Universal law of fractionation for slightly polydisperse systems

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By perturbing about a general monodisperse system, we provide a complete description of two-phase equilibria in any system which is slightly polydisperse in some property (e.g. particle size, charge, etc.). We derive a universal law of fractionation which is corroborated by comprehensive experiments on a model colloidal-polymer mixture. We furthermore predict that phase separation is an effective method of reducing polydispersity only for systems with a skewed distribution of the polydisperse property.

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Complex fluids contain mesoscopic units that are almost inevitably polydisperse, i.e. colloidal or polymeric particles have some characteristic, such as radius, charge, mass or oblateness, which varies quasi-continuously from one to another. A truly polydisperse system contains infinitely many species with a distribution of properties, and could separate into arbitrarily many coexisting phases. The onset of phase separation is at the 'cloud curve', the boundary of coexistence with an infinitesimal amount of a second phase on the 'shadow curve'. In contrast to simple systems, a complete description of phase equilibria entails determining not just these limiting curves, but also the different compositions (described by a distribution) of arbitrary coexisting phases.

Experimentally, phase equilibria have been completely determined for polydisperse polymers [1]. In contrast, most experiments on colloidal phase behavior have ignored polydispersity, despite pragmatic interest in using phase separation to fractionate suspensions [2]; limited data on particulate systems derives only from simulations [3,4]. Many calculations of two-phase equilibria have been attempted for specific polydisperse systems (5 and references therein), especially polymers (which admit mean-field analysis). The popular but arbitrary method of discretizing the distribution [6], though efficacious, gives little insight. The infinity of coexistence conditions hampers the formulation of truly polydisperse statistical mechanics (discussed in [5,7,8]), especially in non-mean-field systems, for which exact phase calculations are consequently scarce [9]. The approach of Gualtieri et al. [5] to calculating two-phase coexistence is applicable to a large class of model systems, but gives rise to formidable non-linear equations. They calculate cloud/shadow curves for a polydisperse van der Waals model, but give no general result. We present a simpler treatment, applicable to real systems, and use it to solve the two-phase coexistence problem completely in the limit of small polydispersities. A universal law of fractionation is derived. We show significant consistency with comprehensive measurements of phase equilibria in a model polydisperse colloid.

Following Gualtieri et al. [5], we divide the total free energy, $F_{\text{tot}} = F_{\text{id}} + F_{\text{ex}}$ into two parts: the free energy of a polydisperse ideal gas of the given species distribution, and the excess due to interactions. The ideal part, $F_{\text{id}}$ is a functional of the distribution $f(\varepsilon)$, where $f(\varepsilon)\,d\varepsilon$ particles have the polydisperse property $\varepsilon$ in the range $\varepsilon$ to $\varepsilon + d\varepsilon$. No matter what the system, $F_{\text{id}}$ always takes the same form (in units of $k_B T$),

$$F_{\text{id}} = \int d\varepsilon \, f(\varepsilon) \left[ \ln \frac{f(\varepsilon)}{V} - 1 \right]$$

(1)

where the integral is the ideal gas free energy of each species in volume $V$. The total number of particles is $N = \int f(\varepsilon)\,d\varepsilon$. Non-trivially, Eq. 1 holds whether or not the members of each species are quantum-mechanically indistinguishable [8]. At equilibrium, the chemical potential for each species, $\mu(\varepsilon) \equiv \delta F_{\text{id}}/\delta f(\varepsilon)$ is equal in any pair of coexisting phases. That is,

$$\Delta\mu(\varepsilon) = \Delta\mu_{\text{id}}(\varepsilon) + \Delta\mu_{\text{ex}}(\varepsilon) = 0 \quad \text{for all} \ \varepsilon,$$

(2)

which is an infinite number of equilibrium conditions (with $\Delta$ indicating difference between phases). From Eq. 1, the ideal part is $\mu_{\text{id}}(\varepsilon) = \ln[Vf(\varepsilon)/V]$. Applying the equilibrium constraints [Eq. 2] allows ratios of densities in coexisting phases $A$ and $B$,

$$\frac{f_A(\varepsilon)/V_A}{f_B(\varepsilon)/V_B} = \exp \Delta\mu_{\text{ex}}(\varepsilon)$$

(3)

to be found from $F_{\text{ex}}$. To reduce the infinity of simultaneous equations for coexistence to a finite set, Gualtieri et al. assumed $F_{\text{ex}}$ was a function of finitely many moments. Recent work [8,10] has clarified the status of the moments neglected in $F_{\text{ex}}$ in this type of approach, but the 'finite moment' assumption remains arbitrary, and is unlikely to be true for real systems.

Our ab initio approach imposes no special structure on $F_{\text{ex}}$. Our starting point is the observation that an almost pure ensemble (one with a narrow normalized distribution $p(\varepsilon) \equiv f(\varepsilon)/V$) should behave similarly to a monodisperse system (for which $p(\varepsilon) = \delta(\varepsilon)$, the Dirac delta function), despite having a very different formal
description (being a mixture of infinitely many species).
Therefore, although a general polydisperse system could
separate into arbitrarily many coexisting phases (see e.g.
[10]), we may restrict our attention to two-phase coex-
istence. We apply thermodynamic perturbation theory
to a monodisperse reference system [11], using \( \varepsilon_i \) as
a small, dimensionless variable assigned to each particle
\( i = 1 \ldots N \). We ascribe no specific physical meaning to
\( \varepsilon_i \) at this stage but it may be, e.g., the fractional
deviation of a particle's radius from the mean. With this
approach, we derive a number of results which are ex-
act and universal in the limit of a narrow distribution.
Such distributions are ubiquitous, being the typical product
of chemical syntheses aimed at producing monodisperse
systems. Details of the calculations are given else-
where [12]. One universal (but perhaps unsurprising)
result to emerge is that, to first order in polydispersity,
the shadow and cloud curves are not perturbed from the
coexistence boundary (the bimodal) of the monodisperse
reference system. Coexisting polydisperse phases in gen-
eral exhibit fractionation and have different distributions
from the parent sample, e.g., the denser phase might fa-
vor larger particles. We derive universal results relating
the moments in the daughter phases to those of the par-
ent, using the location of the monodisperse bimodal as an
input parameter.

We begin by noting [8,10,13] that 'moment densities',
\[
\rho_a \equiv \frac{1}{V} \int_{-\infty}^{\infty} \varepsilon^a f(\varepsilon) \, d\varepsilon ,
\]
being linear combinations of conserved species densities,
are themselves conserved and, accordingly, respect the usual equilibrium conditions. For instance, each 'mo-
ment chemical potential', defined by \( \mu_a \equiv \delta (F/V) / \delta \rho_a \), is equal in coexisting phases. This is clear from expanding
the species chemical potential in partial derivatives
\[
\mu(\varepsilon) \equiv \delta F / \delta f(\varepsilon) = \sum_{a=1}^{\infty} \delta F / \delta \rho_a \delta f(\varepsilon) = \sum_{a=1}^{\infty} \mu_a \varepsilon^a .
\]
Thus, equality of \( \mu(\varepsilon) \) in coexisting phases requires equality
of \( \mu_a \). Note that \( \rho_a = \varepsilon^a \rho \), so that \( \rho_a \) is the overall
particle density \( \rho \). The mean \( a \)th power of \( \varepsilon \), \( \varepsilon^a \), is the
\( a \)th moment of the normalized distribution \( p(\varepsilon) \).

Substituting Eq. 5 into 3 and demanding that, at co-
existence, the distributions in the two phases \( A \) and \( B \)
sum to the parent \( f_p(\varepsilon) \) from which they came yields
\[
f_A(\varepsilon) = \frac{f_P(\varepsilon)}{1 + \sum_{\alpha=(A)}^{\infty} \exp(-\sum_{\alpha=(A)}^{\infty} \Delta \mu_A \varepsilon^a)} \tag{6}
\]
which expresses a daughter distribution in terms of the
parent and excess moment chemical potentials in the
daughter phases. Here \( \Delta \mu_A = \mu_A(A) - \mu_A(B) \), and \( V_A, V_B \)
are the phase volumes. Eq. 6 was expressed in Ref. [5]
in terms of species chemical potentials. It simply en-
sures equality of all chemical potentials in the coexisting
phases. We can obtain all \( \rho(\varepsilon) \) from Eqs. 4 and 6, given
\( \{\mu_A^{\text{ex}}, \mu_B^{\text{ex}}, \ldots\} \), for which we require \( F^{\text{ex}} \).

To find \( F^{\text{ex}} \), let us write the polydisperse Hamiltonian
as \( H_{\text{poly}} = H_{\text{mono}} + H_1 \), the sum [14] of a monodisperse
reference and a perturbation. Using Eq. 1, the free en-
ergy is evaluated from a configurational integral [12] as
\[
F^{\text{ex}}_{\text{poly}} = F^{\text{ex}}_{\text{mono}} - \ln \left( \exp(\Delta H_1)_{\text{mono}} \right) ,
\]
which resembles the usual expression [15] for the free en-
ergy of a perturbed system, but applies only to the ex-
cept parts. Thermal averages with respect to the monodis-
perse reference system are denoted \( \langle \cdot \rangle_{\text{mono}} \). Assuming
the Hamiltonian \( H_{\text{poly}}(\{\varepsilon_i, \varepsilon_i \}) \) is differentiable with respect to the \( \varepsilon_i \), we expand it to first order to find
\[
H_1 = \sum_{j=1}^{N} \varepsilon_j K_j(\{\varepsilon_i \}) \tag{7}
\]
where \( K_j(\{\varepsilon_i \}) = (\partial H_{\text{poly}} / \partial \varepsilon_j) |_{\{\varepsilon_i \}=0} \) and \( \{\varepsilon_i \} \)
are the particle positions. We have not assumed any spe-
cial properties of the Hamiltonian (not even pairwise
additivity of interactions) except differentiability. \( K_j \)
is a property of particle \( j \). Substituting Eq. 8 into 7 gives
\( \sum_{j=1}^{N} \varepsilon_j K_j(\{\varepsilon_i \}) \to 0 \). As the thermal average is taken in the reference system
of identical particles, it is independent of \( j \). We write
\( \langle K_j \rangle_{\text{mono}} = \langle K \rangle_{\text{mono}} \), which is the mean rate of change
of total energy from varying the property \( \varepsilon \) of any parti-

dec. Eq. 7 therefore yields
\[
F^{\text{ex}}_{\text{poly}} / V = F^{\text{ex}}_{\text{mono}} / V + \langle K \rangle_{\text{mono}} \rho_1 + O(\varepsilon^2) .
\]
A strength of the perturbative approach is that high
moments are of higher order in small quantities, so \( F^{\text{ex}}_{\text{poly}} \)
truncates naturally. (We have stopped at first order.) In
previous approaches [5,10] such truncation was necessary
but arbitrary. Since \( \rho_a = \varepsilon^a \rho \), the infinite sum in Eq. 6 becomes finite, and the equations are
tractable. Eq. 9 yields \( \{\mu_A^{\text{ex}}, \mu_B^{\text{ex}}\} \) in terms of \( \{\rho_A, \rho_B\} \),
thus, with Eqs. 4 and 6, solving the problem. The solu-
tion involves linearizing the factor multiplying \( f_P(\varepsilon) \) in
Eq. 6, but \( \varepsilon \) in any way approximating \( f_P(\varepsilon) \) itself.
Consequently, so long as the distribution is narrow, it can
have any shape, including the case of some components
appearing in finite amounts (contributing delta spikes),
which was treated separately in Ref. [5].

Defining the origin of \( \varepsilon \) so its mean vanishes in the par-
ent, \( \varepsilon \equiv 0 \), we find that normalized distributions in the
daughter phases differ by \( \Delta p(\varepsilon) \to -2 p(p(\varepsilon)) \Delta \langle K \rangle_{\text{mono}} \).
Hence their moments differ by
\[
\Delta \varepsilon^{a+1} \to -2 p(\varepsilon) \Delta \langle K \rangle_{\text{mono}} \tag{10}
\]
as \( \varepsilon \) \to 0, which is proportional to the next moment of the
parent distribution. Eq. 10 constitutes a full solution
of the two-phase coexistence problem to lowest order in polydispersity. Taking \( \alpha = 1 \), we find that the separation \( \Delta \bar{\sigma} \) of daughter distributions is proportional not to the width, but to the variance of the parent. For \( \alpha = 2 \), Eq. 10 prescribes the daughters’ variances. (In fact Eq. 10, with closure by conservation of matter [16], gives the second moments of the phases, \( \sigma^2_A, B \). To lowest order, ‘variance’ \( \sigma^2 \equiv \bar{\sigma}^2 - \bar{\sigma}^2 \), and ‘second moment’ are interchangeable since the difference, \( \sigma^2_A, B \), is of fourth order in the width of the parent.) Eq. 10 asserts that purification (which requires one daughter to have a smaller variance than the parent, resulting [16] in a finite \( \Delta \bar{\sigma} \)) of a slightly polydispersed sample by phase separation (\( \varepsilon \) [2]) is ineffective unless the sample’s distribution is strongly skewed, \( \bar{\sigma}_p \neq 0 \).

The system-dependent constant of proportionality \( \Delta \bar{\sigma}_\text{mono} \) in Eq. 10 is a function of properties of the two coexisting phases, but not of \( \alpha [17] \). Hence, to lowest order, the ratio of differences of any two moments in the daughter phases,

\[
\frac{\Delta \bar{\sigma} \sigma_i \bar{\sigma}_p}{\bar{\sigma} \sigma_i \bar{\sigma}_p} = \frac{\varepsilon \sigma_i^3}{\bar{\sigma}_p^3} \quad (11)
\]

is, surprisingly, independent of any properties of the phases themselves (even the nature of the interactions). It relies only on the parent from which they came. As Eq. 11 contains no system-dependent parameters, it constitutes a universal law of slightly polydispersed systems. We restate that the small number \( \varepsilon \) may be any distinguishing property of the particles, such as relative size, charge or mass, expressed dimensionlessly. Having identified no particular physical scale for the property, we must clarify what constitutes ‘small’. Linearization of the distribution’s prefactor in Eq. 6, and of Eq. 7 to produce Eq. 9, is a good approximation over some range of \( \varepsilon \). This range defines the maximum width of \( p(\varepsilon) \) for which the calculation is valid. The extra assumption that the Hamiltonian may be expanded as Eq. 8 is not vital to the final result. Even hard spheres, whose interactions are non-differentiable, have a linearizable excess free energy (as Eq. 9). Their moments therefore obey Eq. 10, but with \( \Delta \bar{\sigma}_\text{mono} \) an unknown [12] constant of proportionality, which still cancels to give Eq. 11.

Eqs. 10 and 11 can, in principle, be tested by observing phase equilibria in any slightly polydispersed system. Here we report results obtained from polydisperse suspensions with polydisperse particle radius \( R \). We define the deviation of the ith particle \( \varepsilon_i \) by \( \varepsilon_i \equiv R_i - \bar{R} \), with \( \bar{R} \) the mean radius in the parent sample. To date, experiments on polydisperse colloids have concentrated on characterisation by static or dynamic light scattering; little data on polydisperse colloidal phase equilibria exist. We have performed comprehensive measurements on two-phase coexistence in a model system: mixtures of sterically-stabilised polymethylmethacrylate (PMMA) particles and random-coil polystyrene (PS) dispersed in cis-decalin.
little fractionation between phases, as all species feel approximately the same driving force. So it is differences in $\Delta \rho^{ex}$ for different $\alpha$ which bias the shapes of the distributions, making one species partition itself differently from another. As all species are similar, these differences are small (of order $\sigma_P d\Delta \rho^{ex}/d\alpha$). This small ‘biasing force’ on the shapes of the distributions is the source of the extra factor $\sigma_P$ in the power law. So Eq. 10 is understood for $\alpha = 1$. Heuristically, Eq. 10 is at least reasonable for $\alpha = 2$. For a symmetric parent, the biasing force happens to leave the variances of both daughters equal to that of the parent. One phase may prefer large particles, accentuating the high-$\alpha$ end of its distribution, but this is exactly balanced by loss from the low-$\alpha$ end. It seems reasonable, however, that the daughters’ widths should differ if one end of the parent distribution is more ‘compact’ than the other. That is, one daughter will be purer than the other if their parent is skewed.

To summarise, we have derived universal laws governing two-phase coexistence in slightly polydisperse systems (Eqs. 10 and 11), which show that skewness in the parent is required for purification and that fractionation, $\Delta \sigma$, is dependent on variance rather than width. Experimental results provide support for Eq. 11, and demonstrate how measurements of one moment permit the inference of another. It is remarkable that the complexities of polydisperse statistical mechanics should give rise to universal results of such mathematical simplicity and experimental utility.

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[14] This should not be confused with the separate dichotomy between ideal and excess free energies.
[16] The value of a moment $\langle r^\alpha \rangle$ in each phase $(A, B)$ is calculated from the difference $\Delta \bar{r}^\alpha$ by appealing to conservation of matter, expressed as $N_A \bar{r}_A^\alpha + N_B \bar{r}_B^\alpha = N \bar{r}_P^\alpha$, with $N_A, B$ the number of particles in each phase [10,12].
[17] For a central, pairwise-additive, isotropic, symmetric interaction potential $U(r, e_1, e_2)$, dependent on the properties $e_1, e_2$ of two particles of separation $r$, $\langle K \rangle_{mono} = 4\pi \rho \int_{0}^{\infty} \frac{\sigma d\sigma}{2\pi} \sum_{\alpha} g(r) dr$ with $g(r)$ the monodisperse radial distribution function [12].
[23] At even higher polymer concentrations, transient gels formed. These samples were excluded from the study.
[24] As our derivation assumed ergodicity, it is probably inapplicable to crystalline coexistence anyway. In a defect-free crystal, once a particle has found a lattice site, it is caged and cannot explore phase space (in contrast to the artificially annealed crystals in [4]).