Chapter 1

Time-dependent Dynamical Energy Analysis

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Abstract. *Dynamical Energy Analysis (DEA) was introduced in 2009 as a novel method for predicting high-frequency acoustic and vibrational energy distributions [13]. In this work we detail how DEA can be reformulated in the time-domain by means of a convolution integral operator and apply the Convolution Quadrature (CQ) method to discretise in time. The CQ method provides a link between the frequency domain and fully time-dependent solutions by means of the $\mathcal{Z}$-transform. The space and momentum variables may be approximated using the same approaches that have previously been implemented in frequency domain DEA. The final result is a fully time-dependent DEA method that can track the propagation of high-frequency transient signals through phase-space.*

1.1 Introduction

Boundary integral methods for modelling time-dependent wave propagation were originally proposed in the 1960s [1, 2]. The considerable increase in available computer power during the latter part of the twentieth century made numerical solutions over longer time intervals feasible, and with this advance long-time instabilities in the numerical solutions also became evident [3, 4, 5]. The cause of these instabilities has been linked to internal resonances of the wave scatterer for exterior problems [4], or the region being modelled for interior problems. For this reason, combined field integral equations, such as the time-dependent Burton and Miller formulation, have been proposed to tackle these stability issues [6, 7]. However, these formulations introduce additional computational overheads and the need to evaluate hypersingular boundary integral operators. An alternative is to apply the CQ method, see for example Refs. [8, 9, 10], which is able to provide stable results based on standard integral equation formulations. The reason for the preferable stability properties of CQ essentially relate to the reconstruction of the time domain solution, or alternatively the time domain boundary integral operator, through a numerical inverse Laplace transform where the contour is taken over Laplace domain frequencies with strictly positive real part. Since the resonances lie on the imaginary axis in the Laplace domain, then they do not effect the result in the time domain.

For high-frequency time-dependent wave problems, such as those arising in seismology or room acoustics, ray-tracing methods are often preferred to full wave models, see for example [11, 12]. Traditional ray based methods work well for applications where only a few reflections need to be considered, but not so well for problems including multiple scattering and chaotic dynamics. In this case, multiple reflections of the rays can give an exponentially growing number of trajectories to track. Dynamical Energy Analysis (DEA) is a phase-space boundary integral method that models wave energy densities [13]. DEA is a frequency domain method formed by seeking solutions of the stationary Liouville equation, circumventing issues regarding the exponentially growing number of rays to track as time increases [14].

Time-domain simulations are important for various applications such as predicting radar cross sections, shock-responses and auralisation of room acoustics. In this proceedings paper we outline a methodology for extending DEA to the time-domain based on the CQ method. The computational cost of time-domain DEA should scale only linearly with the modelled time period regardless of the ray dynamics, comparing favourably with conventional ray-tracers.

1.2 Outline of Methodology

In order to develop a time-dependent DEA method for a domain Ω with associated speed of sound c, the first step is to reformulate the DEA phase-space boundary integral operator to have explicit time dependence. The result is a one-sided convolution (in time t) operator $\mathcal{B}$ given by

$$(\mathcal{B}\rho_0)(t,s,p) = \int_{c|p'|\leq 1}\int_{\Gamma}(k*\rho_0)(t,s',p')\,\mathrm{d}\Gamma(s')\mathrm{d}p', \tag{1.1}$$

which is applied to a specified initial density distribution of rays ρ_0 on the boundary Γ. Here k is the time-dependent kernel of our boundary integral operator, which is given by a multidimensional Dirac delta generalised function specifying the propagation of a ray through time, position and momentum. The variables (s',p') relate respectively to the position and momentum of the starting position of a ray emanating from Γ and (s,p) correspond to the arrival position and momentum on Γ, respectively, following a specular reflection. Note that a damping factor must be applied to obtain convergence in frequency domain DEA, but this is not necessary in the time-dependent formulation owing to the fact that we only model a finite time duration $[0,T]$.

The CQ method can be applied for the time discretisation of one-sided convolution operators such as $\mathcal{B}$ (1.1), see for example [8, 9, 10]. In doing so, the convolution $(k*\rho_0)$ appearing in (1.1) is approximated by a discrete convolution of the form

$$(k*_{\Delta t}\rho_0)(t_n) = \sum_{j=0}^{n} w_{n-j}\,\rho_0(t_j),$$

where $t_j = j\Delta t$ and $\Delta t = T/N$ is the time-step assuming N steps in total. The convolution weights w_j are defined implicitly through the $\mathcal{Z}$-transform as

$$\sum_{j=0}^{\infty} w_j\,\zeta^j = K\left(\frac{\gamma(\zeta)}{\Delta t}\right),$$

where γ is the quotient of generating polynomials for the linear multistep method underlying the CQ discretisation. In this work we will use the second order backward difference formula, for which

$$\gamma(\zeta) = \frac{1}{2}\zeta^2 - 2\zeta + \frac{3}{2}.$$

After applying the time-discretisation, we then need to discretise in the position and momentum variables in order to obtain a fully discrete problem. Here we may use any of the discretisation methods previously applied for frequency domain DEA amongst others, see for example [13, 14, 15] for more details.

1.3 Conclusions and Future Work

In this short paper we have motivated and outlined a methodology to extend the DEA method for time-dependent problems. We will present numerical results based on this work at the conference. For these results we will apply the position and momentum discretisation methods from Ref. [15], since this allows for a verification of the time discretisation method in simple examples for which we can derive an exact solution. We will use these examples to study whether the order of convergence in time is consistent with the expected behaviour of the applied time-stepping approach. We will also present examples for which this choice of position and momentum discretisation is less favourable and it may be preferable to use the methods in Ref. [14], for example, instead.

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