Large Market Games, the Law of One Price, and Market Structure*

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Abstract

This paper introduces a new class of market games featuring multiple posts per commodity, in which trading posts are privately owned. It is demonstrated via three robust counterexamples, that in this setting the law of one price fails, thus showing, contrary to longstanding belief in the literature, that price dispersion in *large* market games is extremely robust. Most importantly, it is established that even in economies with a continuum of small agents and infinitely many atoms (all of whom can arbitrage prices if they so wish), and an infinite number of markets *per commodity*, the set of equilibria—and the resulting market structure—is influenced, both by strategic behaviour, *and* private ownership of posts.

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1 Introduction

The influence of strategic considerations on the process of price formation is a fundamental issue, and one elegant theory capturing both these concepts is that of *strategic market games* (SMG). An SMG, originating in Dubey–Shubik (1978), Shapley–Shubik (1977), and Shubik (1973), is a

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noncooperative trading model in which commodity prices depend on the buy-and-sell decisions of agents. Such strategic decision making by agents has called into question the validity of the *law of one price*[1] (LOP) in SMG with multiple "trading posts" per commodity (MTPC). An important implication of the failure of the LOP is that the set of equilibria depends nontrivially on the structure of trading posts. Regarding the latter, it is interesting to note that in standard SMG models, trading posts are typically assumed to be publicly and costlessly available. Surprisingly, the question never seems to have been asked if (or how) the "privatisation" of trading posts would affect the equilibrium allocations and prices in a meaningful manner. The purpose of this paper is to demonstrate that private ownership of trading posts,[2] alongside strategic behaviour by agents, in MTPC market games is indeed a material issue.

The following fact is established: even in *very large* economies, strategic considerations *still matter* in the determination of the equilibrium market structure. We achieve this by proving, in a new class of MTPC market games, that the LOP, an intimate feature of Walrasian markets, fails to obtain in very general settings. Hence, trading posts cannot be consolidated.[3] This is in stark contrast to the norm in SMG, where provided there are at least countably infinitely many agents (large and/or small), the LOP always prevails. Indeed, Koutsougeras (1999; 2003a; 2003b) shows the failure of the LOP when the number of agents is finite. Koutsougeras (2003a) then proves that as the number of agents increases without bounds in MTPC models with publicly-available posts, the uniformity of prices across trading posts is restored, independently of the characteristics—preferences, endowments, measure, etc.—of agents.[4] In our model, however, the failure of the LOP to obtain effectively stems from the heterogeneity of agents. There are two types of agents

[1]The LOP postulates that *at equilibrium*, there is a single price that clears all markets for a commodity. It is a *central feature* of Walrasian economies, in which markets for a commodity are consolidated and modeled as a single trading spot where transactions take place. See Koutsougeras (1999; 2003b) for numerical examples in which the LOP fails in SMG with finitely many agents, all of whom face no *binding* liquidity constraints.

[2]It must be noted that in the model that we propose, post owners are not given any kind of "extreme" market power. More precisely, they can neither "close down" their post, nor preclude agents from trading at their spots, such that all agents are perfectly free to choose where to trade, and to arbitrage prices should they so wish.

[3]The equilibrium market/trading-post structure is determined by the distribution of prices across posts for each commodity. Consider any commodity k. If for k the support of this distribution is a single point, then there is effectively a single trading post for k. However, if equilibrium prices are not uniform across trading posts for k, then it follows that the equilibrium structure of posts cannot be merged into a single trading platform. So, the LOP—or the failure thereof—is a "tool" that we use to determine the market structure at equilibrium.

[4]Koutsougeras' (2003a) limit economy need not be atomless—even if there exist *finitely* many *non-price-taking* atoms in the limit, the LOP *must still hold.* Thus, note that that limit economy need not be perfectly competitive; indeed, the validity of the LOP is a more general issue than the prevalence of perfect competition.

in this model, pure traders (the only kind of agents that Koutsougeras, 1999; 2003a; 2003b, considers), and trading post owners. The latter not only buy and sell across posts, but also levy a proportional "service charge" on agents who trade at their spots. We show, intriguingly, that even with a continuum of price takers, and only as few as two, or as many as an infinite number of "large" players, the LOP is still violated. This persistent price inequality is driven, both by strategic play by agents, and the trading-post service charge, an intricate concept which looks deceptively trivial.

To the best of our knowledge, none of the existing SMG models analyses the existence and availability of trading posts. Thus far, the SMG literature has been content to assume that trading posts somehow exist, and are somehow made publicly and costlessly available for all agents to trade at, as though by *some other* "invisible hand." In this paper, we depart from the literature and introduce post owners, who in addition to commodities, are endowed with trading posts. These agents also engage in trade, by presenting arrays of buy-and-sell strategies at their own, and/or other post owners' posts, as do the pure traders. Now, in most—if not all—economic models, privately-owned trading platforms are very rarely provided free of charge, and are even more so in real-world economies. In this light, we assume that post owners levy a *proportional* service charge per unit of (monetary) *net* trade on all agents who transact at their post.[5] Thus, an agent whose net trade is zero at a post—and therefore derives no direct monetary benefit from trading there—has no premium to pay. This service charge is reminiscent of the taxes and transactions costs that agents pay and incur in Gabszewicz and Grazzini (1999), Koutsougeras and Ziros (2015), and Rogawski and Shubik (1986). However, differently to these models, in the current paper, it is *agents who charge agents*,[6] and solely on their *net proceeds*, a formulation which is unique to this SMG. We assume that these trading-post service charges are exogenously given. Think of some outside agency as selecting and allocating these charges at the outset, before trading starts.[7]

[5]In the present model, the focus is on how post owners *charge other agents* for trading at platforms that they own. We acknowledge that ideally, setting up a trading post should also be costly. However, since it is assumed that post owners are "endowed" with such posts, there is therefore no cost for them to set up a platform, nor can they *decide* how many trading posts per commodity they would like to open or shut down.

[6]Thus, in addition to how their individual bids and offers directly affect their allocations, agents must also consider how their strategies affect the premia payable that accrue to them. The introduction of this service charge leads to a modification of agents' strategy sets and holdings-surfaces, such that, as opposed to Koutsougeras (1999), but similarly to Koutsougeras (2003a; 2003b), the SMG models considered in this paper are generalised games.

[7]While these charges can be endogenised, in our framework we *choose* to take these as being given, such that

Interestingly, this charge is the *same* across *all* markets for a commodity, but need not be the same across different commodities—see more about the potency of this specification below.

We show that non-uniform prices in equilibrium are much more persistent than has been portrayed in previous models. Indeed, the LOP fails even in cases where conventional wisdom dictates it should not, namely, with a continuum of small agents, and infinitely many atoms. Perhaps it would be helpful at this point to spell out what the failure of the LOP is *not*. An unequal-price equilibrium in our model does not simply mean different market-clearing prices and *similar* effective after-service-charge (ASC) prices across posts for a commodity. As previously remarked, this service charge is *equal at all* posts for the same commodity. Hence, the failure of the LOP as postulated in this paper not only means different market-clearing prices, but *also different* effective ASC prices across different posts for a commodity. This is a strong result.

The intuition behind the failure of the LOP in *every* robust counterexample[8] considered in this paper is the same: what to outside observers seems like an arbitrage opportunity, is actually not for the active market participant. We explain why this is so. The large pure traders and post owners (who also trade) affect market-clearing prices nontrivially. Hence, whenever they try to take advantage of the price difference by altering their bid-and-offer decisions across any two posts, the resulting net change affects them adversely. Consider the "insignificant" agents now. By shifting his orders from one post to another, a negligible individual affects neither the equilibrium price, nor the equilibrium allocation. Yet, he still cannot profit from the price difference, due to the counterbalancing effect that is provided by the trading-post service charge. For clarity, let us contemplate one such very small agent who shifts all of his bids from the more expensive to the cheaper markets, and all of his offers from the cheaper to the more expensive posts. In doing so, he incurs charges on the *full amounts* of: (i) his bids, and; (ii) his receipts from sales, across both markets. The net gain obtained by the shift of orders is thus more than completely offset by the increase in premia payable. So, no insignificant agent has any incentive to deviate, and this unequal-price situation is indeed sustainable as an equilibrium.

extremely little to almost no market power is given to the post owners. Note also that these charges may instead be viewed as taxes imposed by a government. This interpretation was suggested to me by Herakles Polemarchakis.

[8]More precisely, the following is true of all the counterexamples computed in this paper: *any* endowments, *and* utility functions with the same marginal rate of substitution at the consumption allocations as computed in the respective examples, would constitute equilibria with the same properties (such that the LOP still fails). This fact attests to the robustness of our counterexamples in endowment and utility spaces.

In the next section, we construct and show the failure of the LOP in a model with a continuum of small agents, and finitely many large agents and trading posts. In Section 3, we extend this model to include infinitely many trading posts per commodity. In Section 4, we generalise the model in Section 3 to include infinitely many atoms. Our conclusions are summarised in Section 5. The Appendix contains all the technical proofs.

2 The failure of the LOP, Part 1: The model

In this section, we analyse a model featuring: (i) an atomless continuum of small agents; (ii) finitely many atoms, and; (iii) finitely many markets per commodity.

We consider a pure exchange economy with small agents, represented by an atomless continuum, and large agents, represented by atoms. So, we let the set of agents be denoted by $N = N_0 \cup A$, where $N_0 = (0, 1]$, and $A = \{2, ..., H\}$. The collection of all half-open intervals in $(0, 1]$ defined by $\mathscr{S}_0 = \{(a, b] : a, b \in N_0\}$, where $(a, b] = \emptyset$ if $b \leq a$, is a semiring. So, let ν_0 be a measure on $\mathscr{S}_0$ such that $\nu_0 ((a, b]) = b - a$, and denote the Carathéodory extension of ν_0 by μ_0. Let $\mathcal{N}_0$ denote the collection of all μ_0-measurable subsets of N_0 (and recall that μ_0 is in fact the Lebesgue measure when restricted to $\mathcal{N}_0$). Next, define the collection of all the subsets of A by $\mathscr{S}_A = \mathcal{P}(A)$, which is trivially a σ-algebra (and hence, a semiring). Finally, denote by μ_A the counting measure on $\mathscr{S}_A$. We may now introduce the following properties of our set of agents:

The triple $(N, \mathcal{N}, \mu)$—where $\mathcal{N}$ is the collection of all μ-measurable sets of N, and μ is an extended real-valued, σ-additive measure defined on $\mathcal{N}$—is a complete, finite measure space of agents (See Appendix, Lemmata 1 and 2). Let $\mathcal{N}_{N_0}$ denote the restriction of $\mathcal{N}$ to N_0, and $\mathcal{N}_A$ the restriction of $\mathcal{N}$ to A. Then, the measure space $\left(N_0, \mathcal{N}_{N_0}, \mu\right)$, where $\mathcal{N}_{N_0} = \mathcal{N}_0$ and $\mu = \mu_0$ when restricted to $\mathcal{N}_{N_0}$ (See Appendix, Lemma 3), is atomless, while the measure space $\left(A, \mathcal{N}_A, \mu\right)$, where $\mathcal{N}_A = \mathscr{S}_A$ and $\mu = \mu_A$ when restricted to $\mathcal{N}_A$ (See Appendix, Lemma 4), is purely atomic. Moreover, for each $i \in A$, the singleton set $\{i\}$ is an atom of the measure space $(N, \mathcal{N}, \mu)$. Thus, one can describe N_0 and A as being the sets of small and large agents, respectively.

We denote the set of commodities *bought and sold* in this economy by $K = \{1, 2, ..., L\}$. There is also an $(L + 1)^{th}$ commodity, m, which in addition to yielding utility in consumption, acts as money. There are two types of agents in this economy, pure traders, and post owners.

Each pure trader $h \in N$ is characterised by a preference relation, which is representable by a utility function $u_h : \mathbb{R}_+^{L+1} \to \mathbb{R}$, and an initial endowment of commodities $e(h) \in \mathbb{R}_+^{L+1}$.

Post owners, as opposed to pure traders, are also endowed with trading posts. Each post owner $i \in N$ is characterised by a preference relation representable by a utility function $u_i : \mathbb{R}_+^{L+1} \to \mathbb{R}$, and initial endowments of commodities $e(i) \in \mathbb{R}_+^{L+1}$, and trading posts $\Upsilon^i = \{\Upsilon_1^i, \Upsilon_2^i, \ldots, \Upsilon_L^i\}$, i.e., each i is endowed with *only one* post for *each* $k \in K$, where Υ_k^i denotes the post for k owned by i.

We impose the following technical condition on post owners: any post owned by a "small" $w \in N_0$ has capacity[9] $\chi^{small} = 0$, while any post owned by an atom in A has capacity $\chi^{atom} = \mu(N)$.[10]

Before trading starts, an outside agency allocates a service charge to post owners, which *all* agents then take as *given* when making their trading decisions. This proportional service charge $c^k \in (0, 1), k \in K$, is the same across all posts for a good k, but may differ across commodities—i.e., let $|P| < \infty$ denote the total number of (large) post owners, such that $c^{1,k} = \cdots = c^{|P|,k} = c^k$, and $c^{1,l} = \cdots = c^{|P|,l} = c^l$ for all $k, l \in K$, $k \neq l$, but c^k need not be equal to c^l.

Throughout this paper, we will employ the following assumptions:

Assumption (i) *$e(n) > 0$ for each $n \in N$.*

Assumption (ii) *Preferences for each type of agents are strictly convex, C^2, and differentiably strictly monotone,[11] and indifference surfaces through the endowment do not intersect the axes.*

2.1 The strategic market game

Trade in the economy is organised through trading posts, at which agents offer commodities for sale, and place bids for purchases of commodities. Bids, (b), for commodities $1, \ldots, L$, are placed in terms of commodity m, while sales, (q), are made in terms of commodities $1, \ldots, L$. The strategy

[9] i.e., the measure of agents that the trading post can accommodate.

[10] Based on this rule, we may w.l.o.g., view all agents μ-*a.e*, $w \in N_0$, as being (small) pure traders, and view all post owners and large pure traders as lying in $N \setminus N_0$ only. While this condition makes the model more elegant mathematically without any loss of intuition, it is actually also required for the well-definedness of our model—see, e.g., Dubey and Shapley (1994: p. 264). We will therefore use this method throughout this paper.

[11] i.e., if u represents $\succeq$, then for all $\mathbf{x} \in \mathbb{R}_{++}^{L+1}$, $\partial u/\partial x_k > 0$ $\forall k = m, 1, 2, ...L$.

sets of agents are described by a measurable correspondence $S : N \rightrightarrows 2^{\mathbb{R}_+^{2\times|P|\times L}}$ such that

$$S(n) = \left\{(b(n), q(n)) \in \mathbb{R}_+^{|P|\times L} \times \mathbb{R}_+^{|P|\times L} : \sum_{k=1}^{L}\sum_{s=1}^{|P|} b_k^s\left(n\right) + \Lambda(n) \leq e_m\left(n\right); \sum_{s=1}^{|P|} q_k^s\left(n\right) \leq e_k\left(n\right), k \in K\right\},$$

where $\varphi_k^s(n)$, $\varphi = b, q$, denotes the strategies of agent $n \in N$ at the post owned by $s \in A$ for commodity k, and $\Lambda(n)$ is the total premium payable[12] by n (more on how $\Lambda(n)$ is calculated is found below). A strategy profile consists of a pair of measurable mappings $b : N \to \mathbb{R}_+^{|P|\times L}$ and $q : N \to \mathbb{R}_+^{|P|\times L}$, such that $(b(n), q(n)) \in S(n)$ *a.e* in N, i.e., a strategy profile is a measurable choice from the graph of the correspondence S, $Gr(S)$. It is easily seen that $S : N \rightrightarrows 2^{\mathbb{R}_+^{2\times|P|\times L}}$ has a measurable graph, and therefore such mappings exist, by *Aumann's Measurable Selection Theorem* (AMST). For a given strategy profile $(b, q) \in Gr(S)$, we then define $B_k^s = \int_N b_k^s(n)d\mu < \infty$, and $Q_k^s = \int_N q_k^s(n)d\mu < \infty$. Transactions at each post clear through the price $p_k^s = (B_k^s/Q_k^s)$. For $k \in K$, we let $z_k^s(n) = (b_k^s(n)/p_k^s) - q_k^s(n)$ denote the net trade in k of a player $n \in N$ by trading at post Υ_k^s. We also define $B_{-\gamma,k}^s = \int_{N\setminus\{\gamma\}} b_k^s(n)d\mu$, and $Q_{-\gamma,k}^s = \int_{N\setminus\{\gamma\}} q_k^s(n)d\mu$.

Consumption allocations, $x_{h,k}\left(b(h), q(h), B_{-h}, Q_{-h}\right) \equiv x_{h,k}$, for commodities $k = m, 1, \ldots, L$, to pure traders μ-*a.e*, $h \in N$, are as follows:

$$x_{h,k} = \begin{cases} e_k(h) + \sum\limits_{s=1}^{|P|}\left(b_k^s(h) \cdot \dfrac{Q_k^s}{B_k^s} - q_k^s(h)\right) & \text{if } k \neq m; \\ e_k(h) + \sum\limits_{k=1}^{L}\sum\limits_{s=1}^{|P|}\left(q_k^s(h) \cdot \dfrac{B_k^s}{Q_k^s} - b_k^s(h)\right) \cdot \left(1 + t_h^{s,k}c^k\right) & \text{if } k = m, \end{cases} \tag{1}$$

where $t_h^{s,k} : \mathbb{R} \supset z_k^s(h) \to \{-1, +1\}$, c^k is the proportional service charge payable (per unit of monetary net trade) at all trading posts for k, and as is standard in the SMG literature, any division by zero, including $0/0$, is taken to equal zero whenever it appears in the allocation rule above. From here onwards, we will only write $t_h^{s,k}$ instead of $t_h^{s,k}(z_k^s(h))$. The second expression in the allocation rule above incorporates the total premia payable to post owners, in terms of commodity m. The total premium payable at a post depends on the difference between an individual h's total

[12]As mentioned in the introduction, the premium payable at a post Υ_k^s by an agent n depends on his net trade there. His net trade at that post depends on the price at Υ_k^s, which in turn depends on the strategies of all the players. Hence, the term $\Lambda(n)$, and what is feasible depend on the strategies of all the players, such that the market games considered in Sections 2, 3, and 4 of this paper are *generalised games*.

revenue and his bid placed, both at the same post. In this light, we stipulate that $t_h^{s,k} = +1$ if $z_k^s(h) > 0$ (such that $q_k^s(h) \cdot p_k^s - b_k^s(h) < 0$), and $t_h^{s,k} = -1$ if $z_k^s(h) < 0$ (s.t. $q_k^s(h) \cdot p_k^s - b_k^s(h) > 0$). If $z_k^s(h) = 0$, then we use the following rule:[13]

$$t_h^{s,k} = \begin{cases} +1 & \text{if } \exists \theta \in \mathcal{N}, \text{ where } \mu(\theta \cap N_0) > 0, \text{ such that } \mu\text{-}a.e, w \in (\theta \cap N_0), z_k^s(w) \geq 0; \\ -1 & \text{otherwise.} \end{cases}$$

In light of the above, the premium payable at a post Υ_k^s by any agent $n \in N$ may be written explicitly as $-c^k t_n^{s,k}(q_k^s(n) \cdot p_k^s - b_k^s(n))$, such that $\Lambda(n) = \sum_{k=1}^{L} \sum_{s=1}^{|P|} - c^k t_n^{s,k}(q_k^s(n) \cdot p_k^s - b_k^s(n))$.

Consumption allocations, $x_{i,k}(b(i), q(i), B_{-i}, Q_{-i}) \equiv x_{i,k}$, for $k = m, 1, \ldots, L$, to any (large) post owner $i \in A$, are determined as:

$$x_{i,k} = \begin{cases} e_k(i) + \sum_{s=1}^{|P|} \Big(b_k^s(i) \cdot \frac{Q_k^s}{B_k^s} - q_k^s(i) \Big) & \text{if } k \neq m; \\ e_k(i) - \sum_{k=1}^{L} c^k \cdot \Big(\int_N t_n^{i,k} q_k^i(n) d\mu \cdot \frac{B_k^i}{Q_k^i} - \int_N t_n^{i,k} b_k^i(n) d\mu \Big) & \\ \qquad + \sum_{k=1}^{L} \sum_{s=1}^{|P|} \Big(q_k^s(i) \cdot \frac{B_k^s}{Q_k^s} - b_k^s(i) \Big) \cdot \big(1 + t_i^{s,k} c^k\big) & \text{if } k = m. \end{cases} \tag{2}$$

The conditions on $t_i^{s,k}$ are as for pure traders above. The second expression in the above rule also includes total premia receivable at posts i owns, and amounts payable (to other post owners).

An equilibrium for this model is defined as a profile of agents' buy-and-sell decisions across all trading posts and commodities $(b, q) \in Gr(S)$ which forms a Nash equilibrium (N.E). At an equilibrium with positive bids and offers, agents can be viewed as solving the following problem:

$$\max_{(b(n),q(n)) \in S(n)} \Big\{ u_n \Big(\big(x_{n,k}\big(b(n), q(n), B_{-n}, Q_{-n}\big)\big)_{k=1}^{L}, x_{n,m}\big(b(n), q(n), B_{-n}, Q_{-n}\big) \Big) \Big\}. \tag{3}$$

Before we proceed, we define, for a large post owner $i \in A$, the net proceeds of commodity m, from trading at his post Υ_k^i, by $z_{m,k}^i(i) = -c^k \cdot \big(\int_N t_n^{i,k} q_k^i(n) d\mu \cdot \frac{B_k^i}{Q_k^i} - \int_N t_n^{i,k} b_k^i(n) d\mu \big) + \big(q_k^i(i) \cdot \frac{B_k^i}{Q_k^i} -$

[13] Though it may appear abstract, the use of this rule has an important economic meaning. We use this specification because for any $n \in N$ whose net trade is 0 at a post Υ_k^s, we have that $B_{-n,k}^s / Q_{-n,k}^s = B_k^s / Q_k^s$, a condition that for every small $w \in N_0$, regardless of w's net trade being positive, negative, or zero, is *always true*. In other words, if $z_k^s(n) = 0$, then agent n *behaves* like a price taker at post Υ_k^s—but *is not* one, unless $n \in N_0$. We also emphasise at this point that this is a benchmark model, such that other specifications are also possible.

$b^i_k(i)\big) \cdot (1 + t^{i,k}_i c^k)$, while μ-$a.e, n \in N \setminus \{i\}$, $z^i_{m,k}(n) = \big(q^i_k(n) \cdot \frac{B^i_k}{Q^i_k} - b^i_k(n)\big) \cdot (1 + t^{i,k}_n c^k)$.

In the sequel, we will derive properties of interior equilibria[14] for our economy where there are at least two (large) active posts per commodity. A post is *active* if price is *positive* and there is trade. An equilibrium is *interior* if every agent is solving (3) in the interior of $S(n)$.

2.2 Equilibrium Analysis

We proceed with two propositions which characterise equilibrium prices for a commodity between pairs of trading posts. The failure of the LOP is crystallised in Theorem 1.

PROPOSITION 1.1 In an economy with a continuum of agents, and at least two (large) *active* posts, at an interior N.E, the prices for any commodity $k \in K$ between any such pair of trading posts $\Upsilon^i_k, \Upsilon^j_k$ should satisfy the following (no-arbitrage) conditions:[15]

$$\text{For any large pure trader } \tau \in N \setminus N_0: \qquad \big(p^i_k\big)^2 = \frac{B^i_{-\tau,k} Q^j_{-\tau,k} \big(1 + t^{j,k}_\tau c^k\big)}{Q^i_{-\tau,k} B^j_{-\tau,k} \big(1 + t^{i,k}_\tau c^k\big)} \big(p^j_k\big)^2;$$

$$\text{For } \mu\text{-}a.e, w \in N_0: \qquad p^i_k = \frac{\big(1 + t^{j,k}_w c^k\big)}{\big(1 + t^{i,k}_w c^k\big)} p^j_k.$$

PROPOSITION 1.2 In an economy with a continuum of agents, and at least two *active* posts owned by (large) agents $i, j \in A$, $i \neq j$, at an interior N.E, the prices for any commodity $k \in K$ between trading posts $\Upsilon^i_k, \Upsilon^j_k$ should satisfy the following (no-arbitrage) conditions for i and j:

$$\text{For } i: \quad \big(p^i_k\big)^2 = \frac{B^i_{-i,k} Q^j_{-i,k} \big(1 + t^{j,k}_i c^k\big)}{\left[Q^i_{-i,k} + c^k \cdot \left(\int_{N \setminus \{i\}} t^{i,k}_n q^i_k(n) d\mu\right)\right] B^j_{-i,k}} \big(p^j_k\big)^2;$$

$$\text{For } j: \quad \big(p^i_k\big)^2 = \frac{B^i_{-j,k} \left[Q^j_{-j,k} + c^k \cdot \left(\int_{N \setminus \{j\}} t^{j,k}_n q^j_k(n) d\mu\right)\right]}{Q^i_{-j,k} B^j_{-j,k} \big(1 + t^{i,k}_j c^k\big)} \big(p^j_k\big)^2.$$

THEOREM 1 Consider any commodity $k \in K$. Whenever at equilibrium $\exists \theta \in \mathcal{N}$, such that $\mu(\theta \cap N_0) > 0$, and μ-$a.e, w \in (\theta \cap N_0)$, $z^\alpha_k(w) \geq 0$, $z^\beta_k(w) < 0$, $\alpha, \beta \in A$, then $p^\alpha_k \neq p^\beta_k$.

[14] Dubey and Shubik (1978) proved the existence of equilibria—which may fail to be interior—for the species of SMG we use. However, in this paper, we construct *ad hoc* counterexamples, in which non-trivial equilibria do exist.

[15] Recall—see footnote 10—that we may w.l.o.g. view all agents μ-$a.e, w \in N_0$, as being (small) pure traders. Thus, as both conditions below characterise N.E prices for pure traders, they have been presented together.

PROOF: W.l.o.g., let $\alpha = i$ and $\beta = j$, as in Proposition 1.1. Then, μ-$a.e, w \in (\theta \cap N_0)$, $\frac{p^i_k}{p^j_k} = \frac{(1-c^k)}{(1+c^k)} < 1$. $\square$

A well-founded question is whether the LOP would still fail if the service charge became too small, or too large. Theorem 1 gives us the possibility to elaborate on this matter. Take any commodity $k \in K$, and any "sufficiently small" $\epsilon > 0$. If $c^k = \epsilon$, then so long as the conditions in the statement of the theorem hold, it is clear that the LOP still has to fail. A similar argument would be true if $c^k = 1 - \epsilon$. This is because large agents cannot profitably deviate, for their actions affect market-clearing prices adversely. On the other hand, small agents, who themselves "live in a world of infinitesimals," (Dubey and Shapley, 1994: p. 264) are hindered by *any* positive service charge.

Theorem 1 also shows that as c^k decreases (increases), the degree of price dispersion decreases (increases). So, suppose that c^k were very large (close to 1). Would such a high c^k not be prohibitive to trade? It can be easily argued that this need not be the case. Using Theorem 1, we know that a high c^k implies that prices across posts for commodity k must be *vastly unequal*. Thus, if an agent were, say, a net buyer at post Υ^i_k and a net seller at post Υ^j_k, and had no incentive to shift orders across these posts, then this must be because: (i) his total expenditure (inclusive of the premium payable) at Υ^i_k is sufficiently low, and; (ii) his total revenue (net of the premium payable) that he is receiving at Υ^j_k is sufficiently high. (i) and (ii) imply that if $p^i_k < p^j_k$ (or $p^i_k > p^j_k$), then this agent makes high (low) *non-zero* net trades at both posts—while agents on the other sides of the corresponding markets make low (high) net trades. This discussion thus shows that a high c^k does not destroy agents' incentives to trade. An analogous argument is true when c^k is close to zero.

Yet another piece of information that can be gleaned from Theorem 1 is that at equilibrium, only two different prices may be obtained across posts for any commodity $k \in K$ at any time.

2.3 Unequal-price equilibrium: an example

We exemplify the failure of the LOP in a setup with: (i) an atomless continuum of small agents; (ii) finitely many atoms, and; (iii) finitely many posts per commodity.[16]

Let $(N, \mathcal{N}, \mu)$ be a measure space of agents where $N = N_0 \cup A = (0, 1] \cup \{2, 3\}$, and μ is as

[16] For the economy considered in this subsection—and the economies in the subsequent counterexamples as well—equilibria in which the LOP *holds* can be trivially constructed, and are available from the author.

defined in Section 2. For the sake of clarity, we denote agents 2 and 3 by i and j, respectively. There are two commodities $\{k, l\}$, and i and j each own one trading post for each commodity. Each trading post has capacity $\chi^{atom} = \mu(N) = 3$. The service charges allocated to the post owners for each commodity are $(c^k, c^l) = (\frac{1}{5}, \frac{1}{10})$. The commodity endowments of the agents are:

$$\begin{aligned}
\big(e_k(w), e_l(w), e_m(w)\big) &= \left(\tfrac{3999}{2}, \quad \tfrac{21994}{11}, \quad \tfrac{10013}{5}\right), \quad \mu\text{-}a.e, w \in N_0,\\
\big(e_k(i), e_l(i), e_m(i)\big) &= \left(\tfrac{15947}{8}, \quad \tfrac{4451701}{2222}, \quad \tfrac{4038309}{2020}\right),\\
\big(e_k(j), e_l(j), e_m(j)\big) &= \left(\tfrac{16057}{8}, \quad \tfrac{4437511}{2222}, \quad \tfrac{4036439}{2020}\right).
\end{aligned}$$

Consider the markets for k. We look for a profile which simultaneously satisfies the N.E conditions in Propositions 1.1 (for $w \in N_0$), 1.2, and Theorem 1, which the following strategies do:

$$\begin{aligned}
\left(\, b^i_k(w), \quad q^i_k(w), \quad b^j_k(w), \quad q^j_k(w) \,\right) &= \left(\, 2, \quad 2, \quad 5, \quad \tfrac{9}{4} \,\right), \quad \mu\text{-}a.e, w \in N_0,\\
\left(\, b^i_k(i), \quad q^i_k(i), \quad b^j_k(i), \quad q^j_k(i) \,\right) &= \left(\, 8, \quad \tfrac{3}{2}, \quad 20, \quad \tfrac{87}{8} \,\right),\\
\left(\, b^i_k(j), \quad q^i_k(j), \quad b^j_k(j), \quad q^j_k(j) \,\right) &= \left(\, 3, \quad 3, \quad \tfrac{1}{2}, \quad 6 \,\right).
\end{aligned}$$

For the above representation of bids and offers in the two markets for commodity k, the prevailing market-clearing prices are $p^i_k = 2$, and $p^j_k = (4/3)$.[17]

Consider now the markets for commodity l. It can again be verified that the profile of strategies below satisfies Propositions 1.1 (for $w \in N_0$), 1.2, and Theorem 1:

$$\begin{aligned}
\left(\, b^i_l(w), \quad q^i_l(w), \quad b^j_l(w), \quad q^j_l(w) \,\right) &= \left(\, 15, \quad 2, \quad 2, \quad 3 \,\right), \quad \mu\text{-}a.e, w \in N_0,\\
\left(\, b^i_l(i), \quad q^i_l(i), \quad b^j_l(i), \quad q^j_l(i) \,\right) &= \left(\, \tfrac{57}{101}, \quad \tfrac{7}{2}, \quad \tfrac{47602}{303}, \quad 43 \,\right),\\
\left(\, b^i_l(j), \quad q^i_l(j), \quad b^j_l(j), \quad q^j_l(j) \,\right) &= \left(\, \tfrac{1695397}{7676}, \quad \tfrac{16715}{228}, \quad \tfrac{6907}{606}, \quad \tfrac{1}{2} \,\right).
\end{aligned}$$

For the above representation of bids and offers in the two markets for commodity l, the prevailing market-clearing prices are $p^i_l = 3$, and $p^j_l = (11/3)$.

Each agent ends up with consumption $(x_{n,k}, x_{n,l}, x_{n,m}) = (2000, 2000, 2000)$, μ-$a.e, n \in N$. From here, one can proceed as in Koutsougeras (1999) to easily derive utility functions such that the above profile of strategies constitutes an N.E. For concreteness, we present one solution:

[17] We remark that in the class of SMG that we propose in this paper, it is necessary that there be wash-sales for the LOP to fail, consistent with Bloch and Ferrer's (2001) finding. We also point out that agents μ-$a.e, w \in N_0$, need not be identical. We have done this for simplicity, and to ensure that unnecessary matters do not detract from the main message of the paper.

$$
\begin{array}{rcllll}
u_w(\mathbf{x}_w) & = & 16\ln(x_{w,k}) & + & 33\ln(x_{w,l}) & + & 10\ln(x_{w,m}),\ \mu\text{-}a.e, w \in N_0, \\
u_i(\mathbf{x}_i) & = & 779424\ln(x_{i,k}) & + & 769923\ln(x_{i,l}) & + & 243570\ln(x_{i,m}), \\
u_j(\mathbf{x}_j) & = & 117376\ln(x_{j,k}) & + & 366630\ln(x_{j,l}) & + & 104800\ln(x_{j,m}).
\end{array}
$$

3 The failure of the LOP, Part 2: The model

We now analyse the failure of the LOP in a context featuring: (i) an atomless continuum of small agents; (ii) finitely many atoms, and; (iii) infinitely many markets per commodity.

We consider a complete, finite measure space of agents $(N, \mathcal{N}, \mu)$, where $N = N_0 \cup A$, $N_0 = (0, 1]$, and $A = \{2, \dots, H\}$. μ is Lebesgue when restricted to $\mathcal{N}_{N_0}$, and the counting measure when restricted to $\mathcal{N}_A$, where $\mathcal{N}_{N_0}$ and $\mathcal{N}_A$ are as defined in Section 2. The commodity space is defined by $\mathbb{R}^{L+1}$, and the consumption set of each agent by $\mathbb{R}^{L+1}_+$. Each (large) post owner now owns countably *infinitely many trading posts* for each commodity $k \in K = \{1, 2, \dots, L\}$. Now, define $Y = \ell^1 \times \ell^1$, where each factor ℓ^1, the space of absolutely-summable real sequences, is equipped with its 1-norm, i.e., is given its usual norm topology. We supply Y with the product topology, and we equip it with the norm $\|(x, y)\|_Y = \|x\|_{\ell^1} + \|y\|_{\ell^1} < \infty, (x, y) \in Y$. Next, consider $\Theta := \ell^1_+ \times \ell^1_+$, where ℓ^1_+ denotes the positive cone of ℓ^1. The strategy sets of agents can then described by a measurable correspondence $S : N \rightrightarrows 2^{\Theta}$, such that[18]

$$
S(n) = \Big\{(b(n), q(n)) \in \ell^1_+ \times \ell^1_+ : \sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|} b_k^{s,r}(n) + \Lambda(n) \leq e_m(n)\,; \sum_{r=1}^{\infty}\sum_{s=1}^{|P|} q_k^{s,r}(n) \leq e_k(n)\,, k \in K\Big\},
$$

where $(b(n), q(n)) = \Big\{\big\{\{b_k^{s,r}(n), q_k^{s,r}(n)\}_{s=1}^{|P|}\big\}_{k=1}^{L}\Big\}_{r=1}^{\infty}$, and $|P|$ is the number of large post owners in A. Note that in this section, $\Upsilon_k^{s,r}$ represents the r^{th} trading post owned by post owner $s \in A$ for commodity k. As before, posts owned by atoms have capacity $\chi^{atom} = \mu(N)$, while $\chi^{small} = 0$.

A strategy profile consists of a pair of measurable mappings $b : N \to \ell^1_+$, and $q : N \to \ell^1_+$, such that $(b(n), q(n)) \in S(n)$ $a.e$ in N. ℓ^1_+ is a separable Banach (complete normed vector) space. Y, equipped with the norm as described above, is also a Banach space, and therefore, so is Θ. Now, it is well known that the countable Cartesian product of separable spaces is separable, and hence, we

[18] Since for agents μ-$a.e$, $n \in N$, $e_k(n) < \infty$ $\forall k \in (\{m\} \cup K)$, we have that bids and offers must converge for every individual, hence our choice of sequence space.

have that the product vector space Θ is a separable Banach space. Considering $S : N \rightrightarrows 2^{\Theta}$ has a measurable graph, we have that the measurable mappings b and q exist, by AMST. A mapping $f : N \to X$ is said to be μ-measurable if there exists a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that μ-$a.e$, $\lim_{n\to\infty} \|f_n - f\|_X = 0$. Seeing that every measurable mapping can be obtained as the limit of some sequence of simple functions, it follows that b and q are μ-measurable. A μ-measurable function $f : N \to X$ is called Bochner-integrable if there exists a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \int_N \|f_n - f\|_X \, d\mu = 0$. It can equally be shown that a μ-measurable function $f : N \to X$ is Bochner-integrable iff its norm function $\|f\|$ is Lebesgue-integrable, i.e., $\int_N \|f\|_X \, d\mu < \infty$ (see, e.g., Diestel and Uhl, 1977: p. 45). As $(N, \mathcal{N}, \mu)$ is a finite measure space, it is easy to see that $\int_N \|b\|_{\ell^1} \, d\mu < \infty$ and $\int_N \|q\|_{\ell^1} \, d\mu < \infty$, and thus, we have that b and q are Bochner-integrable. Furthermore, since the μ-measurable mappings b and q take values in a separable Banach space, they are also weakly measurable, by Pettis' Measurability Theorem (see, e.g., Diestel and Uhl, 1977: p. 42). This therefore implies that the coordinate functionals $\{\{\{b_k^{s,r}(n), q_k^{s,r}(n)\}_{s=1}^{|P|}\}_{k=1}^{L}\}_{r=1}^{\infty}$, where $b_k^{s,r}, q_k^{s,r} : \mathbb{N} \times \mathbb{R}_+ \supset N \to \mathbb{R}_+$, are also measurable. Hence, for a given strategy profile $(b, q) \in Gr(S)$, we define, for each $k \in K$, $s \in \{1, 2, \ldots, |P|\}$ and $r \in \mathbb{N}$, $B_k^{s,r} = \int_N b_k^{s,r}(n) d\mu < \infty$ and $Q_k^{s,r} = \int_N q_k^{s,r}(n) d\mu < \infty$. Transactions at each post clear through the price $p_k^{s,r} = (B_k^{s,r}/Q_k^{s,r})$. The allocation rule for pure traders μ-$a.e, h \in N$, is now:

$$x_{h,k} = \begin{cases} e_k(h) + \displaystyle\sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\Big(b_k^{s,r}(h) \cdot \frac{Q_k^{s,r}}{B_k^{s,r}} - q_k^{s,r}(h)\Big) & \text{if } k \neq m; \\ e_k(h) + \displaystyle\sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|}\Big(q_k^{s,r}(h) \cdot \frac{B_k^{s,r}}{Q_k^{s,r}} - b_k^{s,r}(h)\Big) \cdot \big(1 + t_h^{s,r,k} c^k\big) & \text{if } k = m, \end{cases} \tag{4}$$

while the distribution rule in (2) for any (large) post owner $i \in A$, must now be rewritten as

$$x_{i,k} = \begin{cases} e_k(i) + \displaystyle\sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\Big(b_k^{s,r}(i) \cdot \frac{Q_k^{s,r}}{B_k^{s,r}} - q_k^{s,r}(i)\Big) & \text{if } k \neq m; \\ e_k(i) - \displaystyle\sum_{r=1}^{\infty}\sum_{k=1}^{L} c^k \cdot \Big(\int_N t_n^{i,r,k} q_k^{i,r}(n) d\mu \cdot \frac{B_k^{i,r}}{Q_k^{i,r}} - \int_N t_n^{i,r,k} b_k^{i,r}(n) d\mu\Big) & \\ \qquad + \displaystyle\sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|}\big(q_k^{s,r}(i) \cdot \frac{B_k^{s,r}}{Q_k^{s,r}} - b_k^{s,r}(i)\big) \cdot \big(1 + t_i^{s,r,k} c^k\big) & \text{if } k = m, \end{cases} \tag{5}$$

where as before, any division by zero, including $0/0$, is taken to equal zero. The conditions on $t_n^{s,r,k}$

for each $n \in N$ and $s \in \{1, 2, \ldots, |P|\}$ are as in Section 2. We impose that the total endowment and allocation of any commodity $k \in (\{m\} \cup K)$ in the economy be such that $0 < \int_N x_{n,k} d\mu \leq \int_N e_k(n) d\mu < \infty$.

At an N.E with positive bids and offers, agents are viewed as solving the same programme as in (3). By Assumption (ii), $u(\cdot)$ is (twice-) continuously differentiable, and since our commodity space is still finite-dimensional, we have that $u(\cdot)$ is in fact (twice-) continuously Fréchet-differentiable[19] (in x). As we show in the Appendix, Lemma 6, $x_{h,k}$ and $x_{i,k}$, as (well-) defined above, are in turn, Gâteaux-differentiable[19] (in b and q). Hence, we are able to prove that:

PROPOSITION 2 The (no-arbitrage) equilibrium conditions in Propositions 1.1, 1.2 and Theorem 1 are true for this section as well, *mutatis mutandis*.[20]

3.1 Unequal-price equilibrium: an example

We demonstrate the invalidity of the LOP in a framework featuring: (i) an atomless continuum of small agents; (ii) finitely many atoms; and; (iii) infinitely many markets per commodity.

We let $(N, \mathcal{N}, \mu)$ be a complete, finite measure space as defined in Section 3, where $N = N_0 \cup A = (0,1] \cup \{2,3\}$. As before, we denote agents 2 and 3 by i and j, respectively. There are two commodities $\{k, l\}$, and i and j each own countably infinitely many trading posts for each commodity. The capacity of each trading post is $\chi^{atom} = \mu(N) = 3$. The service charges allocated to the post owners for each commodity are $(c^k, c^l) = (\frac{1}{5}, \frac{1}{10})$. The commodity endowments of the agents are as follows:

$$
\begin{array}{lcllll}
 & & k & l & m & \\
e(w) & = & (1999, & \frac{21988}{11}, & \frac{10026}{5}), & \mu\text{-}a.e, w \in N_0, \\
e(i) & = & (\frac{7947}{4}, & \frac{2229701}{1111}, & \frac{2018309}{1010}), & \\
e(j) & = & (\frac{8057}{4}, & \frac{2215511}{1111}, & \frac{2016439}{1010}). &
\end{array}
$$

At this point, it is helpful to note that in Propositions 1.1 (for $w \in N_0$) and 1.2, the prices, along with the fractions $\frac{B^i_{-i,k}}{Q^i_{-i,k} + c^k \cdot \left(\int_{N \setminus \{i\}} t_n^{i,k} q_k^i(n) d\mu\right)}$, $\frac{Q^j_{-i,k}}{B^j_{-i,k}}$, $\frac{B^i_{-j,k}}{Q^i_{-j,k}}$, $\frac{Q^j_{-j,k} + c^k \cdot \left(\int_{N \setminus \{j\}} t_n^{j,k} q_k^j(n) d\mu\right)}{B^j_{-j,k}}$, are *homogeneous of degree zero* in bids and offers, a fact which will be used throughout this example.

[19] Please refer to Luenberger (1969: pp. 171-172) for a formal definition.

[20] ℓ^1_+ has an empty interior. Thus, in the statements of these results, the equilibria must be recharacterised, instead of *interior*, as N.E with positive bids and offers, and no binding liquidity and offer constraints.

In particular, using the above fact, we may easily reformulate the strategy profile in Section 2.3 to let i and j both own countably infinitely many markets for each commodity, by simply replicating posts as we next show. So first, we relabel the posts owned by i for commodity k as $\{\Upsilon_k^{i,1}, \Upsilon_k^{i,2}, \ldots\} = \{\Upsilon_k^{i,r}\}_{r=1}^{\infty}$, and similarly for posts owned by j, $\{\Upsilon_k^{j,r}\}_{r=1}^{\infty}$. It can then be verified, for *any pair* of trading posts $\Upsilon_k^{i,\alpha}, \Upsilon_k^{j,\beta}$, $\alpha, \beta \in \mathbb{N}$, that the following strategies satisfy Propositions 1.1, 1.2 and Theorem 1, *mutatis mutandis*, thus constituting an (unequal-price) N.E:

For commodity k, at posts $\Upsilon_k^{i,r}$ and $\Upsilon_k^{j,r}$, $r \in \mathbb{N}$:

	Bids at $\Upsilon_k^{i,r}$	Offers at $\Upsilon_k^{i,r}$	Bids at $\Upsilon_k^{j,r}$	Offers at $\Upsilon_k^{j,r}$
μ-$a.e, w \in N_0$	$\left(\frac{1}{2^{r-1}}\right) \cdot 2$	$\left(\frac{1}{2^{r-1}}\right) \cdot 2$	$\left(\frac{1}{2^{r-1}}\right) \cdot 5$	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{9}{4}$
i	$\left(\frac{1}{2^{r-1}}\right) \cdot 8$	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{3}{2}$	$\left(\frac{1}{2^{r-1}}\right) \cdot 20$	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{87}{8}$
j	$\left(\frac{1}{2^{r-1}}\right) \cdot 3$	$\left(\frac{1}{2^{r-1}}\right) \cdot 3$	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{1}{2}$	$\left(\frac{1}{2^{r-1}}\right) \cdot 6$

For the above representation of bids and offers in the markets for commodity k, the prevailing market-clearing prices are $p_k^{i,r} = 2$, and $p_k^{j,r} = (4/3), r \in \mathbb{N}$.

For commodity l, at posts $\Upsilon_l^{i,r}$ and $\Upsilon_l^{j,r}$, $r \in \mathbb{N}$:

	Bids at $\Upsilon_l^{i,r}$	Offers at $\Upsilon_l^{i,r}$	Bids at $\Upsilon_l^{j,r}$	Offers at $\Upsilon_l^{j,r}$
μ-$a.e, w \in N_0$	$\left(\frac{1}{2^{r-1}}\right) \cdot 15$	$\left(\frac{1}{2^{r-1}}\right) \cdot 2$	$\left(\frac{1}{2^{r-1}}\right) \cdot 2$	$\left(\frac{1}{2^{r-1}}\right) \cdot 3$
i	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{57}{101}$	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{7}{2}$	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{47602}{303}$	$\left(\frac{1}{2^{r-1}}\right) \cdot 43$
j	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{1695397}{7676}$	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{16715}{228}$	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{6907}{606}$	$\left(\frac{1}{2^{r-1}}\right) \cdot \frac{1}{2}$

For the above representation of bids and offers in the markets for commodity l, the prevailing market-clearing prices are $p_l^{i,r} = 3$, and $p_l^{j,r} = (11/3)$, $r \in \mathbb{N}$.

It is also interesting to note that for both k and l, the LOP would still fail if instead of $\frac{1}{2^{r-1}}, r \in \mathbb{N}$, we multiplied every b and q by $\frac{1}{\pi^{r-1}}, r \in \mathbb{N}$, where $\pi \in (1, \infty)$.

Each agent ends up with consumption $(x_{n,k}, x_{n,l}, x_{n,m}) = (2000, 2000, 2000)$, μ-$a.e, n \in N$. Utility functions, such that the above profile of strategies constitutes an N.E, are as in Section 2.3.

4 The failure of the LOP, Part 3: The model

In this section, we look at the failure of the LOP in a context featuring: (i) an atomless continuum of small agents; (ii) infinitely many atoms, and; (iii) infinitely many markets per commodity.

For this section, we let the set of agents be denoted by $N = N_0 \cup A \cup C$, where as before, $N_0 = (0,1]$ and $A = \{2, ..., H\}$, and we define $C = \{H+1, H+2, \ldots\}$. We let $\mathscr{S}_0$, $\mathscr{S}_A$, $\mathcal{N}_{N_0}$, $\mathcal{N}_A$, μ_0 and μ_A be defined as in Section 2, and we define $\mathscr{S}_C = \mathcal{P}(C)$. Trivially, $\mathscr{S}_C$ is a σ-algebra. So, let μ_C be a measure on $\mathscr{S}_C$ such that for each agent $(H+g) \in C$, $g \in \mathbb{N}$, we have $\mu_C(H+g) = \big(1/2^g\big)$. Then, as in the previous sections, the triple $(N, \mathcal{N}, \mu)$ is a complete, finite measure space of agents (see Appendix, Lemma 7). There are finitely many commodities $k \in K = \{1, 2, \ldots, L\}$, and *each* large post owner is endowed with *infinitely many posts* for each commodity, the capacity of each post being equal to $\mu(N)$. To keep notation tractable, we choose to let *post owners lie in A only*.[21] The commodity space is $\mathbb{R}^{L+1}$, and the consumption set of every individual is $\mathbb{R}^{L+1}_+$. Let the product vector space Y be defined as in Section 3, and consider $\Theta \coloneqq \ell^1_+ \times \ell^1_+$. The strategy sets of agents are described by a measurable correspondence $S : N \rightrightarrows 2^{\Theta}$, such that

$$S(n) = \Big\{(b(n), q(n)) \in \ell^1_+ \times \ell^1_+ : \sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|} b_k^{s,r}(n) + \Lambda(n) \le e_m(n)\, ; \sum_{r=1}^{\infty}\sum_{s=1}^{|P|} q_k^{s,r}(n) \le e_k(n), k \in K\Big\},$$

where $(b(n), q(n)) = \Big\{\big\{\{b_k^{s,r}(n), q_k^{s,r}(n)\}_{s=1}^{|P|}\big\}_{k=1}^{L}\Big\}_{r=1}^{\infty}$, and $|P|$ is the number of post owners in A.

A strategy profile consists of a pair of measurable mappings $b : N \to \ell^1_+$, and $q : N \to \ell^1_+$, such that $(b(n), q(n)) \in S(n)$ *a.e* in N. By the same argument as in the preceding section, we have that the μ-measurable mappings b and q exist, by AMST. Since $(N, \mathcal{N}, \mu)$ as defined in this section is still a finite measure space, it easily follows that $\int_N \|b\|_{\ell^1} d\mu < \infty$ and $\int_N \|q\|_{\ell^1} d\mu < \infty$, such that b and q are Bochner-integrable. As in the previous section, b and q are also weakly measurable, such that the component functions $\Big\{\big\{\{b_k^{s,r}(n), q_k^{s,r}(n)\}_{s=1}^{|P|}\big\}_{k=1}^{\infty}\Big\}_{r=1}^{\infty}$, where $b_k^{s,r}, q_k^{s,r} : \mathbb{N} \times \mathbb{R}_+ \supset N \to \mathbb{R}_+$, are measurable. For a given strategy profile $(b, q) \in Gr(S)$, we then define, for each $k \in \mathbb{N}$, $s \in \{1, 2, \ldots |P|\}$, and $r \in \mathbb{N}$, $B_k^{s,r} = \int_N b_k^{s,r}(n) d\mu < \infty$ and $Q_k^{s,r} = \int_N q_k^{s,r}(n) d\mu < \infty$. Transactions at each post clear through the price $p_k^{s,r} = (B_k^{s,r}/Q_k^{s,r})$.

Consumption assignments for $k = m, 1, \ldots, L$, to pure traders μ-*a.e*, $h \in N$, are defined as in (4), while for any (large) post owner $i \in A$, the allocation rule is as in (5). As before, the total endowment and allocation of every commodity $k \in (\{m\} \cup K)$ in the economy are such that $0 < \int_N x_{n,k} d\mu \le \int_N e_k(n) d\mu < \infty$.

[21] We could, with no major technical difficulty, let post owners lie in $A \cup C$. However, this comes at the cost of an even more cumbersome notation, with no significant gain in intuition, and no change whatsoever in our conclusions.

At an N.E with positive bids and offers, agents are viewed as solving the same problem as in (3).

PROPOSITION 3 The results in Propositions 1.1 through to 2 and Theorem 1, are true for this section as well, *mutatis mutandis.*

PROOF: The proof is identical to that of Proposition 2, and is therefore omitted. $\square$

4.1 Unequal-price equilibrium: an example

We show the failure of the LOP in a framework featuring: (i) an atomless continuum of small agents; (ii) infinitely many atoms, and; (iii) infinitely many markets per commodity.

Let $(N, \mathcal{N}, \mu)$ be a measure space of agents as defined in Section 4 above. Consider an economy where $N = N_0 \cup A \cup C$, $N_0 = (0,1]$, $A = \{2,3\}$, and $C = \{4,5,\ldots\}$, where agents 2 and 3 are a large post owner and a large pure trader, respectively. We denote agent 2 by i, agent 3 by τ, and the agents in C by $3+g$, $g \in \mathbb{N}$. The set of commodities is $K = \{1,2,\ldots,10\}$, and i owns infinitely many markets for each commodity. The capacity of each post is $\mu(N) = 4$. The service charges allocated to i for commodities $a \in K_O = \{1,3,5,7,9\}$ and $b \in K_E = \{2,4,6,8,10\}$ are:

$$(c^a, c^b) = (\tfrac{1}{5}, \tfrac{1}{10}).$$

The commodity endowments of the agents are as follows:[22]

$$\begin{array}{llcllll}
 & & & \forall a \in K_O & \forall b \in K_E & m & \\
\mu\text{-}a.e, w \in N_0, & e(w) & = & \Big(\big(\frac{1}{2^{a-1}}\big)\cdot 1999.92, & \big(\frac{1}{2^{b-2}}\big)\cdot 1999.92, & 2003.17\Big), \\
 & e(i) & = & \Big(\big(\frac{1}{2^{a-1}}\big)\cdot 2000.62, & \big(\frac{1}{2^{b-2}}\big)\cdot 2000.34, & 1980.88\Big), \\
 & e(\tau) & = & \Big(\big(\frac{1}{2^{a-1}}\big)\cdot 1999.46, & \big(\frac{1}{2^{b-2}}\big)\cdot 1999.74, & 2015.95\Big), \\
 & e(3+g),\ g \in \mathbb{N} & = & \Big(\big(\frac{1}{2^{a-1}}\big)\cdot 2000.00, & \big(\frac{1}{2^{b-2}}\big)\cdot 2000.00, & 2000.00\Big).
\end{array}$$

Consider first a pair of commodities $a \in K_O$ and $b \in K_E$. It can be verified that the strategies below satisfy the conditions as in Theorem 1 and Propositions 1.1 and 1.2, *mutatis mutandis*,[23]

[22] For clarity of exposition, the numbers in this example have been rounded off. The *exact* figures (rational numbers), which were used in all computations throughout this example, are available from the author.

[23] There is now only one post owner i. So, at an equilibrium with positive bids and offers, and no binding liquidity and offer constraints, the prices for any $k \in K$ between *any* two trading posts $\Upsilon_k^{i,\alpha}, \Upsilon_k^{i,\beta}$, $\alpha, \beta \in \mathbb{N}$, should satisfy the following (no-arbitrage) condition for i:

and therefore constitute an (unequal-price) N.E:

For any commodity $a \in K_O$, at posts $\Upsilon_a^{i,2g-1}$ and $\Upsilon_a^{i,2g}$, $g \in \mathbb{N}$:

Agent	Bids at $\Upsilon_a^{i,2g-1}$	Offers at $\Upsilon_a^{i,2g-1}$	Bids at $\Upsilon_a^{i,2g}$	Offers at $\Upsilon_a^{i,2g}$
μ-$a.e, w \in N_0$,	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 1.000$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.000$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.000$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.001$
i	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.429$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.656$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 1.000$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.058$
τ	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 9.000$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.207$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.100$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.001$
Every agent $\in C$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 9.240$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.765$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.008$	$\left(\frac{1}{2^{a-1+g}}\right) \cdot 0.000$

For the above representation of bids and offers in the markets for commodity a, the prevailing market-clearing prices are $p_a^{i,2g-1} = 12.08$, and $p_a^{i,2g} = 18.13$, $g \in \mathbb{N}$.

For any commodity $b \in K_E$, at posts $\Upsilon_b^{i,2g-1}$ and $\Upsilon_b^{i,2g}$, $g \in \mathbb{N}$:

Agent	Bids at $\Upsilon_b^{i,2g-1}$	Offers at $\Upsilon_b^{i,2g-1}$	Bids at $\Upsilon_b^{i,2g}$	Offers at $\Upsilon_b^{i,2g}$
μ-$a.e, w \in N_0$,	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 1.100$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.000$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.000$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.001$
i	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.429$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.368$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 1.125$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.067$
τ	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 8.273$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.327$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.100$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.003$
Every agent $\in C$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 11.60$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.824$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.008$	$\left(\frac{1}{2^{b-2+g}}\right) \cdot 0.000$

For the above representation of bids and offers in the markets for commodity b, the prevailing market-clearing prices are $p_b^{i,2g-1} = 14.09$, and $p_b^{i,2g} = 17.22$, $g \in \mathbb{N}$.

Certainly, the same can be done for every other pair of commodities $c \in K_O \setminus \{a\}$ and $d \in K_E \setminus \{b\}$. In this light, the final consumption of all agents μ-$a.e, n \in N$, is $(x_{n,a}, x_{n,b}, x_{n,m}) = (2000/2^{a-1}, 2000/2^{b-2}, 2000)$, $\forall a \in K_O$, $\forall b \in K_E$. From here, one can follow Koutsougeras (1999) to derive utility functions such that the above strategies form an N.E.

Before we close this section, we highlight a few striking features of this example. First, as in the examples in Subsections 2.3 and 3.1, equal-price equilibria can also be easily constructed for the economy considered in this example. Second, note that the allocated service charge is the same

$$\left(p_k^{i,\alpha}\right)^2 = \frac{B_{-i,k}^{i,\alpha} \cdot \left[Q_{-i,k}^{i,\beta} + c^k \cdot \left(\int_{N\setminus\{i\}} t_n^{i,\beta,k} q_k^{i,\beta}(n) d\mu\right)\right]}{\left[Q_{-i,k}^{i,\alpha} + c^k \cdot \left(\int_{N\setminus\{i\}} t_n^{i,\alpha,k} q_k^{i,\alpha}(n) d\mu\right)\right] \cdot B_{-i,k}^{i,\beta}} \left(p_k^{i,\beta}\right)^2.$$

This result follows as a simple corollary of Proposition 1.2, and is readily seen from (6) in the proof of Proposition 1.2. Proposition 1.1 still holds, with the only difference being that posts are now also indexed by $\alpha, \beta \in \mathbb{N}$.

at all posts for a commodity $k \in K$, and yet, the LOP still fails across infinitely many posts, all owned by the *same* agent.[24] It is also interesting to note that $\forall a \in K_O$, and $\forall b \in K_E$, we have that $\left|p_a^{i,2g-1} - p_a^{i,2g}\right| > \left|p_b^{i,2g-1} - p_b^{i,2g}\right| > \left|p_k^{i,r} - p_k^{j,r}\right| = \left|p_l^{i,r} - p_l^{j,r}\right|$, $g, r \in \mathbb{N}$, where p_k^i, p_k^j, p_l^i and p_l^j are as in Subsections 2.3 and 3.1. In other words, increasing the number of atoms denumerably has, intriguingly, not decreased the degree of price dispersion. There is thus no convergence of *any kind* to no-arbitrage equilibria, and definitely not to Walrasian (price-taking) equilibria. The main message of this example (and of this paper, as a matter of fact) is that it is not who owns the trading posts that matters, nor how many posts one owns—finitely or infinitely many. What truly counts is merely that posts be privately owned, such that service charges are imposed, for strategic behaviour to influence the resulting equilibrium market structure.

5 Conclusion

In this paper, we have presented a framework in which strategic behaviour by agents, and private ownership of trading posts, affect the set of equilibria of large economies non-trivially. Indeed, while Koutsougeras (2003a) shows that the structure of trading posts becomes immaterial as the number of agents increases without bounds, this is not true for our model. This is evidenced by the failure of the LOP even in economies with *many* agents *and* markets. Thus, we can conclude that in large frictionless (in the sense that no money and/or commodity leaves the system or is "lost") SMG, it is strategic behaviour, alongside the private ownership of trading posts, that constitutes the *only* source of equilibria with arbitrage.

In light of the above, we believe that the model proposed in this paper can be easily applied to other fields of study, to provide a theoretical underpinning for the failure of the LOP that is based on strategic behaviour by agents. In particular, fields such as differential-information, international, macro-, and even public economics[25] would be suitable avenues, for these are areas

[24]The presence of only one post owner does not weaken our conclusions in any way; if anything, it strengthens them. This is because at a glance, it would seem that this post owner would have the *strongest* incentives to shift his orders across posts (all of which he owns) in a way such that the aggregate net trade across the markets remains unchanged. However, this unilateral deviation will not be profitable for him—see, for instance, Koutsougeras (2003b) and Toraubally (2017b).

[25]The interested reader is referred to Goenka et al. (1998) for a pioneering and rigorous application of SMG to macroeconomics. For an application of SMG to differential-information economics, see, e.g., Faias et al. (2010). For a recent application of SMG to international trade, see, e.g., Toraubally (2017b). For the use of the SMG

in which there are, not only large numbers of agents and markets, but also many well-documented cases of inconsistent prices. In future research, given the crucial role that the trading-post service charges play in this model, it would be worthwhile to analyse in greater detail how changes in those charges affect the behaviour of net buyers and net sellers, and in particular, the direction of trade.

Appendix

LEMMA 1 The set function $\nu : \mathscr{S} \to [0, \infty]$, where $\mathscr{S} = \{W \subseteq N : W = K \cup L; K \in \mathscr{S}_0; L \in \mathscr{S}_A\}$, such that $\nu(W) = \nu_0 (W \cap N_0) + \mu_A (W \cap A)$ for each $W \in \mathscr{S}$, is a measure.

PROOF: It can be easily shown, given $\mathscr{S}_0$ and $\mathscr{S}_A$ are semirings, that $\mathscr{S}$ is also a semiring. So, for each countable family $\{W_n\}_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathscr{S}$, with $\bigcup_{n=1}^{\infty} W_n \in \mathscr{S}$, we have

$$\begin{aligned}
\nu\left(\bigcup_{n=1}^{\infty} W_n\right) &= \nu_0\left(\left(\bigcup_{n=1}^{\infty} W_n\right) \cap N_0\right) + \mu_A\left(\left(\bigcup_{n=1}^{\infty} W_n\right) \cap A\right) \\
&= \nu_0\left(\bigcup_{n=1}^{\infty} (W_n \cap N_0)\right) + \mu_A\left(\bigcup_{n=1}^{\infty} (W_n \cap A)\right) \\
&= \sum_{n=1}^{\infty} \nu_0 (W_n \cap N_0) + \sum_{n=1}^{\infty} \mu_A (W_n \cap A) \\
&= \sum_{n=1}^{\infty} \left(\nu_0 (W_n \cap N_0) + \mu_A (W_n \cap A)\right) = \sum_{n=1}^{\infty} \nu (W_n) .
\end{aligned}$$

So ν is σ-additive. Finally, by construction, we have that $\nu(\emptyset) = \nu_0(\emptyset) + \mu_A(\emptyset) = 0$. □

Since ν is a measure on $\mathscr{S}$, it generates a nonnegative extended real-valued set function μ, the Carathéodory extension of ν, defined on $\mathcal{P}(N)$. μ is, as is well known, an outer measure. So, a set $W \subseteq N$ is said to be μ-measurable if $\mu(Y) = \mu(Y \cap W) + \mu(Y \cap W^c)$, for each $Y \subseteq N$. Next, denote by $\mathcal{N}$ the collection of all the μ-measurable subsets of N. Thus, we have that $\mathcal{N}$ is a σ-algebra, and μ is a complete measure (see, e.g., Aliprantis and Border, 2006: p. 387) when restricted to $\mathcal{N}$. We then have that:

LEMMA 2 The triple $(N, \mathcal{N}, \mu)$ is a complete, finite measure space of agents. Moreover, μ is the unique extension of ν to a measure on $\mathcal{N}$.

PROOF: Since $\mu(N) = H < \infty$, we have that the measure space $(N, \mathcal{N}, \mu)$ is finite. This implies that the measure ν on $\mathscr{S}$ is also finite, and therefore, σ-finite. Since $\mathcal{N}$ is a semiring with $\mathscr{S} \subseteq \mathcal{N}$,

mechanism in public economics, see, e.g., Faias et al. (2014).

our result follows (see, e.g., Aliprantis and Border, 2006: pp. 382-383). $\square$

Consider now the set function $\nu_0 : \mathscr{S}_0 \to [0, \infty]$. Since ν_0 is a measure on $\mathscr{S}_0$, it also generates a nonnegative extended real-valued set function μ_0, the Carathéodory extension of ν_0, defined on $\mathcal{P}(N_0)$. Denote by $\mathcal{N}_0$ the collection of all the μ_0-measurable subsets of N_0. Then, we have that $\mathcal{N}_0$ is a σ-algebra, and μ_0 is a measure when restricted to $\mathcal{N}_0$. Thus, the triple $(N_0, \mathcal{N}_0, \mu_0)$ is a complete, finite measure space, and μ_0 is the unique extension of ν_0 to a measure on $\mathcal{N}_0$. Let $\mathcal{N}_{N_0} = \{N_0 \cap W : W \in \mathcal{N}\}$ denote the restriction of $\mathcal{N}$ to N_0. We then have the following result.

LEMMA 3 The triple $\left(N_0, \mathcal{N}_{N_0}, \mu\right)$ is a measure space such that $\mathcal{N}_{N_0} = \mathcal{N}_0$, and $\mu = \mu_0$ when restricted to $\mathcal{N}_{N_0}$.

PROOF: We first prove the second part of our lemma, which involves appropriately manipulating, for every W in N_0, the following formula for μ, the Carathéodory extension of ν, thus:

$$\begin{aligned}
\mu(W) &= \inf\left\{\textstyle\sum_{n=1}^{\infty} \nu(W_n) : \{W_n\}_{n\in\mathbb{N}} \subseteq \mathscr{S}; W \subseteq \bigcup_{n=1}^{\infty} W_n\right\} \\
&= \inf\left\{\textstyle\sum_{n=1}^{\infty} \left(\nu_0(W_n \cap N_0) + \mu_A(W_n \cap A)\right) : \{W_n\}_{n\in\mathbb{N}} \subseteq \mathscr{S}; W \subseteq \bigcup_{n=1}^{\infty} W_n\right\} \\
&= \inf\left\{\textstyle\sum_{n=1}^{\infty} \left(\nu_0(W_n \cap N_0)\right) : \{W_n\}_{n\in\mathbb{N}} \subseteq \mathscr{S}; W_n \cap A = \emptyset, n \in \mathbb{N}; W \subseteq \bigcup_{n=1}^{\infty} (W_n \cap N_0)\right\} \\
&= \inf\left\{\textstyle\sum_{n=1}^{\infty} \nu_0(W_n) : \{W_n\}_{n\in\mathbb{N}} \subseteq \mathscr{S}; W_n \cap A = \emptyset, n \in \mathbb{N}; W \subseteq \bigcup_{n=1}^{\infty} (W_n \cap N_0)\right\} = \mu_0(W).
\end{aligned}$$

Now, since N_0 is μ-measurable, $\mathcal{N}_{N_0}$ is then, by definition, a collection of μ-measurable subsets of N_0. So, let $W \in \mathcal{N}_0$, and consider any $Y \subseteq N$. By the σ-subadditivity of μ, and given $\mu(\emptyset) = 0$, we have

$$\begin{aligned}
\mu(Y) &\leq \mu\big(Y \cap W\big) + \mu\big(Y \cap W^c\big) \\
&\leq \mu\big(N_0 \cap Y \cap W\big) + \mu\big(A \cap Y \cap W\big) + \mu\big(N_0 \cap Y \cap W^c\big) + \mu\big(A \cap Y \cap W^c\big) \\
&= \mu\big(N_0 \cap Y \cap W\big) + \mu\big(N_0 \cap Y \cap W^c\big) + \mu\big(A \cap Y\big).
\end{aligned}$$

Since $\mu = \mu_0$ for every $W \subseteq N_0$, and using the μ_0-measurability of W, we may show that

$$\begin{aligned}
&\mu\big(N_0 \cap Y \cap W\big) + \mu\big(N_0 \cap Y \cap W^c\big) + \mu\big(A \cap Y\big) \\
&= \mu_0\big(N_0 \cap Y \cap W\big) + \mu_0\big(N_0 \cap Y \cap W^c\big) + \mu\big(A \cap Y\big) \\
&= \mu_0\big(N_0 \cap Y\big) + \mu\big(A \cap Y\big) \\
&= \mu\big(N_0 \cap Y\big) + \mu\big(A \cap Y\big) \\
&= \mu(Y).
\end{aligned}$$

This implies that $\mu(Y \cap W)+\mu(Y \cap W^c) = \mu(Y)$, which then implies that W is also μ-measurable, thereby showing that $\mathcal{N}_0 \subseteq \mathcal{N}_{N_0}$. To show the reverse inclusion, we first let $W \in \mathcal{N}_{N_0}$, then use the μ-measurability of W, and subsequently, the σ-additivity of μ on $\mathcal{N}$, to show that W is also μ_0-measurable, such that the result that $\mathcal{N}_{N_0} \subseteq \mathcal{N}_0$ easily follows. Thus, we have $\mathcal{N}_0 = \mathcal{N}_{N_0}$. $\square$

Consider now the triple $\left(A, \mathscr{S}_A, \mu_A\right)$. Since $\mathscr{S}_A$ is the collection of all the μ_A-measurable subsets of A, and μ_A is a measure defined on $\mathcal{P}(A)$, we have that the triple $\left(A, \mathscr{S}_A, \mu_A\right)$ is a complete measure space. Let $\mathcal{N}_A = \{A \cap W : W \in \mathcal{N}\}$ denote the restriction of $\mathcal{N}$ to A. The following fact characterises the restriction of the measure space $(N, \mathcal{N}, \mu)$ to A.

LEMMA 4 The triple $\left(A, \mathcal{N}_A, \mu\right)$ is a measure space such that $\mathcal{N}_A = \mathscr{S}_A$, and $\mu = \mu_A$ when restricted to $\mathcal{N}_A$.

PROOF: The result that $\mu = \mu_A$ when restricted to $\mathcal{N}_A$ can be easily obtained by using a similar line of reasoning as in the proof of Lemma 3. To prove that $\mathcal{N}_A = \mathscr{S}_A$, note that because $\mathscr{S}_A \coloneqq \mathcal{P}(A)$, we only need to show that $\mathscr{S}_A \subseteq \mathcal{N}_A$, which is trivial, since $\mathscr{S}_A \subset \mathscr{S} \subseteq \mathcal{N} \Rightarrow \mathscr{S}_A = \{A \cap L : L \in \mathscr{S}\} \subseteq \{A \cap W : W \in \mathcal{N}\} = \mathcal{N}_A$. $\square$

From Lemmata 3 and 4 respectively, it is easy to show that the measure space $\left(N_0, \mathcal{N}_{N_0}, \mu\right)$ is atomless, while the measure space $\left(A, \mathcal{N}_A, \mu\right)$ is purely atomic. Moreover, for each $i \in A$, the singleton set $\{i\}$ is an atom of the measure space $(N, \mathcal{N}, \mu)$.

The following lemma shows that the set of all feasible net trades is convex. We will use this result in the proofs of Propositions 1.1, 1.2, and 2.

LEMMA 5 For each $n \in N$, the set of all feasible net trades is convex.

PROOF: To avoid repetition and to save space, we prove the result for when there are countably many trading posts per commodity. This way, the result automatically holds true for all of the constructions in Sections 2 through to 4. For ease of exposition, we will adopt a different notation to the one that has been used thus far for the purposes of the first part of the proof only.

So, let Z_n denote the set of all feasible net trades for some $n \in N$. Let $\tilde{z}_n$ and $\hat{z}_n \in Z_n$, where $\tilde{z}_n$ and $\hat{z}_n$ are obtained by the vectors of strategies $(\tilde{b}_n, \tilde{q}_n)$ and $(\hat{b}_n, \hat{q}_n)$, respectively, i.e., $\tilde{z}_n \coloneqq z(\tilde{b}_n, \tilde{q}_n)$ and $\hat{z}_n \coloneqq z(\hat{b}_n, \hat{q}_n)$. We need to show that, given the strategies of all the other agents, for any

$\lambda \in [0,1]$, $\lambda \tilde{z}_n + (1-\lambda)\,\hat{z}_n \in Z_n$. In other words, if $\overset{\star}{z}_n := \lambda \tilde{z}_n + (1-\lambda)\,\hat{z}_n$, then we want to show that $\exists(\overset{\star}{b}_n, \overset{\star}{q}_n)$ feasible such that $z(\overset{\star}{b}_n, \overset{\star}{q}_n) = \overset{\star}{z}_n$. So, first, fix a $\lambda \in [0,1]$. Now, we may rewrite $\tilde{z}_n$ and $\hat{z}_n$ explicitly as $\tilde{z}_n = \left(\tilde{z}_{n,m}, \tilde{z}_{n,1}, \ldots, \tilde{z}_{n,k}, \ldots, \tilde{z}_{n,L}\right)^T$, and $\hat{z}_n = \left(\hat{z}_{n,m}, \hat{z}_{n,1}, \ldots, \hat{z}_{n,k}, \ldots, \hat{z}_{n,L}\right)^T$, where T denotes transposition. Hence, $\overset{\star}{z}_n = \lambda\left(\tilde{z}_{n,m}, \tilde{z}_{n,1}, \ldots, \tilde{z}_{n,L}\right)^T + (1-\lambda)\left(\hat{z}_{n,m}, \hat{z}_{n,1}, \ldots, \hat{z}_{n,L}\right)^T = \left(\lambda \tilde{z}_{n,m} + (1-\lambda)\,\hat{z}_{n,m}, \lambda \tilde{z}_{n,1} + (1-\lambda)\,\hat{z}_{n,1}, \ldots, \lambda \tilde{z}_{n,L} + (1-\lambda)\,\hat{z}_{n,L}\right)^T \equiv \left(\overset{\star}{z}_{n,m}, \overset{\star}{z}_{n,1}, \ldots, \overset{\star}{z}_{n,L}\right)^T$. But there are multiple posts for each $k \in K$. So, for every $k \in K$, we have that $\lambda \tilde{z}_{n,k} + (1-\lambda)\,\hat{z}_{n,k} = \lambda\left(\tilde{z}^{s,1}_{n,k} + \tilde{z}^{s,2}_{n,k} + \ldots\right)^{|P|}_{s=1} + (1-\lambda)\left(\hat{z}^{s,1}_{n,k} + \hat{z}^{s,2}_{n,k} + \ldots\right)^{|P|}_{s=1} = \Big(\left[\lambda \tilde{z}^{s,1}_{n,k} + (1-\lambda)\,\hat{z}^{s,1}_{n,k}\right] + \left[\lambda \tilde{z}^{s,2}_{n,k} + (1-\lambda)\,\hat{z}^{s,2}_{n,k}\right] + \ldots\Big)^{|P|}_{s=1} = \sum_{r=1}^{\Gamma}\sum_{s=1}^{|P|}\left[\lambda \tilde{z}^{s,r}_{n,k} + (1-\lambda)\,\hat{z}^{s,r}_{n,k}\right] = \sum_{r=1}^{\Gamma}\sum_{s=1}^{|P|} \overset{\star}{z}{}^{s,r}_{n,k} = \overset{\star}{z}_{n,k}$. Here, $\Gamma \leq \infty$ denotes the number of posts that an agent $s \in A$ owns, and $|P|$ is the number of post owners in A. So, our strategy will be to proceed post by post, and show that each $\overset{\star}{z}{}^{s,r}_{n,k}$, $s \in \{1, \ldots, |P|\}$, $r \in \{1, \ldots, \Gamma\}$, is feasible, such that $\overset{\star}{z}_{n,k}$ has to be feasible. One can then repeat the algorithm for every other commodity $l \in K \setminus \{k\}$.

With these preliminary remarks out of the way, consider a commodity $k \in K$, any one trading post r for k owned by an atom i, and the net trade of an agent $n \in N$ there, $z^{i,r}_{n,k}$,[26] and define,

$$
\begin{aligned}
\tilde{z}^{i,r}_{n,k} &= z\big(\tilde{b}^{i,r}_{n,k}, \tilde{q}^{i,r}_{n,k}\big);\\
\hat{z}^{i,r}_{n,k} &= z\big(\hat{b}^{i,r}_{n,k}, \hat{q}^{i,r}_{n,k}\big).
\end{aligned}
$$

Next, define

$$
\begin{aligned}
\underline{q}^{i,r}_{n,k} &= \min\big(\tilde{q}^{i,r}_{n,k}, \hat{q}^{i,r}_{n,k}\big);\\
\bar{q}^{i,r}_{n,k} &= \max\big(\tilde{q}^{i,r}_{n,k}, \hat{q}^{i,r}_{n,k}\big).
\end{aligned}
$$

Since $z^{i,r}_{n,k}$ is decreasing in $q^{i,r}_{n,k}$ (holding $b^{i,r}_{n,k}$ constant), we have that

$$
\begin{aligned}
\underline{\tilde{z}}^{i,r}_{n,k} &\equiv z\big(\tilde{b}^{i,r}_{n,k}, \bar{q}^{i,r}_{n,k}\big) \leq z\big(\tilde{b}^{i,r}_{n,k}, \tilde{q}^{i,r}_{n,k}\big) \leq z\big(\tilde{b}^{i,r}_{n,k}, \underline{q}^{i,r}_{n,k}\big) \equiv \bar{\tilde{z}}^{i,r}_{n,k};\\
\underline{\hat{z}}^{i,r}_{n,k} &\equiv z\big(\hat{b}^{i,r}_{n,k}, \bar{q}^{i,r}_{n,k}\big) \leq z\big(\hat{b}^{i,r}_{n,k}, \hat{q}^{i,r}_{n,k}\big) \leq z\big(\hat{b}^{i,r}_{n,k}, \underline{q}^{i,r}_{n,k}\big) \equiv \bar{\hat{z}}^{i,r}_{n,k}.
\end{aligned}
$$

But $z^{i,r}_{n,k}$ is increasing and concave in $b^{i,r}_{n,k}$ (holding $q^{i,r}_{n,k}$ constant), such that we have

$$
\begin{aligned}
z\big(0, \bar{q}^{i,r}_{n,k}\big) &\leq \lambda \underline{\tilde{z}}^{i,r}_{n,k} + (1-\lambda)\,\underline{\hat{z}}^{i,r}_{n,k}\\
&\leq \overset{\star}{z}{}^{i,r}_{n,k} = \lambda \tilde{z}^{i,r}_{n,k} + (1-\lambda)\,\hat{z}^{i,r}_{n,k}\\
&\leq \lambda \bar{\tilde{z}}^{i,r}_{n,k} + (1-\lambda)\,\bar{\hat{z}}^{i,r}_{n,k}\\
&\leq z\big(\lambda \tilde{b}^{i,r}_{n,k} + (1-\lambda)\,\hat{b}^{i,r}_{n,k}, \underline{q}^{i,r}_{n,k}\big) \equiv z\big({}^{\lambda}b^{i,r}_{n,k}, \underline{q}^{i,r}_{n,k}\big).
\end{aligned}
$$

[26] As noted before, for the first part of the proof, we will use $z^{i,r}_{n,k}$ to denote the more elaborate $z^{i,r}_{k}(n)$.

$z^{i,r}_{n,k}$ is continuous and increasing in $b^{i,r}_{n,k}$ (holding $q^{i,r}_{n,k}$ constant), while $z^{i,r}_{n,k}$ is continuous and decreasing in $q^{i,r}_{n,k}$ (holding $b^{i,r}_{n,k}$ constant). So, define two continuous surjections $\mathscr{B} : \left[0, {}^{\lambda}b^{i,r}_{n,k}\right] \rightarrow z\left(\cdot, \underline{q}^{i,r}_{n,k}\right)$, and $\mathscr{Q} : \left(\underline{q}^{i,r}_{n,k}, \bar{q}^{i,r}_{n,k}\right] \rightarrow z(0, \cdot)$, and let us consider two cases: (i) $\overset{\star}{z}^{i,r}_{n,k} \geq z\left(0, \underline{q}^{i,r}_{n,k}\right)$, and; (ii) $\overset{\star}{z}^{i,r}_{n,k} < z\left(0, \underline{q}^{i,r}_{n,k}\right)$. If (i) is true, then note that $\mathscr{B}(0) \leq \overset{\star}{z}^{i,r}_{n,k} \leq \mathscr{B}\left({}^{\lambda}b^{i,r}_{n,k}\right)$ implies that $\exists \overset{\star}{b}^{i,r}_{n,k} \in \left[0, {}^{\lambda}b^{i,r}_{n,k}\right]$ such that $\mathscr{B}\left(\overset{\star}{b}^{i,r}_{n,k}\right) = \overset{\star}{z}^{i,r}_{n,k}$, by the *Intermediate Value Theorem* (IVT). If (ii) is true, then observe that $\mathscr{Q}\left(\bar{q}^{i,r}_{n,k}\right) \leq \overset{\star}{z}^{i,r}_{n,k} < \mathscr{Q}\left(\underline{q}^{i,r}_{n,k}\right)$, such that by the IVT, $\exists \overset{\star}{q}^{i,r}_{n,k} \in \left(\underline{q}^{i,r}_{n,k}, \bar{q}^{i,r}_{n,k}\right]$, with $\mathscr{Q}\left(\overset{\star}{q}^{i,r}_{n,k}\right) = \overset{\star}{z}^{i,r}_{n,k}$. Notice that the above procedure can now be repeated for each post $\Upsilon^{s,r}_{k}$, $s \in \{1, \ldots, |P|\}$, $r \in \{1, \ldots, \Gamma\}$, such that $\overset{\star}{z}_{n,k}$ is budget-feasible. Certainly, the same can also be done for all commodities $k \in K$ to get $\overset{\star}{z}_{n}$ feasible. Hence, the set of all budget-feasible net trades in commodities $k \in K$ is convex. Denote this set by Z_K.

Now, the net proceeds of commodity m to any μ-*a.e*, $w \in N_0$, $z_m(w)$, can be written as a function of $\left\{\left\{\left\{z^{s,r}_{k}(w)\right\}^{|P|}_{s=1}\right\}^{L}_{k=1}\right\}^{\Gamma}_{r=1}$ as shown below:

$$z_m(w) = \sum_{r=1}^{\Gamma}\sum_{k=1}^{L}\sum_{s=1}^{|P|} - \frac{B^{s,r}_{-w,k} z^{s,r}_{k}(w)}{Q^{s,r}_{-w,k}} \cdot \left(1 + t^{s,r,k}_{w} c^{k}\right).$$

For large pure traders $\tau \in N \setminus N_0$, $z_m(\tau)$ can be written as a function of $\left\{\left\{\left\{z^{s,r}_{k}(\tau)\right\}^{|P|}_{s=1}\right\}^{L}_{k=1}\right\}^{\Gamma}_{r=1}$ as follows:

$$z_m(\tau) = \sum_{r=1}^{\Gamma}\sum_{k=1}^{L}\sum_{s=1}^{|P|} \frac{B^{s,r}_{-\tau,k} z^{s,r}_{k}(\tau)}{\left(z^{s,r}_{k}(\tau) \cdot \mu(\tau) - Q^{s,r}_{-\tau,k}\right)} \cdot \left(1 + t^{s,r,k}_{\tau} c^{k}\right),$$

while for large post owners $i \in A$, $z_m(i)$ is expressible as a function of $\left\{\left\{\left\{z^{s,r}_{k}(i)\right\}^{|P|}_{s=1}\right\}^{L}_{k=1}\right\}^{\Gamma}_{r=1}$ thus:

$$\begin{aligned} z_m(i) = \sum_{r=1}^{\Gamma}\sum_{k=1}^{L} - c^{k} \cdot \Bigg(\int_{N\setminus\{i\}} t^{i,r,k}_{n} q^{i,r}_{k}(n) d\mu \cdot \frac{B^{i,r}_{-i,k}}{\left(Q^{i,r}_{-i,k} - z^{i,r}_{k}(i) \cdot \mu(i)\right)} - \int_{N\setminus\{i\}} t^{i,r,k}_{n} b^{i,r}_{k}(n) d\mu\Bigg) \\ + \sum_{r=1}^{\Gamma}\sum_{k=1}^{L} \frac{B^{i,r}_{-i,k} z^{i,r}_{k}(i)}{\left(z^{i,r}_{k}(i) \cdot \mu(i) - Q^{i,r}_{-i,k}\right)} + \sum_{r=1}^{\Gamma}\sum_{k=1}^{L}\sum_{s\neq i} \frac{B^{s,r}_{-i,k} z^{s,r}_{k}(i)}{\left(z^{s,r}_{k}(i) \cdot \mu(i) - Q^{s,r}_{-i,k}\right)} \cdot \left(1 + t^{s,r,k}_{i} c^{k}\right), \end{aligned}$$

where, as in the main body of the paper, $\varphi^{s,r}_{k}(n)$, $\varphi = b, q$, denotes the strategies of agent $n \in N$ at the r^{th} post owned by $s \in A$ for commodity k. Now, over the convex set Z_K, it can be verified that $z_m(w)$, being the sum of countably many concave functions is concave, while $z_m(\tau)$ and $z_m(i)$, being the sum of countably many strictly concave functions, are both strictly concave functions.

Given that the hypograph of a concave function is a convex set, we finally have that for every agent μ-$a.e, n \in N$, the set of *all* feasible net trade bundles, Z_n, is indeed convex. $\square$

PROOF OF PROPOSITION 1.1: The Lagrangian to the programme in (3) for large pure traders $\tau \in N \setminus N_0$ can be written as

$$\mathcal{L}_\tau = u_\tau(\mathbf{x}) - \lambda^b \cdot \left(\sum_{k=1}^{L}\sum_{s=1}^{|P|} b_k^s(\tau) + \Lambda(\tau) - e_m(\tau)\right) - \sum_{k=1}^{L} \lambda_k^q \cdot \left(\sum_{s=1}^{|P|} q_k^s(\tau) - e_k(\tau)\right),$$

where $\mathbf{x} = (x_{\tau,m}, x_{\tau,1}, \ldots, x_{\tau,L})$, and λ^b and $\{\lambda_k^q\}_{k=1}^L$ are the Lagrange multipliers associated with the constraints in (3). Next, consider any post Υ_k^i, for any $k \in K$. Solving for the first-order necessary and sufficient[27] conditions, we have, at an interior equilibrium, that[28]

$$\begin{aligned} \frac{\partial u_\tau}{\partial x_{\tau,k}} \frac{Q_k^i B_{-\tau,k}^i}{\left(B_k^i\right)^2} - \frac{\partial u_\tau}{\partial x_{\tau,m}} \frac{Q_{-\tau,k}^i}{Q_k^i} \cdot \left(1 + t_\tau^{i,k} c^k\right) &= 0; \\ -\frac{\partial u_\tau}{\partial x_{\tau,k}} \frac{B_{-\tau,k}^i}{B_k^i} + \frac{\partial u_\tau}{\partial x_{\tau,m}} \frac{B_k^i Q_{-\tau,k}^i}{\left(Q_k^i\right)^2} \cdot \left(1 + t_\tau^{i,k} c^k\right) &= 0. \end{aligned}$$

It is easy to see that both equations above imply that

$$\frac{\partial u_\tau / \partial x_{\tau,k}}{\partial u_\tau / \partial x_{\tau,m}} = \frac{\left(B_k^i\right)^2 Q_{-\tau,k}^i}{\left(Q_k^i\right)^2 B_{-\tau,k}^i} \cdot \left(1 + t_\tau^{i,k} c^k\right).$$

Now, consider another post for commodity k, say Υ_k^j. Then, by the same argument as above, at an interior N.E, it must be true that

$$\frac{\partial u_\tau / \partial x_{\tau,k}}{\partial u_\tau / \partial x_{\tau,m}} = \frac{\left(B_k^j\right)^2 Q_{-\tau,k}^j}{\left(Q_k^j\right)^2 B_{-\tau,k}^j} \cdot \left(1 + t_\tau^{j,k} c^k\right).$$

The conclusion for large pure traders $\tau \in N \setminus N_0$ is implied by the last two equations above.

Next, consider small pure traders and post owners in N_0. Since by construction $\chi^{small} = 0$, one may view all agents in N_0 as being small pure traders. So, it is easy to see that since N_0 is an atomless set of agents, we have that $B_{-w,k}^s = \int_{N \setminus \{w\}} b_k^s(n) d\mu = \int_N b_k^s(n) d\mu = B_k^s$, $s = i, j$. Similarly, $Q_{-w,k}^s = \int_{N \setminus \{w\}} q_k^s(n) d\mu = \int_N q_k^s(n) d\mu = Q_k^s$, $s = i, j$. Hence, μ-$a.e, w \in N_0$, it has to be true that

[27] By construction, $x_k : z_k \to e_k + z_k$, $\forall k \in (\{m\} \cup K)$. So, define a new function $v = u \circ x : \mathbb{R}^{L+1} \supset z \to \mathbb{R}$. u is a smooth strictly concave function of x; trivially therefore, v is a smooth strictly concave function of z. By Lemma 5, we have that the set of all feasible/attainable net trades is convex. Hence, by the *Supporting Hyperplane Theorem*, a unique optimum exists, i.e., there is a unique "net trade bundle" that is a best response to the strategies played by other agents.

[28] To simplify exposition, we make use of the fact that $z_k^i(n) \cdot (\partial t_n^{i,k} / \partial z_k^i(n)) \cdot (\partial z_k^i(n) / \partial b_k^i(n)) = z_k^i(n) \cdot (\partial t_n^{i,k} / \partial z_k^i(n)) \cdot (\partial z_k^i(n) / \partial q_k^i(n)) = 0$ in what will follow. We do this in the proof of Proposition 1.2 as well.

$$\left(p_k^i\right)^2 = \frac{B^i_{-w,k} Q^j_{-w,k}\left(1+t_w^{j,k}c^k\right)}{Q^i_{-w,k} B^j_{-w,k}\left(1+t_w^{i,k}c^k\right)}\left(p_k^j\right)^2 \Leftrightarrow p_k^i = \frac{\left(1+t_w^{j,k}c^k\right)}{\left(1+t_w^{i,k}c^k\right)}\, p_k^j \ . \qquad \square$$

PROOF OF PROPOSITION 1.2: The Lagrangian to the programme in (3) for large post owners $i \in A$ (for whom $\mu(i) = 1$) can be written as

$$\mathcal{L}_i = u_i(\mathbf{x}) - \lambda^b \cdot \left(\sum_{k=1}^L \sum_{s=1}^{|P|} b_k^s(i) + \Lambda(i) - e_m(i)\right) - \sum_{k=1}^L \lambda_k^q \cdot \left(\sum_{s=1}^{|P|} q_k^s(i) - e_k(i)\right).$$

The first-order conditions of (3) at an interior equilibrium, at i's post Υ_k^i, for any $k \in K$, are:

$$\begin{aligned}
\frac{\partial u_i}{\partial x_{i,k}} \frac{B^i_{-i,k} Q^i_k}{(B^i_k)^2} + \frac{\partial u_i}{\partial x_{i,m}} \cdot \Big(-\frac{c^k \int_N t_n^{i,k} q_k^i(n) d\mu}{Q_k^i} + t_i^{i,k} c^k - \frac{Q^i_{-i,k}}{Q^i_k} \cdot \left(1 + t_i^{i,k} c^k\right)\Big) &= 0;\\
\frac{\partial u_i}{\partial x_{i,k}} \frac{B^i_{-i,k}}{B^i_k} - \frac{\partial u_i}{\partial x_{i,m}} \cdot B^i_k \cdot \Big(-\frac{Q_k^i t_i^{i,k} c^k - c^k \int_N t_n^{i,k} q_k^i(n) d\mu}{(Q_k^i)^2} + \frac{Q^i_{-i,k}\left(1 + t_i^{i,k} c^k\right)}{(Q_k^i)^2}\Big) &= 0.
\end{aligned}$$

Both equations above imply that

$$\frac{\partial u_i / \partial x_{i,k}}{\partial u_i / \partial x_{i,m}} = \frac{(B_k^i)^2}{(Q_k^i)^2} \cdot \Big(\frac{Q^i_{-i,k} + c^k \int_{N \setminus \{i\}} t_n^{i,k} q_k^i(n) d\mu}{B^i_{-i,k}}\Big). \tag{6}$$

Consider now another post for k, owned by some other large agent $j \in A \setminus \{i\}$. Then, at an interior equilibrium, it has to be true that

$$\begin{aligned}
\frac{\partial u_i}{\partial x_{i,k}} \frac{B^j_{-i,k} Q^j_k}{(B^j_k)^2} - \frac{\partial u_i}{\partial x_{i,m}} \frac{Q^j_{-i,k}\left(1 + t_i^{j,k} c^k\right)}{Q^j_k} &= 0;\\
\frac{\partial u_i}{\partial x_{i,k}} \frac{B^j_{-i,k}}{B^j_k} - \frac{\partial u_i}{\partial x_{i,m}} \frac{Q^j_{-i,k} B^j_k \left(1 + t_i^{j,k} c^k\right)}{(Q^j_k)^2} &= 0,
\end{aligned}$$

both of which are equivalent to

$$\frac{\partial u_i / \partial x_{i,k}}{\partial u_i / \partial x_{i,m}} = \frac{(B_k^j)^2 Q^j_{-i,k}}{(Q_k^j)^2 B^j_{-i,k}} \cdot \left(1 + t_i^{j,k} c^k\right). \tag{7}$$

The condition for $i \in A$ is implied by equations (6) and (7) above. The condition for $j \in A \setminus \{i\}$ can be obtained by following an argument analogous to the one above. $\square$

LEMMA 6 The distribution rules, as defined in Section 3, are Gâteaux-differentiable.

PROOF: Let a strategy profile $(b, q) \in Gr(S)$ be given. We first prove that $x_{n,k}$, $k \neq m$, is Gâteaux-differentiable at (b, q).

Gâteaux-differentiability of $x_{n,k}$, $k \neq m$:

For clarity of exposition, we introduce some notation. Let $\ell^1_+ \supset h_b(n) := \big\{\big\{\big\{h^{s,r,k}_{n,b}\big\}_{s=1}^{|P|}\big\}_{k\in K}\big\}_{r=1}^{\infty}$, and $\ell^1_+ \supset \hat{h}_b(n) := \big\{\big\{\big\{\hat{h}^{s,r,k}_{n,b}\big\}_{s=1}^{|P|}\big\}_{k\in K}\big\}_{r=1}^{\infty}$, where each $\hat{h}^{s,r,k}_{n,b} = \dfrac{h^{s,r,k}_{n,b}}{\|h_b\|_{\ell^1}}$, such that $\|\hat{h}_b\|_{\ell^1} = 1$. We define the terms $h_q, \hat{h}_q$ and $\hat{h}^{s,r,k}_{n,q}$ analogously. Next, let $\ell^1_+ \supset b(n) := \big\{\big\{\big\{b^{s,r}_k(n)\big\}_{s=1}^{|P|}\big\}_{k\in K}\big\}_{r=1}^{\infty}$, and $\ell^1_+ \supset \hat{b}(n) := \big\{\big\{\big\{\hat{b}^{s,r}_k(n)\big\}_{s=1}^{|P|}\big\}_{k\in K}\big\}_{r=1}^{\infty}$, where each $\hat{b}^{s,r}_k(n) = \dfrac{b^{s,r}_k(n)}{\|b\|_{\ell^1}}$. $q(n)$ and $\hat{q}(n)$ are defined analogously. Finally, let $\hat{\mathcal{B}}_k(n) := \big\{\hat{b}(n) : \sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\hat{b}^{s,r}_k(n) = 1;\ \sum_{r=1}^{\infty}\sum_{s=1}^{|P|} b^{s,r}_k(n) \leq e_m(n)\big\}$, and $\hat{\mathcal{Q}}(n) := \big\{\hat{q}(n) : \sum_{r=1}^{\infty}\sum_{s=1}^{|P|} q^{s,r}_k(n) \leq e_k(n),\ k \in K\big\}$. We may now proceed.

Using the definition of the Gâteaux differential as in Luenberger (1969: p. 171) and taking limits, we have that the Gâteaux differential of $x_{n,k}$, which we denote by $\delta_{x_{n,k}}((b,q),h)$, is

$$\sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\frac{B^{s,r}_{-n,k}Q^{s,r}_k}{(B^{s,r}_k)^2}\cdot\hat{h}^{s,r,k}_{n,b} - \sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\frac{B^{s,r}_{-n,k}}{B^{s,r}_k}\cdot\hat{h}^{s,r,k}_{n,q},$$

where $\hat{h}^{s,r,k}_{n,b}$ and $\hat{h}^{s,r,k}_{n,q}$ are the increments associated with $b^{s,r}_k(n)$ and $q^{s,r}_k(n)$, respectively. Clearly, $\delta_{x_{n,k}}((b,q),h)$ is linear in its increments h_b and h_q. We need to prove that $-\infty < \delta_{x_{n,k}}((b,q),h) < \infty$. So let us begin by noting that

$$\sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\frac{B^{s,r}_{-n,k}Q^{s,r}_k}{(B^{s,r}_k)^2}\cdot\hat{h}^{s,r,k}_{n,b} \leq \sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\frac{Q^{s,r}_k}{B^{s,r}_k}\cdot\hat{h}^{s,r,k}_{n,b}.$$

Since $\int_N e_k(n)d\mu < \infty\ \forall k \in (\{m\}\cup K)$, we have that $\int_N x_{n,k}d\mu < \infty$. So, bearing in mind that $q(n)$, and hence $\hat{q}(n)$, lie in ℓ^1_+, it has to be true that for any $\hat{b}(n) \in \hat{\mathcal{B}}_k(n)$, $\sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\hat{b}^{s,r}_k(n)\cdot\frac{Q^{s,r}_k}{B^{s,r}_k}$ is finite $a.e$ in N. Now, since $\hat{h}_b$ is normalised, we may choose some $\bar{\hat{b}}(n) \in \hat{\mathcal{B}}_k(n)$, such that $\hat{h}^{s,r,k}_{n,b} = \bar{\hat{b}}^{s,r}_k(n)$ for each $s \in \{1,2,\ldots,|P|\}$ and $r \in \mathbb{N}$. Hence,

$$\sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\frac{Q^{s,r}_k}{B^{s,r}_k}\cdot\hat{h}^{s,r,k}_{n,b} = \sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\frac{Q^{s,r}_k}{B^{s,r}_k}\cdot\bar{\hat{b}}^{s,r}_k(n) < \infty.$$

We next show that $\sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\frac{B^{s,r}_{-n,k}}{B^{s,r}_k}\cdot\hat{h}^{s,r,k}_{n,q} < \infty$. Easily, $\sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\frac{B^{s,r}_{-n,k}}{B^{s,r}_k}\cdot\hat{h}^{s,r,k}_{n,q} \leq \sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\hat{h}^{s,r,k}_{n,q} < \infty$, since $\big\{\big\{\hat{h}^{s,r,k}_{n,q}\big\}_{s=1}^{|P|}\big\}_{r=1}^{\infty} \in \ell^1_+$. Combining these two results yields $-\infty < \delta_{x_{n,k}}((b,q),h) < \infty$. Finally, note that $\hat{h}_b, \hat{h}_q \in \ell^1_+$ were arbitrarily chosen. Hence, $x_{n,k}$ is indeed Gâteaux-differentiable.

Gâteaux-differentiability of $x_{n,k}$, $k = m$:

We now prove that $x_{i,m}$ for some post owner $i \in A$ (such that $\mu(i) = 1$) is Gâteaux-differentiable at (b, q). That $x_{n,m}$, μ-$a.e, n \in N \setminus \{i\}$, is also Gâteaux-differentiable at (b, q) then follows as a direct consequence of this result. The Gâteaux differential of $x_{i,m}$ at (b, q), $\delta_{x_{i,m}}((b, q), h)$, is

$$\sum_{r=1}^{\infty}\sum_{k=1}^{L}\Big(c^k \frac{t_i^{i,r,k}Q_{-i,k}^{i,r} - \int_{N\setminus\{i\}} t_n^{i,r,k}q_k^{i,r}(n)d\mu}{Q_k^{i,r}} \cdot \hat{h}_{i,b}^{i,r,k} - \sum_{s=1}^{|P|}\frac{Q_{-i,k}^{s,r}\big(1 + t_i^{s,r,k}c^k\big)}{Q_k^{s,r}} \cdot \hat{h}_{i,b}^{s,r,k}\Big) +$$

$$\sum_{r=1}^{\infty}\sum_{k=1}^{L}\Big(B_k^{i,r}c^k \frac{-t_i^{i,r,k}Q_{-i,k}^{i,r} + \int_{N\setminus\{i\}} t_n^{i,r,k}q_k^{i,r}(n)d\mu}{(Q_k^{i,r})^2} \cdot \hat{h}_{i,q}^{i,r,k} + \sum_{s=1}^{|P|}\frac{Q_{-i,k}^{s,r}B_k^{s,r}\big(1 + t_i^{s,r,k}c^k\big)}{(Q_k^{s,r})^2} \cdot \hat{h}_{i,q}^{s,r,k}\Big),$$

where as before, both sequences of increments have been normalised. Clearly, $\delta_{x_{i,m}}((b, q), h)$ is also linear in its increments h_b and h_q. For ease of exposition, let the summation in the first row in the expression just above be represented by $\mathscr{W}$, and the summation in the second row by $\mathscr{Y}$. We first show that $\mathscr{W}$ is finite. We begin by noting that

$$\sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|}\frac{Q_{-i,k}^{s,r}\big(1 + t_i^{s,r,k}c^k\big)}{Q_k^{s,r}} \cdot \hat{h}_{i,b}^{s,r,k} \leq \sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|}\big(1 + t_i^{s,r,k}c^k\big) \cdot \hat{h}_{i,b}^{s,r,k} < 2\sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|}\hat{h}_{i,b}^{s,r,k} < \infty,$$

where the last inequality follows from the fact that $\big\{\big\{\big\{\hat{h}_{i,b}^{s,r,k}\big\}_{s=1}^{|P|}\big\}_{k\in K}\big\}_{r=1}^{\infty} \in \ell_+^1$. Next,

$$-\infty < -2\sum_{r=1}^{\infty}\sum_{k=1}^{L}\hat{h}_{i,b}^{i,r,k} < \sum_{r=1}^{\infty}\sum_{k=1}^{L}c^k \frac{t_i^{i,k,r}Q_{-i,k}^{i,r} - \int_{N\setminus\{i\}} t_n^{i,r,k}q_k^{i,r}(n)d\mu}{Q_k^{i,r}} \cdot \hat{h}_{i,b}^{i,r,k}$$

$$< \sum_{r=1}^{\infty}\sum_{k=1}^{L}c^k \frac{Q_k^{i,r} + Q_k^{i,r}}{Q_k^{i,r}} \cdot \hat{h}_{i,b}^{i,r,k} = \sum_{r=1}^{\infty}\sum_{k=1}^{L}2c^k \cdot \hat{h}_{i,b}^{i,r,k} < 2\sum_{r=1}^{\infty}\sum_{k=1}^{L}\hat{h}_{i,b}^{i,r,k} < \infty.$$

So $\mathscr{W}$ is finite. Looking at $\mathscr{Y}$ next, we have that

$$-\mathscr{A} < \sum_{r=1}^{\infty}\sum_{k=1}^{L}\Big(B_k^{i,r}c^k \frac{-t_i^{i,r,k}Q_{-i,k}^{i,r} + \int_{N\setminus\{i\}} t_n^{i,r,k}q_k^{i,r}(n)d\mu}{(Q_k^{i,r})^2} \cdot \hat{h}_{i,q}^{i,r,k} + \sum_{s=1}^{|P|}\frac{Q_{-i,k}^{s,r}B_k^{s,r}\big(1 + t_i^{s,r,k}c^k\big)}{(Q_k^{s,r})^2} \cdot \hat{h}_{i,q}^{s,r,k}\Big)$$

$$< \sum_{r=1}^{\infty}\sum_{k=1}^{L}\Big(2c^k \frac{B_k^{i,r}}{Q_k^{i,r}} \cdot \hat{h}_{i,q}^{i,r,k} + \sum_{s=1}^{|P|}\frac{B_k^{s,r}\big(1 + t_i^{s,r,k}c^k\big)}{Q_k^{s,r}} \cdot \hat{h}_{i,q}^{s,r,k}\Big) < \mathscr{A},$$

where $\mathscr{A} := 2\sum_{r=1}^{\infty}\sum_{k=1}^{L}\big(\frac{B_k^{i,r}}{Q_k^{i,r}} \cdot \hat{h}_{i,q}^{i,r,k} + \sum_{s=1}^{|P|}\frac{B_k^{s,r}}{Q_k^{s,r}} \cdot \hat{h}_{i,q}^{s,r,k}\big)$. Now, by construction, since $\int_N e_m(n)d\mu < \infty$, we have that $\int_N x_{n,m}d\mu < \infty$. Thus, as $b(i)$, and hence $\hat{b}(i)$, lie in ℓ_+^1, it has to be true that $\sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|}\hat{q}_k^{s,r}(i) \cdot \frac{B_k^{s,r}}{Q_k^{s,r}}$ is finite for any $\hat{q}(i) \in \hat{\mathcal{Q}}(i)$. Since $\hat{h}_q$ is normalised, we may therefore pick some $\bar{\hat{q}}(i) \in \hat{\mathcal{Q}}(i)$, such that $\hat{h}_{i,q}^{s,r,k} = \bar{\hat{q}}_k^{s,r}(i)$ for each $s \in \{1, 2, \ldots, |P|\}$, $r \in \mathbb{N}$, and each

$k \in K$. Thus, $\sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|}\frac{B_k^{s,r}}{Q_k^{s,r}}\cdot\hat{h}_{i,q}^{s,r,k}$ is finite, implying that $\sum_{r=1}^{\infty}\sum_{k=1}^{L}\frac{B_k^{i,r}}{Q_k^{i,r}}\cdot\hat{h}_{i,q}^{i,r,k}$ is also finite. Hence, $-\infty < \mathscr{Y} < \infty$, such that $-\infty < \delta_{x_{i,m}}((b,q),h) < \infty$. Finally, recall that in this part of our proof as well, $\hat{h}_b, \hat{h}_q \in \ell_+^1$ were arbitrarily chosen. Hence, $x_{i,m}$ is Gâteaux-differentiable. That $x_{n,m}, \mu$-$a.e, n \in N \setminus \{i\}$, is also Gâteaux-differentiable at (b,q), follows as a corollary. $\square$

PROOF OF PROPOSITION 2: Let the positive bids and offers of all agents other than some $n \in N$ be given. By Assumption (ii), $u(\cdot)$ is strictly concave in x. Define a new mapping $v = u \circ x : \mathbb{R}^{L+1} \supset Z_n \ni z(n) \to \mathbb{R}$, where $x : \mathbb{R}^{L+1} \supset Z_n \ni z(n) \to \mathbb{R}_+^{L+1}$, and Z_n is agent n's set of all feasible net trades, which by Lemma 5 is convex. Trivially therefore, $v(\cdot)$ is strictly concave. So, at an interior equilibrium, each agent μ-$a.e, n \in N$, solves $\inf_{z(n)\in Z_n}\{-v(\cdot)\}$. Since $-v(\cdot)$ is strictly convex, if this problem has a solution, then it is unique. We may therefore move back to strategy space to derive the first-order necessary and sufficient conditions. This will be achieved by using a Generalised Kuhn–Tucker theorem (GKTT) as in Luenberger (1969: p. 249). Thus, we need to show that all of the conditions of that theorem hold for our construction in Section 3. To help the reader, we relate each piece of notation from the GKTT to the symbols that we use in our model. In particular, X will now be represented by Y (for conciseness, $X \equiv Y$), $Z \equiv \mathbb{R}^{L+1}$, $P \equiv \mathbb{R}_+^{L+1}$, $f \equiv -\zeta$, $G \equiv G$, $x_0 \equiv \underline{s}$, $z_0^* \equiv \psi^*$, and $Z^* \equiv \mathbb{R}^{L+1}$.

We proceed by noting that $Y \coloneqq \ell^1 \times \ell^1$ (equipped with the norm $\|\cdot\|_Y = \|\cdot\|_{\ell^1} + \|\cdot\|_{\ell^1}$, as defined in Section 3), is a normed vector space, and so is $\mathbb{R}^{L+1}$ (endowed with its usual Euclidean norm), whose positive cone $\mathbb{R}_+^{L+1}$ contains an interior point. By Assumption (ii), $u(\cdot)$ is Fréchet-differentiable in x. By Lemma 6, $x(\cdot)$ is Gâteaux-differentiable in b and q, and is linear in its increments (h_b and h_q). Thus, $\zeta \coloneqq u \circ x : Y \supset \ell_+^1 \times \ell_+^1 \ni (b,q) \to \mathbb{R}$, is a Gâteaux-differentiable functional possessing linear increments, and hence, so is $-\zeta(\cdot)$. Let $G(\cdot)$ be a mapping $G : Y \supset \ell_+^1 \times \ell_+^1 \ni (b,q) \to \mathbb{R}^{L+1}$, such that

$$\begin{bmatrix} \sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|} b_k^{s,r}(n) + \Lambda(n) - e_m(n) \\ \sum_{r=1}^{\infty}\sum_{s=1}^{|P|} q_1^{s,r}(n) - e_1(n) \\ \vdots \\ \sum_{r=1}^{\infty}\sum_{s=1}^{|P|} q_L^{s,r}(n) - e_L(n) \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$\Lambda(n)$ can be explicitly written as $\sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|} -c^k t_n^{s,r,k}\big(q_k^{s,r}(n)\cdot p_k^{s,r} - b_k^{s,r}(n)\big)$. In this regard, it can be easily shown that $G(\cdot)$ is a Gâteaux-differentiable mapping with linear increments (see

Luenberger, 1969: p. 171, and, e.g., Lemma 6). Next, suppose that the strategy $\underline{s}(n) := (\underline{b}(n), \underline{q}(n))$ minimises $-\zeta(\cdot)$ subject to $G(s(n)) \leq \mathbf{0}$, where $\mathbf{0}$ is a null vector. We need to show that $\underline{s}(n)$ is a regular point (see Luenberger, 1969: p. 248). This is straightforward. Choose an increment $h = h(\epsilon) \in Y$ (see Luenberger, 1969: pp. 171-172) such that for a given and sufficiently small $\epsilon \in \mathbb{R}_{++}$, we have

$$\begin{bmatrix} \sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|}\underline{b}_k^{s,r}(n) + \Lambda(n) - e_m(n) \\ \sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\underline{q}_1^{s,r}(n) - e_1(n) \\ \sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\underline{q}_2^{s,r}(n) - e_2(n) \\ \vdots \\ \sum_{r=1}^{\infty}\sum_{s=1}^{|P|}\underline{q}_L^{s,r}(n) - e_L(n) \end{bmatrix} + \begin{bmatrix} \sum_{r=1}^{\infty}\sum_{k=1}^{L}\sum_{s=1}^{|P|} - \frac{\epsilon}{2^{(s+r+k)}} \cdot \left(1 + \frac{c^k t_n^{s,r,k} Q_{-n,k}^{s,r}}{Q_k^{s,r}}\right) \\ \sum_{r=1}^{\infty}\sum_{s=1}^{|P|} - \frac{\epsilon}{2^{(s+r+1)}} \\ \sum_{r=1}^{\infty}\sum_{s=1}^{|P|} - \frac{\epsilon}{2^{(s+r+2)}} \\ \vdots \\ \sum_{r=1}^{\infty}\sum_{s=1}^{|P|} - \frac{\epsilon}{2^{(s+r+L)}} \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the first, second, and third matrices correspond to $G(\underline{s}(n))$, $\delta G(\underline{s}(n); h) := G'(\underline{s}(n)) \cdot h$, and $\mathbf{0}$, respectively, with $G'(\underline{s}(n))$ denoting the Gâteaux derivative of $G(\cdot)$ at $\underline{s}(n)$. Note that $-1 < (c^k t_n^{s,r,k} Q_{-n,k}^{s,r}/Q_k^{s,r}) < 1$, such that $[1 + (c^k t_n^{s,r,k} Q_{-n,k}^{s,r}/Q_k^{s,r})] > 0$. So, it is easy to see, by considering one row at a time, that each sum above is strictly negative, whether the constraints are binding or not. Hence, $\underline{s}(n)$ is a regular point, i.e., $G(\underline{s}(n)) \leq \mathbf{0}$, and $\exists h \in Y$, such that $G(\underline{s}(n)) + \delta G(\underline{s}(n); h) < \mathbf{0}$.

Since all the conditions of the GKTT are met, we have, at an N.E with positive bids and offers, that there exists $\psi^* \geq \mathbf{0}$ in $(\mathbb{R}^{L+1}, \|\cdot\|_2)$, where $\psi^* = \left(\psi_b^*, \{\psi_{q,k}^*\}_{k=1}^{L}\right)$, such that

$$\begin{aligned} &\frac{\partial u_n}{\partial x_{n,k}} \cdot \left.\frac{\partial x_{n,k}}{\partial b_k^{s,r}(n)}\right|_{\underline{s}(n)} + \frac{\partial u_n}{\partial x_{n,m}} \cdot \left.\frac{\partial x_{n,m}}{\partial b_k^{s,r}(n)}\right|_{\underline{s}(n)} - \psi_b^* \cdot \left(1 + \left.\frac{\partial \Lambda(n)}{\partial b_k^{s,r}(n)}\right|_{\underline{s}(n)}\right) &&= 0, \ k \in K; \\ &\frac{\partial u_n}{\partial x_{n,k}} \cdot \left.\frac{\partial x_{n,k}}{\partial q_k^{s,r}(n)}\right|_{\underline{s}(n)} + \frac{\partial u_n}{\partial x_{n,m}} \cdot \left.\frac{\partial x_{n,m}}{\partial q_k^{s,r}(n)}\right|_{\underline{s}(n)} - \psi_{q,k}^* &&= 0, \ k \in K; \\ &\psi^* \cdot G(\underline{s}(n)) &&= 0. \end{aligned}$$

Propositions 1.1 and 1.2, for Section 3, *mutatis mutandis*, can be easily derived by plugging into the above system of equations the respective allocation rules for each type of agents—and by noting that at an N.E with positive bids and offers, and no binding liquidity and offer contraints, $\psi^* = \mathbf{0}$. The corresponding version of Theorem 1 can be subsequently retrieved. □

LEMMA 7 The set function $\nu : \mathscr{S} \to [0,\infty]$, where $\mathscr{S} = \{W \subseteq N : W = K \cup L \cup M; K \in \mathscr{S}_0; L \in \mathscr{S}_A; M \in \mathscr{S}_C\}$, such that $\nu(W) = \nu_0(W \cap N_0) + \mu_A(W \cap A) + \mu_C(W \cap C)$ for each $W \in \mathscr{S}$, is a measure. Moreover, the triple $(N, \mathcal{N}, \mu)$—where $\mathcal{N}$ is the collection of all the μ-measurable subsets of N, and μ is the unique extension of ν to a measure on $\mathcal{N}$—is a complete, finite measure space of agents.

PROOF: The first part of the lemma is easily proved by following the workings in Lemma 1. Now, since ν is a measure on $\mathscr{S}$, it generates a nonnegative extended real-valued set function μ, the Carathéodory extension of ν, defined on $\mathcal{P}(N)$. In this regard, note that $\mu(N) = H + 1 < \infty$, such that the measure space $(N, \mathcal{N}, \mu)$ is finite (and complete). This implies that the measure ν on $\mathscr{S}$ is also finite, and therefore, σ-finite. Since $\mathcal{N}$ is a semiring with $\mathscr{S} \subseteq \mathcal{N}$, our result follows. $\square$

Remark. The other properties of the new measure μ, together with those of the measure spaces $(N_0, \mathcal{N}_{N_0}, \mu)$, $(A, \mathcal{N}_A, \mu)$, and $(C, \mathcal{N}_C, \mu)$, where $\mathcal{N}_C$ denotes the restriction of $\mathcal{N}$ to C, can be derived by following the workings in Lemmata 3 and 4.

References

-Aliprantis, C. D. & Border K. C. (2006) *Infinite Dimensional Analysis*. Berlin, Springer.

-Bloch, F. & Ferrer, H. (2001) Trade fragmentation and coordination in strategic market games. *Journal of Economic Theory*, 101, 301-316.

-Codognato, G. & Ghosal, S. (2000) Oligopoly à la Cournot-Nash in markets with a continuum of traders. Discussion Paper No 2000-5, CEPET, Institute of Public Economics, Graz University.

-Diestel, J. & Uhl, J. J. (1977) *Vector Measures.* Mathematical surveys and monographs, 15, American Mathematical Society, Providence, Rhode Island.

-Dubey, P. & Shubik, M. (1978) The non-cooperative equilibria of a closed trading economy with market supply and bidding strategies. *Journal of Economic Theory*, 17, 1-20.

-Dubey, P. & Shapley, L. S. (1994) Noncooperative general exchange with a continuum of traders: two models. *Journal of Mathematical Economics*, 23, 253-293.

-Faias, M., Hervés-Beloso, C. & Moreno-García, E. (2010) Equilibrium price formation in markets with differentially informed agents. *Economic Theory*, 48, 205-218.

-Faias, M., Moreno-Garcia, E. & Wooders, M. (2014) A strategic market game approach for the private provision of public goods. *Journal of Dynamics and Games*, 1 (2), 283-298.

-Gabszewicz, J. J. & Grazzini, L. (1999) Taxing market power. *Journal of Public Economic Theory*, 1 (4), 475-497.

-Goenka, A., Kelly, D. L. & Spear, S. (1998) Endogenous strategic business cycles. *Journal of Economic Theory*, 81, 97-125.

-Koutsougeras, L. C. (1999) A remark on the number of trading posts in strategic market games. Tilburg University Discussion Paper No 9904, C.O.R.E Discussion Paper No 9905.

-Koutsougeras, L. C. (2003a) Convergence to no-arbitrage equilibria in market games. *Journal of Mathematical Economics*, 39, 401-420.

-Koutsougeras, L. C. (2003b) Non-Walrasian equilibria and the law of one price. *Journal of Economic Theory*, 108, 169-175.

-Koutsougeras, L. C. & Ziros, N. (2015) The second welfare theorem in economies with non-Walrasian markets. *Journal of Public Economic Theory*, 17 (3), 415-432.

-Luenberger, D. G. (1969) *Optimization by vector space methods*. New York, John Wiley & Sons.

-Peck, J., Shell, K. & Spear, S. (1992) The market game: existence and structure of equilibrium. *Journal of Mathematical Economics*, 21, 271-299.

-Rogawski, J. & Shubik, M. (1986) A strategic market game with transactions costs. *Mathematical Social Sciences*, 11 (2), 139-160.

-Shapley, L. S. & Shubik, M. (1977) Trade using one commodity as a means of payment. *Journal of Political Economy*, 85, 937-968.

-Shubik, M. (1973) Commodity money, oligopoly, credit and bankruptcy in a general equilibrium model. *Western Economic Journal*, 11 (1), 24-38.

-Toraubally, W. A. (2017a) *Essays in strategic market games*, Ph.D., University of Manchester.

-Toraubally, W. A. (2017b) Strategic market games and Ricardo. *Economics Bulletin,* 37 (4), 2517-2525.