721319	1955775	4.88E+11	-0.00486	-2221149	1	0.004866	0.037156
0.80227	1656390	4.14E+11	-0.00411	-2472616	1	0.004117	0.038501
0.885278	1229174	3.07E+11	-0.00305	-2731562	1	0.003051	0.014457
0.968224	388699.2	9.74E+10	-0.00096	-2993541	1	0.000962	-0.00882
0.968135	-390480	-9.8E+10	0.000964	-2998616	1	-0.00097	-0.00772
0.887952	-1218336	-3.1E+11	0.003003	-2755674	1	-0.00301	-0.01853
0.807543	-1646310	-4.1E+11	0.004053	-2508923	1	-0.00406	-0.02656
0.727182	-1955347	-4.9E+11	0.004809	-2261253	1	-0.00482	-0.03517
0.645815	-2198838	-5.5E+11	0.005404	-2009796	1	-0.00541	-0.03138
0.566104	-2390059	-6E+11	0.00587	-1762948	1	-0.00588	-0.04801
0.484835	-2547713	-6.4E+11	0.006253	-1510848	1	-0.00626	-0.04542
0.404173	-2673387	-6.8E+11	0.006557	-1260260	1	-0.00657	-0.05032
0.322796	-2773250	-7E+11	0.006798	-1007114	1	-0.00681	-0.0464
0.242787	-2847752	-7.2E+11	0.006976	-757917	1	-0.00699	-0.05936
0.161002	-2901469	-7.3E+11	0.007104	-502890	1	-0.00712	-0.05041
0.079205	-2933676	-7.4E+11	0.007178	-247536	1	-0.00719	-0.04132
-0.0014	-2944992	-7.5E+11	0.007202	4374.33	1	-0.00721	-0.04693
-0.08257	-2936082	-7.4E+11	0.007176	258347.3	1	-0.00719	-0.0455
-0.16375	-2906546	-7.4E+11	0.0071	512610.1	1	-0.00711	-0.04402
-0.24435	-2856129	-7.3E+11	0.006973	765322.8	1	-0.00698	-0.04978
-0.32651	-2781843	-7.1E+11	0.006787	1023248	1	-0.0068	-0.0362
-0.4072	-2684474	-6.8E+11	0.006546	1276885	1	-0.00656	-0.04072
-0.4882	-2559533	-6.5E+11	0.006237	1531830	1	-0.00625	-0.04142
-0.56923	-2402947	-6.1E+11	0.005852	1787232	1	-0.00586	-0.0418
-0.65127	-2205209	-5.6E+11	0.005366	2046253	1	-0.00537	-0.02974
-0.73313	-1956571	-5E+11	0.004757	2305265	1	-0.00476	-0.01982
-0.81505	-1632896	-4.2E+11	0.003967	2565178	1	-0.00397	-0.0092
-0.81838	1629216	4.19E+11	-0.00393	2595233	1	0.003934	0.031916
-0.73753	1959246	5.04E+11	-0.00472	2340914	1	0.004726	0.034409
-0.65667	2214318	5.7E+11	-0.00533	2085893	1	0.005337	0.036966
-0.57609	2417509	6.23E+11	-0.00581	1831183	1	0.005823	0.042814
-0.49453	2583485	6.66E+11	-0.00621	1572939	1	0.006218	0.03659
-0.41386	2715427	7.01E+11	-0.00652	1317154	1	0.006531	0.041418
-0.33343	2819499	7.28E+11	-0.00677	1061805	1	0.006778	0.049248

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.25149	2900310	7.49E+11	-0.00696	801312.1	1	0.006968	0.038331
-0.17048	2957136	7.64E+11	-0.00709	543488.6	1	0.0071	0.038934
-0.08998	2992124	7.74E+11	-0.00717	287026.2	1	0.00718	0.045913
-0.00828	3006454	7.78E+11	-0.0072	26441.21	1	0.00721	0.038022
0.07347	2999678	7.76E+11	-0.00718	-234605	1	0.00719	0.029437
0.154131	2972093	7.7E+11	-0.00711	-492434	1	0.00712	0.03436
0.234857	2923051	7.57E+11	-0.00699	-750757	1	0.006998	0.03847
0.316674	2850266	7.39E+11	-0.00681	-1012868	1	0.00682	0.029121
0.397673	2753451	7.14E+11	-0.00657	-1272671	1	0.006584	0.029868
0.479075	2628501	6.82E+11	-0.00627	-1534105	1	0.006281	0.025634
0.560135	2472280	6.42E+11	-0.0059	-1794809	1	0.005904	0.025633
0.641981	2275578	5.91E+11	-0.00542	-2058474	1	0.00543	0.015924
0.722464	2032871	5.29E+11	-0.00484	-2318257	1	0.004847	0.023039
0.804055	1715448	4.47E+11	-0.00408	-2582301	1	0.004086	0.016476
0.886324	1268705	3.31E+11	-0.00301	-2849633	1	0.003019	0.001552
0.968686	371277.5	9.7E+10	-0.00088	-3120615	1	0.000881	-0.01452
0.968978	-363460	-9.5E+10	0.00086	-3126397	1	-0.00086	-0.01812
0.888067	-1263107	-3.3E+11	0.002984	-2870969	1	-0.00299	-0.01995
0.807084	-1714286	-4.5E+11	0.004046	-2611983	1	-0.00405	-0.02089
0.72568	-2040583	-5.4E+11	0.004812	-2350556	1	-0.00482	-0.01664
0.645598	-2290340	-6E+11	0.005396	-2092700	1	-0.00541	-0.0287
0.565037	-2491849	-6.6E+11	0.005867	-1832788	1	-0.00588	-0.03484
0.483295	-2656842	-7E+11	0.006251	-1568633	1	-0.00626	-0.02642
0.402648	-2787430	-7.3E+11	0.006555	-1307646	1	-0.00657	-0.0315
0.321297	-2891108	-7.6E+11	0.006795	-1044047	1	-0.00681	-0.02792
0.240758	-2968822	-7.8E+11	0.006973	-782765	1	-0.00699	-0.03434
0.159586	-3023931	-8E+11	0.007099	-519132	1	-0.00711	-0.03294
0.078427	-3056925	-8.1E+11	0.007172	-255258	1	-0.00718	-0.03171
-0.0027	-3068469	-8.1E+11	0.007195	8780.251	1	-0.00721	-0.03092
-0.0838	-3058771	-8.1E+11	0.007169	273030.1	1	-0.00718	-0.03039
-0.16432	-3027876	-8E+11	0.007092	535660.7	1	-0.0071	-0.03703
-0.24596	-2974174	-7.9E+11	0.006963	802240.9	1	-0.00697	-0.02983
-0.32745	-2896793	-7.7E+11	0.006778	1068623	1	-0.00679	-0.02451
-0.40803	-2794915	-7.4E+11	0.006536	1332342	1	-0.00655	-0.03042
-0.48942	-2663391	-7.1E+11	0.006224	1599022	1	-0.00623	-0.02645
-0.57169	-2496446	-6.6E+11	0.00583	1869022	1	-0.00584	-0.01143
-0.65254	-2291813	-6.1E+11	0.005349	2134771	1	-0.00536	-0.01404
-0.73373	-2033394	-5.4E+11	0.004742	2402183	1	-0.00475	-0.01239
-0.81472	-1698655	-4.5E+11	0.003958	2669636	1	-0.00396	-0.01326
-0.818	1694484	4.53E+11	-0.00392	2699891	1	0.003926	0.027129
-0.73652	2041139	5.47E+11	-0.00472	2433087	1	0.004725	0.022023
-0.65613	2305318	6.18E+11	-0.00532	2169117	1	0.005332	0.030288
-0.5761	2515763	6.75E+11	-0.0058	1905777	1	0.005815	0.042891
-0.49363	2690624	7.22E+11	-0.0062	1634007	1	0.006215	0.025558

$\frac{\partial F}{\partial A}$	$\frac{\partial F}{\partial \theta}$	$\frac{\partial F}{\partial \theta'}$	$\frac{\partial F}{\partial B}$	$\frac{\partial F}{\partial \lambda}$	$\frac{\partial F}{\partial Q}$	B_{ii}	b
-0.41354	2827015	7.59E+11	-0.00651	1369677	1	0.006526	0.037488
-0.33209	2936633	7.89E+11	-0.00676	1100519	1	0.006775	0.032641
-0.25075	3019922	8.12E+11	-0.00695	831413.7	1	0.006963	0.029181
-0.17094	3078205	8.28E+11	-0.00708	567105.2	1	0.007093	0.044693
-0.08878	3115284	8.38E+11	-0.00716	294684.2	1	0.007175	0.031069
-0.00831	3129778	8.43E+11	-0.00719	27608.72	1	0.007204	0.038374
0.073373	3122661	8.41E+11	-0.00717	-243794	1	0.007184	0.030635
0.153966	3093876	8.34E+11	-0.0071	-511841	1	0.007114	0.03639
0.235758	3041868	8.2E+11	-0.00698	-784165	1	0.00699	0.027351
0.316373	2966905	8E+11	-0.0068	-1052864	1	0.006814	0.032832
0.397304	2866041	7.74E+11	-0.00657	-1322924	1	0.006579	0.034423
0.478637	2735887	7.39E+11	-0.00627	-1594665	1	0.006276	0.031045
0.559626	2573180	6.95E+11	-0.00589	-1865629	1	0.005899	0.031908
0.642279	2365873	6.4E+11	-0.00541	-2142597	1	0.00542	0.012243
0.722202	2114214	5.72E+11	-0.00483	-2410924	1	0.00484	0.026263
0.802993	1786646	4.84E+11	-0.00408	-2682836	1	0.004086	0.029579
0.884774	1325127	3.59E+11	-0.00302	-2959159	1	0.003027	0.020677
0.967973	380223.5	1.03E+11	-0.00087	-3243667	1	0.000866	-0.00572
0.968288	-371280	-1E+11	0.000844	-3249477	1	-0.00085	-0.00961
0.888554	-1304501	-3.6E+11	0.00296	-2987509	1	-0.00297	-0.02595
0.806811	-1781098	-4.9E+11	0.004036	-2715529	1	-0.00404	-0.01753
0.72674	-2116523	-5.8E+11	0.004793	-2448030	1	-0.0048	-0.02972
0.646504	-2378296	-6.5E+11	0.005381	-2179301	1	-0.00539	-0.03987
0.564717	-2591776	-7.1E+11	0.005861	-1904852	1	-0.00587	-0.0309
0.483565	-2762493	-7.6E+11	0.006243	-1632101	1	-0.00625	-0.02975
0.403553	-2897719	-7.9E+11	0.006544	-1362816	1	-0.00656	-0.04267

		A^TA			
346.9127	-4676801	-1.3E+12	0.002388	-6E+08	97.77561
-4676801	2.73E+15	5.63E+20	-9316825	1.56E+13	13475755
-1.3E+12	5.63E+20	1.23E+26	-1.7E+12	4.41E+18	3.65E+12
0.002388	-9316825	-1.7E+12	0.042788	-34755.9	0.007768
-6E+08	1.56E+13	4.41E+18	-34755.9	1.36E+15	-1.7E+08
97.77561	13475755	3.65E+12	0.007768	-1.7E+08	1165

$(A^TA)^{-1}$					
0.012411	3.43E-10	-1.2E-15	0.030087	5.45E-09	-0.00025
3.43E-10	2.38E-14	-8.2E-20	1.87E-06	1.89E-16	-3E-11
-1.2E-15	-8.2E-20	3.04E-25	-5.8E-12	-7.1E-22	3.56E-17
0.030087	1.87E-06	-5.8E-12	199.9246	1.49E-08	-0.00528
5.45E-09	1.89E-16	-7.1E-22	1.49E-08	3.14E-15	1.41E-12
-0.00025	-3E-11	3.56E-17	-0.00528	1.41E-12	0.000879

A^Tb
-8.4E-13
-2.9E-05
-5.62878
9.28E-14
1.85E-06
-1.2E-12

$\hat{x} = -(A^TA)^{-1}A^Tb$
-2.7E-16
-4.3E-20
1.16E-25
-2.6E-12
1.21E-21
-6.9E-16

A	12.33664	σ_A	0.009132
θ	0.000597	σ_θ	1.26E-08
θ'	3.9E-12	$\sigma_{\theta'}$	4.52E-14
B	9056.718	σ_B	1.159053
λ	9.3E-08	σ_λ	4.6E-09
Q	-0.93581	σ_Q	0.002431

Maximum residuals	0.273701
$\hat{\sigma}_0$	0.081973