Analysis of Partial Diffusion LMS for Adaptive Estimation Over Networks with Noisy Links

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Abstract—In partial diffusion-based least mean square (PDLMS) scheme, each node shares a part of its intermediate estimate vector with its neighbors at each iteration. In this paper, besides being involved in more general PDLMS scheme, we figure out how the noisy links affect deterioration of network performance during the exchange of weight estimates. We investigate the steady state mean square deviation (MSD) and derive a theoretical expression for it. We demonstrate that the PDLMS algorithm is stable and convergent in both mean and mean-square sense under non-ideal links. However, unlike the established statements on PDLMS scheme under ideal links, the trade-off between MSD performance and the number of selected entries of the intermediate estimate vectors as a sign of communication cost is mitigated. Strictly speaking, considering non-ideal links condition adds a new complexity to MSD relation that has a noticeable effect on its performance. This term violates the trade-off between communication cost and estimation performance of the networks in comparison to noise-free condition on the links. Our simulation results substantiate the effect of noisy links on PDLMS algorithm and verify the theoretical findings. They match well with theory.

Index Terms—Adaptive networks, distributed estimation, least mean-square, noisy links, partial diffusion.

I. INTRODUCTION

W e consider the problem of distributed estimation in the diffusion adaptive networks context, where the spatially-scattered nodes have adaptation and learning capabilities. In such networks, the nodes are linked together through a topology and exchange information through localized inter-network processing to perform decentralized information processing and optimization in a cooperative and online manner. The local interactions and diffusion of information across the network enable the nodes to respond in real-time to the drifts in statistical properties of the data and to the changes in network topology [1]–[4]. Several strategies for distributed estimation over adaptive networks have been reported in the literature. Diffusion strategies [5]–[10] are among the most popular propositions, in the literature. They are scalable as well as robust to link/node failure and have good adaptability and tracking performance with respect to other strategies [2]. In adaptive diffusion implementations, the nodes communicate with their immediate neighbors and the information is processed locally and simultaneously at all nodes across the network.

It is obvious that the weight estimates that are exchanged among the nodes can be subject to different perturbations such as quantization errors, noisy input data, additive noise over the communication links and wireless link impairments. Studying the degradation in performance that results from the mentioned perturbations can be found in [11]–[17].

Due to the limited power and bandwidth resources for communication among nodes over a distributed networks (such as wireless sensor networks), the most expensive part of realizing a cooperative task is the data transmission through the links. Generally speaking, although the benefits of diffusion strategies achieved by increasing internode communications, they compromised by the communication cost. As the consequence, since different nodes can have different numbers of neighbors, they may require disparate hardware or consume power differently. Therefore, reducing the amount of internode communications, while maintaining the benefits of cooperation is of practical importance. There have been several efforts to achieve the mentioned objective such as reducing the dimension of the estimates [18]–[20], selecting a subset of the entries of the intermediate estimate vectors [21], [22], set-membership filtering [23]–[25] and partial updating [26].

Among these methods, we focus on [21] where the LMS algorithm for adaptive distribute estimation has been formulated and analyzed by utilizing partial-diffusion. In [21], an adapt-then-combine (ATC) PDLMS algorithm has been reported for distributed estimation over adaptive networks with ideal links. In the mentioned algorithm, at each iteration, each node transmits a subset of the entries of intermediate estimate vector to its neighbors. However, as we mentioned earlier, in practice the weight estimates that are exchanged among the nodes can be subject to additive noise over communication links. In this paper, besides being involved in more general PDLMS scheme, we figure out how the noisy links affect deterioration of network performance during the exchange of weight estimates. Among other results, our analysis provides some useful insights on the communication cost and estimation performance trade-off for general PDLMS scheme under non-ideal links. Our main contributions in this paper include:

(i) Focusing on [21] which involves transmission of a subset of the entries of the internode estimate vectors named partial diffusion, we provide a more general algorithmic structure of which [21] is just a special case. To achieve this, we consider the fact that weight estimates exchanged among the nodes can be subject to quantization errors and additive noise over communication links. Like
[21], we also consider two different schemes for selecting
the weight vector entries for transmission at each
iteration. We allow for noisy exchange just during the
two combination steps. It should be noted that since our
objective is to minimize the internode communication,
the nodes only exchange their intermediate estimates
with their neighbors;
(ii) Using the energy conservation argument [27] we analyze
the stability of algorithms in mean and mean square
senses under certain statistical conditions.
(iii) We illustrate the comparable convergence performance
of PDLMS algorithm with noisy links in different numerical
examples.

The main aim of this paper is that the noisy links are the
main factor in performance degradation of a partial diffusion
least mean squares (PLMS) algorithm running in a network
with noisy links. In other words, considering noisy links adds
an extra term to MSD relation. This term seriously upset
the balance of the trading off between communication cost
and the estimation performance, in comparison with the ideal
case. Because, the more entries are communicated at each
iteration, the more perturbed weight estimates are interred in
the consultation phase.

This work is organized as follows. In Section II, we for-
mulate the PDLMS under noisy information exchange. The
performance analyses are examined in Section III. We provide
simulation results in Section IV and draw the conclusions in
Section V.

A. Notation

We use the lowercase letters to denote vectors, uppercase
letter for matrices, plain letter for deterministic variables, and
the boldface letters for random variables. We also use \((\cdot)^*\)
to denote conjugate transposition, \(\text{tr}(\cdot)\) for the trace of matrix,
\(\otimes\) for Kronecker product, and \(\text{vec}\{\cdot\}\) for a vector formed by
stacking the columns of its matrix argument. We further use
\(\text{diag}\{\cdot\}\) to denote a (block) diagonal matrix formed from its
argument, and \(\text{col}\{\cdot\}\) to denote a column vector formed by
stacking its arguments on top of each other. All vectors in our
treatment are column vectors, with the exception of regression
vectors, \(u_{k,i}\).

II. PARTIAL DIFFUSION ALGORITHMS WITH NOISY
INFORMATION EXCHANGE

Consider a connected network consisting of \(N\) nodes. Each
node \(k\) collects scalar measurements \(d_{k,i}\) and \(1 \times M\) regression
data vectors \(u_{k,i}\) over successive time instants \(i \geq 0\). Note
that we use parenthesis to refer to the time-dependence of scalar
variables, as in \(d_{k,i}\), and subscripts to refer to the time-
dependence of vector variables, as in \(u_{k,i}\). The measurements
across all nodes are assumed to be related to an unknown
\(M \times 1\) vector \(w^o\) via linear regression model of the form [27]:

\[
d_{k,i} = u_{k,i}w^o + v_{k,i} \tag{1}
\]

where \(v_{k,i}\) denotes the measurement or model noise. We are
now interested in solving optimization problems of the type:

\[
w^o = \min_w \sum_{k=1}^{N} \mathbb{E} \left[ (d_{k,i} - u_{k,i}w)^2 \right] \tag{2}
\]

The nodes in the network would like to cooperate with
each other in order to estimate \(w^o\) by solving the equation
above in an adaptive manner. Putting an accurate interpretation
on solution vector \(w^o\) from (2) depends on application under
consideration. One possible interpretation is that the entries of
\(w^o\) represent the location coordinates of a flying object (such
as tracking a projectile) that agents are trying to find. In other
applications, the entries of \(w^o\) describes an underlying tapped-
delay-line model also known as finite-impulse-response (FIR)
that agents are interested in estimating the parameters of an
FIR model, such as taps of a communication channel or the
parameters of some (approximate) model of interest in finance
or biology [2]. We review the diffusion adaptation strategies
with noisy links below.

A. Diffusion Adaptation with Noisy Information Exchange

Consider the following general adaptive diffusion strategies
corresponding to the case in which the nodes only share weight
estimates for \(i \geq 0\):

\[
\phi_{k,i} = \sum_{l \in N_k} c_{1,lk} w_{l,i-1} \tag{3}
\]

\[
\psi_{k,i} = \phi_{k,i-1} + \mu_k u_{k,i}^* \left[ d_{k,i} - u_{k,i} \phi_{k,i-1} \right] \tag{4}
\]

\[
w_{k,i} = \sum_{l \in N_k} c_{2,lk} \psi_{l,i} \tag{5}
\]

Local estimators of \(w^o\), that node \(k\) computes based on
observations \(\{d_{l,j}, u_{k,j}\} | j \leq i\) in addition to intermediate
estimators up to and including time \(i\), are denoted by \(M \times 1\)
matrices \(\{\phi_{k,i}, \psi_{k,i}, w_{k,i}\}\). The physical meanings of these
vectors and \(w^o\) are exactly the same. The scalars \(\{c_{1,lk}, c_{2,lk}\}\)
are non-negative real coefficients corresponding to the \(\\{l,k\}\)
entries of \(N \times N\) combination matrices \(\{C_1, C_2\}\), respectively.
They are zero whenever node \(l \notin N_k\), where \(N_k\) denotes the
neighborhood of node \(k\). These matrices are assumed to satisfy
the conditions:

\[
C_1^T = 1_N, \quad C_2^T = 1_N \tag{6}
\]

where the notation \(1_N\) denotes an \(N \times 1\) column vector with
all its entries equal to one.

We model the noisy data received by node \(k\) from its
neighbor \(l\) as follows:

\[
w_{lk,i-1} = w_{l,i-1} + v_{lk,i}^{(w)} \tag{7}
\]

\[
\psi_{lk,i} = \psi_{l,i} + v_{lk,i}^{(w)} \tag{8}
\]

where \(v_{lk,i}^{(w)} (M \times 1)\) and \(v_{lk,i}^{(s)} (M \times 1)\) are the noise
observations. It should be noted that the subscript \(lk\) indicates
that \(l\) is the source and \(k\) is the sink and the flow of information
is from \(l\) to \(k\).
Using the perturbed data (7) and (8), the adaptive strategy (3)-(5) becomes
\[
\phi_{k,i} = \sum_{l \in N_k} c_{1,lk} w_{lk,i-1}
\]
\[
\psi_{k,i} = \phi_{k,i-1} + \mu_k u_{k,i}^* [d_{k,i} - u_{k,i} \phi_{k,i-1}]
\]
\[
w_{k,i} = \sum_{l \in N_k} c_{2,lk} \psi_{lk,i}
\]
\[
(9)
\]
\[
(10)
\]
\[
(11)
\]

**B. Partial Diffusion with Noisy Links**

In this paper, we adopt a similar approach proposed in [21] and build up our algorithm upon it. Selecting and scattering \( L \) out of \( M \), \( 0 \leq L \leq M \), entries of the intermediate estimate vector of each node \( k \) at time instant \( i \), make the realization of reducing internode communication possible. According to this scheme, the selection of to be scattered elements could be realized by a diagonal selection matrix, \( K_{k,i} \). Multiplication of \( \{ w_{lk,i-1}, \dot{\psi}_{lk,i} \} \) by \( K_{k,i} \) that have \( L \) ones and \( M - L \) zeros on its diagonal replaces its non-selected entries with zero. The positions of the ones on diagonal of \( K_{k,i} \) determine the entries of node \( k \) that are selected to diffused at time \( i \). Note that, the integer \( L \) is fixed and pre-specified [21]. According to (9) and (11)
\[
\phi_{k,i} = c_{1,kk} w_{k,i-1} + \sum_{l \in N_k \setminus \{k\}} c_{1,lk} [K_{l,i-1} w_{lk,i-1}]
\]
\[
+ (I_M - K_{l,i-1}) w_{lk,i-1}
\]
\[
(12)
\]
\[
\psi_{k,i} = \phi_{k,i-1} + \mu_k u_{k,i}^* [d_{k,i} - u_{k,i} \phi_{k,i-1}]
\]
\[
w_{k,i} = c_{2,kk} \psi_{k,i} + \sum_{l \in N_k \setminus \{k\}} c_{2,lk} [K_{l,i} \dot{\psi}_{lk,i}]
\]
\[
+ (I_M - K_{l,i}) \dot{\psi}_{lk,i}
\]
\[
(13)
\]
\[
(14)
\]
where \( I_M \) is the identity matrix of size \( M \times M \).

The most fundamental problem, we are faced with, hinges on ambiguities in non-diffused elements of nodes in combination phase. When intermediate estimate are partially transmitted, the non-communicated entries are not available to take part in this phase. However, each node requires all entries of intermediate estimate vectors of its neighbors for combination. To avoid this ambiguity, nodes can replace the entries of their own intermediate estimates instead of the ones from the neighbors that are not available. substitute
\[
(I_M - K_{l,i-1}) w_{lk,i-1}, \quad \forall l \in N_k \setminus \{k\}
\]
\[
(15)
\]
for
\[
(I_M - K_{l,i-1}) w_{lk,i-1}, \quad \forall l \in N_k \setminus \{k\}
\]
\[
(16)
\]
and
\[
(I_M - K_{l,i}) \dot{\psi}_{lk,i}, \quad \forall l \in N_k \setminus \{k\}
\]
\[
(17)
\]
for
\[
(I_M - K_{l,i}) \dot{\psi}_{lk,i}, \quad \forall l \in N_k \setminus \{k\}
\]
\[
(18)
\]
Based on this approach, we formulate general PDLMS under noisy information exchange as follows:
\[
\phi_{k,i} = c_{1,kk} w_{k,i-1} + \sum_{l \in N_k \setminus \{k\}} c_{1,lk} [K_{l,i-1} w_{lk,i-1}]
\]
\[
+ (I_M - K_{l,i-1}) w_{lk,i-1}
\]
\[
(19)
\]
\[
\psi_{k,i} = \phi_{k,i-1} + \mu_k u_{k,i}^* [d_{k,i} - u_{k,i} \phi_{k,i-1}]
\]
\[
w_{k,i} = c_{2,kk} \psi_{k,i} + \sum_{l \in N_k \setminus \{k\}} c_{2,lk} [K_{l,i} \dot{\psi}_{lk,i}]
\]
\[
+ (I_M - K_{l,i}) \dot{\psi}_{lk,i}
\]
\[
(20)
\]
\[
(21)
\]
**Remark.** The probability of transmission for all the entries at each node is equal and expressed as
\[
\rho = L/M
\]
\[
(22)
\]
Moreover, the entry selection matrices, \( K_{k,i} \), do not depend on any data/parameter other than \( L \) and \( M \).

From (7), (8), expression (19)-(21) can be written as:
\[
\phi_{k,i-1} = c_{1,kk} w_{k,i-1} + \sum_{l \in N_k \setminus \{k\}} c_{1,lk} [K_{l,i-1} w_{lk,i-1}]
\]
\[
+ (I_M - K_{l,i-1}) w_{lk,i-1}
\]
\[
(23)
\]
\[
\psi_{k,i} = \phi_{k,i-1} + \mu_k u_{k,i}^* [d_{k,i} - u_{k,i} \phi_{k,i-1}]
\]
\[
w_{k,i} = c_{2,kk} \psi_{k,i} + \sum_{l \in N_k \setminus \{k\}} c_{2,lk} [K_{l,i} \dot{\psi}_{lk,i}]
\]
\[
+ (I_M - K_{l,i}) \dot{\psi}_{lk,i}
\]
\[
(24)
\]
\[
(25)
\]
Introducing the following aggregate \( M \times 1 \) zero mean noise signals:
\[
v^{(w)}_{k,i-1} = \sum_{l \in N_k \setminus \{k\}} c_{1,lk} K_{l,i-1} v^{(w)}_{lk,i-1}
\]
\[
v^{(w)}_{k,i} = \sum_{l \in N_k \setminus \{k\}} c_{2,lk} K_{l,i} v^{(w)}_{lk,i}
\]
\[
(26)
\]
\[
(27)
\]
where \( \{ v^{(w)}_{k,i-1}, v^{(w)}_{k,i} \} \) represent the aggregate effect on node \( k \) of all selected exchange noises from its neighbors while exchanging the estimates \( \{ w_{lk,i-1}, \psi_{lk,i} \} \) during the two combination steps. The \( M \times M \) covariance matrices of these noises are given by:
\[
R^{(w)}_{v,k} = \sum_{l \in N_k \setminus \{k\}} c_{1,lk}^2 \beta^2 R^{(w)}_{v,lk}
\]
\[
(28)
\]
\[
R^{(w)}_{v,k} = \sum_{l \in N_k \setminus \{k\}} c_{2,lk}^2 \beta^2 R^{(w)}_{v,lk}
\]
\[
(29)
\]
C. Entry Selection Methods

To select \( L \)-subset of a set on \( M \) elements containing exactly \( L \) elements, we employ a similar approach proposed in [21]. Doing so, there exist two different scheme named sequential and stochastic partial-diffusion. These methods are analogous to the selection processes in sequential and stochastic partial-update schemes [26], [28]–[30]. In sequential partial-diffusion the entry selection matrices, \( K_{k,i} \), is diagonal matrix:

\[
K_{k,i} = \begin{bmatrix}
\kappa_{1,i} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \kappa_{M,i}
\end{bmatrix}, \quad \kappa_{\ell,i} = \begin{cases} 1 & \text{if } \ell \in \mathcal{J}_{i} \mod B + 1 \\ 0 & \text{otherwise} \end{cases}
\]

(30)

with \( B = \lceil M/L \rceil \). The number of selection entries at each iteration is limited by \( L \). The coefficient subsets \( \mathcal{J}_i \) are not unique as long as they meet the following requirements [26]:

1) Cardinality of \( \mathcal{J}_i \) is between 1 and \( L \);

2) \( \bigcup_{\tau=1}^{B} \mathcal{J}_\tau = \mathcal{S} \) where \( \mathcal{S} = \{1, 2, \ldots, M\} \);

3) \( \mathcal{J}_\tau \cap \mathcal{J}_\eta = \emptyset \) for \( \forall \tau, \eta \in \{1, \ldots, B\} \) and \( \tau \neq \eta \).

The description of entry selection matrices, \( K_{k,i} \), in stochastic partial-diffusion approach is similar to that of sequential one. The only difference is as follows. At a given iteration \( i \), the sequential case one of the set \( \mathcal{J}_\tau \) is chosen in a predetermined fashion, whereas for stochastic case, one of the sets \( \mathcal{J}_\tau \) is sampled at random from \( \{ \mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_B \} \). One might ask why these methods are considered to organize mentioned selection matrices. To answer this question, it is worth mentioning that the nodes need to know which entries of their neighbors’ intermediate estimates have been transmitted at each iteration. These schemes are not subject to such requirements.

III. PERFORMANCE ANALYSIS

We now move on to examine the behavior of the general PDLMS implementations (23)-(25), and the influence of the mentioned perturbations on convergence and steady-state performance. For this reason, we shall study the convergence of the weight estimates both in the mean and mean-square senses.

Assumptions. In order to make the analysis tractable, we consider the following assumptions on statistical properties of the measurement data and noise signals.

(i) The regression data \( u_{k,i} \) are temporally white and spatially independent random variables with zero mean and covariance matrix \( R_{u,k} \triangleq \mathbb{E} \left[ u_{k,i}^* u_{k,i} \right] \geq 0 \) where \( k = \{1, \ldots, N\} \).

(ii) The noise signals \( u_{k,i} \) and \( v_{k,i} \) are temporally white and spatially independent random variables with zero mean and co-variance \( \sigma_v^2, \sigma_{u}^2, R_v, R_u \) respectively. In addition, the quantities \( \{ R_v, R_u \} \) are all zero if \( l \in N_k \) or when \( l = k \).

(iii) The regression data \( \{ u_{m,i} \} \), the model noise signal \( v_{n} \), and the link noise signals \( v_{l_1,k_{1,j_1}} \) and \( v_{l_2,k_{2,j_2}} \) are mutually independent random variables for all indexes \( \{ i_1, i_2, j_1, j_2, k_{1,2}, l_{1,2}, m, n \} \).

(iv) The step-sizes, \( \mu_k \), \( \forall k \), are small enough such that their squared values are negligible.

We are interested in examining the evolution of the weight-error vectors. To do so, we introduce the error vectors:

\[
\hat{\phi}_{k,i} \triangleq w^{\circ} - \phi_{k,i} \quad (31)
\]

\[
\hat{\psi}_{k,i} \triangleq w^{\circ} - \psi_{k,i} \quad (32)
\]

\[
\hat{w}_{k,i} \triangleq w^{\circ} - w_{k,i} \quad (33)
\]

Substituting the linear model (1) into adaptation step (24) and subtraction of both sides from \( w^{\circ} \) give:

\[
\hat{\psi}_{k,i} = (I_M - \mu_k u_{k,i}^* u_{k,i}) \hat{\phi}_{k,i-1} - \mu_k u_{k,i}^* v_{k,i} \quad (34)
\]

Using conditions (6), we can rewrite (23) and (25) as

\[
\Phi_{k,i-1} = \left( I_M - \sum_{l \in N_k \setminus \{k\}} c_{1,lk} K_{l,i-1} \right) \Phi_{k,i-1} + \sum_{l \in N_k \setminus \{k\}} c_{1,lk} K_{l,i-1} \Phi_{l,i-1} \quad (35)
\]

\[
w_{k,i} = \left( I_M - \sum_{l \in N_k \setminus \{k\}} c_{2,lk} K_{l,i} \right) \psi_{k,i} + \sum_{l \in N_k \setminus \{k\}} c_{2,lk} K_{l,i} \psi_{l,i} \quad (36)
\]

Subtracting (35) from

\[
w^{\circ} = \left( I_M - \sum_{l \in N_k \setminus \{k\}} c_{1,lk} K_{l,i-1} \right) w^{\circ} + \sum_{l \in N_k \setminus \{k\}} c_{1,lk} K_{l,i-1} w^{\circ} \quad (37)
\]

and (36) from

\[
w^{\circ} = \left( I_M - \sum_{l \in N_k \setminus \{k\}} c_{2,lk} K_{l,i} \right) w^{\circ} + \sum_{l \in N_k \setminus \{k\}} c_{2,lk} K_{l,i} w^{\circ} \quad (38)
\]

gives

\[
\hat{\phi}_{k,i-1} = \left( I_M - \sum_{l \in N_k \setminus \{k\}} c_{1,lk} K_{l,i-1} \right) \hat{w}_{k,i-1} + \sum_{l \in N_k \setminus \{k\}} c_{1,lk} K_{l,i-1} \hat{w}_{l,i-1} - \hat{v}_{k,i-1} \quad (39)
\]
\[ \tilde{w}_{k,i} = \left( I_M - \sum_{l \in \mathcal{N}_k \backslash \{k\}} c_{2,l,i} K_{l,i} \right) \tilde{\psi}_{k,i} + \sum_{l \in \mathcal{N}_k \backslash \{k\}} c_{2,l,k} K_{l,i} \tilde{\psi}_{l,i} - v^{(w)}_{k,i} \]  

(40)

To describe these relations in more closed form, we collect the information from across the network into block vectors and matrices. Stacking the error vectors from across all nodes into the following \( N \times 1 \) block vectors, whose individual entries are of size \( M \times 1 \) each, we have:

\[ \tilde{\phi}_i = \text{col} \{ \tilde{\phi}_{1,i}, \ldots, \tilde{\phi}_{N,i} \} \]  

(41)

\[ \tilde{\psi}_i = \text{col} \{ \tilde{\psi}_{1,i}, \ldots, \tilde{\psi}_{N,i} \} \]  

(42)

\[ \tilde{\omega}_i = \text{col} \{ \tilde{\omega}_{1,i}, \ldots, \tilde{\omega}_{N,i} \} \]  

(43)

Also, collecting the noise signal (26) and (27), and their covariances from across the network into \( N \times 1 \) block vectors and \( N \times N \) block diagonal matrices as follows:

\[ \psi^{(w)}_i = \text{col} \{ \psi^{(w)}_{1,i}, \ldots, \psi^{(w)}_{N,i} \} \]  

(44)

\[ \psi^{(v)}_i = \text{col} \{ \psi^{(v)}_{1,i}, \ldots, \psi^{(v)}_{N,i} \} \]  

(45)

\[ R^{(w)}_i = \text{col} \{ R^{(w)}_{1,1}, \ldots, R^{(w)}_{N,N} \} \]  

(46)

\[ R^{(v)}_i = \text{col} \{ R^{(v)}_{1,1}, \ldots, R^{(v)}_{N,N} \} \]  

(47)

Subsequently, we can verify that

\[ \tilde{\phi}_{i-1} = \mathcal{A}_{1,i-1} \tilde{\omega}_{i-1} - \psi^{(w)}_{i-1} \]  

(48)

\[ \tilde{\psi}_i = (I_{NM} - M \mathcal{R}_{\mathcal{U},i}) \mathcal{M} s_i \]  

(49)

\[ \tilde{\omega}_i = \mathcal{A}_{2,i} \tilde{\psi}_{i-1} - v^{(v)}_i \]  

(50)

where

\[ M \triangleq \text{diag} \{ \mu_1 M_1, \ldots, \mu_N M_N \} \]  

(51)

\[ \mathcal{R}_{\mathcal{U},i} \triangleq \text{diag} \{ R_{u,1}, \ldots, R_{u,\mathcal{U}} \} \]  

(52)

and

\[ \mathcal{M} = \text{diag} \{ \mu_1 I_M, \ldots, \mu_N I_M \} \]  

(53)

\[ s_i \triangleq \text{diag} \{ u^*_{1,i}, v_{1,i}, \ldots, u^*_{N,i}, v_{N,i} \} \]  

(54)

Here, \( s_i \) denotes \( N \times 1 \) block column vector, whose entries are of size \( M \times 1 \) each. Following Assumption (i), we have

\[ \mathbb{E} [s_i] = 0 \]  

(55)

The covariance matrix of \( s_i \) is \( N \times N \) block diagonal with blocks of size \( M \times M \):

\[ S = \mathbb{E} [s_i s_i^*] = \text{diag} \{ \sigma^2_{v,1} R_{u,1}, \ldots, \sigma^2_{v,N} R_{u,N} \} \]  

(56)

\( I_{NM} \) is also the identity matrix of size \( MN \times MN \). Moreover,

\[ \mathcal{A}_{r,i} = \begin{bmatrix} A_{1,1,i} & \cdots & A_{1,N,i} \\ \vdots & \ddots & \vdots \\ A_{N,1,i} & \cdots & A_{N,N,i} \end{bmatrix} \]  

\( \forall r \in \{1, 2\} \)  

(57)

where

\[ A_{p,q,i} = \begin{cases} I_M - \sum_{l \in \mathcal{N}_p \backslash \{p\}} c_{r,l,p} K_{l,i} & \text{if } p = q \\ c_{r,q,p} K_{q,i} & \text{if } q \in \mathcal{N}_p \backslash \{p\} \\ 0_M & \text{otherwise} \end{cases} \]  

(58)

So that the network weight error vector, \( \tilde{\omega}_i \), evolves according to the following stochastic recursion:

\[ \tilde{\omega}_i = \mathcal{A}_{2,i} (I_{NM} - M \mathcal{R}_{\mathcal{U},i}) \mathcal{A}_{1,i-1} \tilde{\omega}_{i-1} - \mathcal{A}_{2,i} (I_{NM} - M \mathcal{R}_{\mathcal{U},i}) \tilde{\psi}_i - \mathcal{A}_{2,i} \mathcal{M} s_i - v^{(v)}_i \]  

(59)

A. Convergence in Mean

Taking expectation of both sides of (59) under Remark and Assumptions, we find that the mean error vector evolves according to the following recursion:

\[ \mathbb{E} [\tilde{\omega}_i] = Q_2 (I_{NM} - M \mathcal{R}_u) Q_1 \mathbb{E} [\tilde{\omega}_{i-1}] \]  

(60)

where

\[ Q_1 = \mathbb{E} [\mathcal{A}_{1,i-1}] \]  

(61)

\[ Q_2 = \mathbb{E} [\mathcal{A}_{2,i}] \]  

(62)

Like [21], \( Q_r, r \in \{1, 2\} \) can be obtained for both stochastic and sequential partial-diffusion using the definition of \( \mathcal{A}_{r,i}, r \in \{1, 2\} \), see (61) and (62). What is most noteworthy here is to find the value of each \( Q_r, r \in \{1, 2\} \) entries after applying expectation operator. Therefore, we can write

\[ \mathbb{E} [A_{p,q,i}] = \begin{cases} (1 - \rho + \rho c_{r,l,p}) I_M & \text{if } p = q \\ \rho c_{r,q,p} I_M & \text{if } q \in \mathcal{N}_p \backslash \{p\} \\ 0_M & \text{otherwise} \end{cases} \]  

(63)

All the entries of \( Q_r, r \in \{1, 2\} \) are real and non-negative and all the rows of \( Q_r, r \in \{1, 2\} \) add up to unity. This property [21] can be established for both stochastic and sequential partial-diffusion schemes and for any value of \( L \).

**Theorem 1 (Convergence in Mean).** Consider the problem of optimizing the global cost (2), Pick \( Q_1 \) and \( Q_2 \) with are real non-negative entries and all their rows add up to unity. Assume each node in the network measures data that satisfy conditions described in Assumptions, and run adaptive diffusion algorithm (23)-(25). Assume further that the exchange of the variables \( \{w_{i-1}, \psi_{i,1}\} \) is subject to additive noise as (7) and (8). Moreover, the regressors and desired signals are assumed not to exchange among the nodes. Then, all estimates \( \{w_{i,1}\} \) across the network converge in the mean to optimal solution \( w^* \) if the step-size parameters \( \{\mu_k\} \) satisfy

\[ 0 < \mu_k < \frac{2}{\lambda_{\max}} \{R_{u,k}\}, \forall k \]  

(64)

**Proof:** The weight error vectors \( \{\tilde{\omega}_i\} \) converge to zero if, and only if, the matrix \( Q_2 (I_{NM} - M \mathcal{R}_u) Q_1 \) in (60) is a stable matrix. Matrix stability means that all its eigenvalues should lie inside the unit circle. From the established statement on [21], all the entries of \( Q_1 \) and \( Q_2 \) are real non-negative and all the rows of \( Q_1 \) and \( Q_2 \) add up to unity, we know that
$Q_2 \left( I_{NM} - M R_u \right) Q_1$ is stable if the matrix \((I_{NM} - M R_u)\). or
\[
|\lambda_{max} \{ I_{NM} - M R_u \}| < 1 \tag{65}
\]
It is known simple and easy to understand confirm that condition (65) ensures the stability of \((I_{NM} - M R_u)\)

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where
\[ \mathcal{U} \triangleq \mathbb{E} \left[ I_{NM} - \mathcal{M} \mathcal{R}_{u,i} \right] = \left( I_{NM} - \mathcal{M} \mathcal{R}_u \right) \] (86)

Last term on RHS of (69):
\[ \mathbb{E} \left[ \mathbf{v}_i^{(\psi)} \Sigma^{\psi} \mathbf{v}_i^{(\psi)} \right] = \mathbb{E} \left[ \text{tr} \left( \Sigma^{\psi} \mathbf{v}_i^{(\psi)} \right) \right] \]
\[ = \text{vec}^T \left\{ \mathbf{R}_u^{(\psi)} \right\} \sigma \] (87)

The variance relation becomes
\[ \mathbb{E} \left[ \| \mathbf{w}_i \|^2 \right] = \mathbb{E} \left[ \| \mathbf{w}_i \|_{F, \sigma}^2 \right] \]
\[ + \left( \text{vec}^T \{ \mathcal{G} \} \mathcal{D}_2 + \text{vec}^T \{ \mathcal{U} R_u^{(\psi)} \mathcal{U}^* \} \mathcal{D}_2 \right. \]
\[ + \left. \text{vec}^T \{ \mathbf{R}_u^{(\psi)} \} \right) \sigma \] (88)

**Theorem 2** (Mean-Square Stability). Consider the same setting of Theorem 1. Assume sufficiently small step-sized to justify ignoring terms that depend on higher power of the step-sizes. The perturbed adaptive partial diffusion algorithm (23)-(25) is mean-square stable if, and only if, the matrix \( \mathcal{F} \) defined by (81), or its approximate defines further (89), is stable. This condition is satisfied small for step-sizes \( \mu_k \) as is (64).

**Proof:** A resonable approximat expression for \( \mathcal{F} \) for sufficiently small step-sizes is
\[ \mathcal{F} \approx \mathcal{D}_1 \left[ \left( I_{NM} - \mathcal{M} \mathcal{R}_u \right)^T \otimes \left( I_{NM} - \mathcal{M} \mathcal{R}_u \right)^T \right] \mathcal{D}_2 \] (89)

Recall that, in the Kronecker product case \( C = B \otimes A \), eigenvalues are the outer product of the eigenvalues of the two matrices. Therefore, using expression (89), we have that \( \zeta (\mathcal{F}) = \zeta (I_{NM} - \mathcal{M} \mathcal{R}_u) \), where \( \zeta (A) \) is the spectral radius \( A \). It follows that \( \mathcal{F} \) is stable if, and only if, \( (I_{NM} - \mathcal{M} \mathcal{R}_u) \) is stable. we already mentioned in (1) that (65) ensures the stability of \( (I_{NM} - \mathcal{M} \mathcal{R}_u) \).

**Corollary** (Steady-State Variance Relation). Consider the same setting of Theorem 2. The weight-error vector, \( \mathbf{w}_i \), of the (PDLMS) (23)-(25) satisfies the following equation in steady-state:

\[ \mathbb{E} \left[ \| \mathbf{w}_i \| \right]_{(I_{NM}^2 - \mathcal{F})} = \left( \text{vec}^T \{ \mathcal{G} \} \mathcal{D}_2 + \text{vec}^T \{ \mathcal{U} R_u^{(\psi)} \mathcal{U}^* \} \mathcal{D}_2 \right. \]
\[ + \left. \text{vec}^T \{ \mathbf{R}_u^{(\psi)} \} \right) \sigma \] (90)

for any Hermitian nonnegative-definite matrix \( \Sigma \) which follows (76).

**C. Mean-Square Performance**

Expression (90) prove a very useful relation; it allows us to evaluate the network MSD through appropriate selection of the weighting matrix \( \Sigma \). The network MSD is defined as the average value:

\[ \text{MSD}_{\text{network}} \triangleq \lim_{i \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{E} \left[ \| \mathbf{w}_{k,i} \|^2 \right] \] (91)

which amounts to averaging the MSDs of the individual nodes. Therefore,

\[ \text{MSD}_{\text{network}} = \lim_{i \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[ \| \mathbf{w}_i \|^2 \right] = \lim_{i \rightarrow \infty} \mathbb{E} \left[ \| \mathbf{w}_{i} \|^2 \right]_{1/N} \] (92)

This means that in order to recover the network MSD from (90), we should select the weighting vector \( \sigma \) such that
\[ (I_{NM}^2 - \mathcal{F}) \sigma = \frac{1}{N} \text{vec} \left\{ I_{NM} \right\} \] (93)

Solving for \( \sigma \) and substituting back into (90) we arrive at the following expression for the network MSD

\[ \text{MSD}_{\text{network}} = \frac{1}{N} \left( \text{vec}^T \{ \mathcal{G} \} \mathcal{D}_2 + \text{vec}^T \{ \mathcal{U} R_u^{(\psi)} \mathcal{U}^* \} \mathcal{D}_2 \right. \]
\[ + \left. \text{vec}^T \{ \mathbf{R}_u^{(\psi)} \} \right) \times (I_{NM}^2 - \mathcal{F})^{-1} \text{vec} \left\{ I_{NM} \right\} \] (94)

When links are ideal, the last two terms of (90) do not arise. So, we can conclude that the network MSD deteriorates as follows:

\[ \text{MSD}_{\text{network}} = \text{MSD}_{\text{network}, \text{ideal}} + \frac{1}{N} \left( \text{vec}^T \{ \mathcal{U} R_u^{(\psi)} \mathcal{U}^* \} \mathcal{D}_2 + \text{vec}^T \{ \mathbf{R}_u^{(\psi)} \} \right) \times \]
\[ (I_{NM}^2 - \mathcal{F})^{-1} \text{vec} \left\{ I_{NM} \right\} \] (95)

**IV. DETAILED DISCUSSION ON THE NETWORK MSD**

So far we have mentioned based on Theorems 1 on 2 that the partial diffusion LMS strategy does not diverge due to noisy links. But, it is the main factor on performance degradation of steady-state network MSD. Moreover, focus on (95) and compare it with that stated at [21], there exist an additional term, denoted as channel noise term, that plays a crucial rule on the performance degradation of network MSD performance. It is abundantly clear that this term has been arose from channel noise condition. Here we concentrate on (95) to explicitly highlight characterization of how convergence and performance of PDLMS affects by the presence of noisy channels. We analyze steady-state network MSD under the following assumptions:

**Assumptions.**

(v) Nodes run ATC PDLMS at each iteration, i.e., \( C = I_N \).

For the sake of simplicity of notation we consider, \( C = C \) and \( D = D \).

(vi) The step size, noise variance, input covariance matrix and channel noise covariance matrix all are the same in the network, i.e., \( \mu_k = \mu, \sigma^2_{v,k} = \sigma^2_v, R_{u,k} = R_u, \) and \( R_{v,k} = R_v \), \( \forall k \in 1, \ldots, N \).

(vii) During any \( M \) consecutive iterations, the intermediate estimate vector does not differ considerably at each node \( k \) [21].

Considering Assumptions (v), (vii), and using the results of analysis mentioned in [21], we have

\[ D = (1 - \rho) I_{NM}^2 + \rho C \otimes C \] (96)

where, \( C = C \otimes I_M \).
To specify the network performance in steady-state, we consider the global MSD describe at (95) and denote it by \( \eta_L \). We have
\[
\eta_L = \frac{1}{N} \left( \text{vec}^T \{ G \} D + \text{vec}^T \{ R_v^{(\psi)} \} \right) \left( \sum_{n=0}^{\infty} F_n \right) \text{vec} \{ I_{NM} \}
\]
\[
= \frac{1}{N} \sum_{n=0}^{\infty} \left( \text{vec}^T \{ G \} D + \text{vec}^T \{ R_v^{(\psi)} \} \right) \times \left[ (U^T \otimes U^T) D \right]^n \text{vec} \{ I_{NM} \}
\]
Substituting (96) into (97) results in
\[
\eta_L = \frac{1}{N} \sum_{n=0}^{\infty} (1 - \rho)^{n+1} \text{vec}^T \{ G \} [(U)^n \otimes (U)^n]^T \times \text{vec} \{ I_{NM} \} + \frac{1}{N} \sum_{n=0}^{\infty} \rho^{n+1} \text{vec}^T \{ G \} \left[ C (U^T C)^n \otimes C (U^T C)^n \right] \times \text{vec} \{ I_{NM} \} + \frac{1}{N} \sum_{n=0}^{\infty} (1 - \rho)^n \text{vec}^T \{ R_v^{(\psi)} \} [(U)^n \otimes (U)^n]^T \times \text{vec} \{ I_{NM} \} + \frac{1}{N} \sum_{n=0}^{\infty} \rho^n \text{vec}^T \{ R_v^{(\psi)} \} \left[ C (U^T C)^n \otimes C (U^T C)^n \right] \times \text{vec} \{ I_{NM} \}
\]
Utilizing \( \text{vec} \{ \cdot \} \) property, (98) can be rewrite as
\[
\eta_L = \frac{1}{N} \sum_{n=0}^{\infty} (1 - \rho)^{n+1} \text{vec}^T \left\{ [(U)^n G^T (U)^n]^T \right\} \times \text{vec} \{ I_{NM} \} + \frac{1}{N} \sum_{n=0}^{\infty} \rho^{n+1} \text{vec}^T \left\{ C (U^T C)^n \otimes C (U^T C)^n \right\} \times \text{vec} \{ I_{NM} \} + \frac{1}{N} \sum_{n=0}^{\infty} (1 - \rho)^n \text{vec}^T \left\{ (U)^n \left( R_v^{(\psi)} \right)^T (U)^n \right\} \times \text{vec} \{ I_{NM} \} + \frac{1}{N} \sum_{n=0}^{\infty} \rho^n \text{vec}^T \left\{ C (U^T C)^n R_v^{(\psi)} C (U^T C)^n \right\} \times \text{vec} \{ I_{NM} \}
\]
Using trace properties, we have
\[
\eta_L = \frac{1}{N} \sum_{n=0}^{\infty} (1 - \rho)^{n+1} \text{tr} \left\{ [(U)^n G^T (U)^n]^T \right\} + \frac{1}{N} \sum_{n=0}^{\infty} \rho^{n+1} \text{tr} \left\{ C (U^T C)^n \otimes C (U^T C)^n \right\} + \frac{1}{N} \sum_{n=0}^{\infty} (1 - \rho)^n \text{tr} \left\{ (U)^n \left( R_v^{(\psi)} \right)^T (U)^n \right\} + \frac{1}{N} \sum_{n=0}^{\infty} \rho^n \text{tr} \left\{ C (U^T C)^n R_v^{(\psi)} C (U^T C)^n \right\}
\]
In view of Assumption (v), we have
\[
\begin{align*}
\mathcal{M} &= \mu I_N \otimes I_M, \\
\mathcal{U} &= I_N \otimes (I_M - \mu R_u), \\
\mathcal{G} &= \mu^2 \sigma^2 I_M \otimes R_v
\end{align*}
\]
and
\[
R_v^{(\psi)} = R_v
\]
Calculating the trace terms at (100), we get
\[
\text{tr} \left\{ \left[ (U)^n G^T (U)^n \right]^T \right\} = N \mu^2 \sigma^2
\]
\[
\text{tr} \left\{ (U)^n \left( R_v^{(\psi)} \right)^T (U)^n \right\} = N \times \text{tr} \left\{ \left[ (I_M - \mu R_u)^n R_v (I_M - \mu R_u)^n \right]^T \right\}
\]
\[
\begin{align*}
\text{tr} \left\{ \left[ C (U^T C)^n \otimes C (U^T C)^n \right] \right\} &= \mu^2 \sigma^2 \text{tr} \left\{ (C^T)^{n+1} C^{n+1} \right\} \\
\text{tr} \left\{ (C^T)^{n+1} C^{n+1} \right\} &= \text{tr} \left\{ (C^T)^{n+1} R_v (C^T)^{n+1} \right\} = \sum_{k=1}^{N} \| c_{n+1,k} \|^2, \ n \geq 0
\end{align*}
\]
In equation above, \( c_{n+1,k} \) is the \( k \)-th column of \( C^{n+1} \). Considering a connected network holds that,
\[
\| c_{n+1,k} \|^2 < 1, \ n \geq 0, \forall k.
\]
This fact yields
\[
\begin{align*}
\text{tr} \left\{ C (U^T C)^n \otimes C (U^T C)^n \right\} < \text{tr} \left\{ (U)^n G^T (U)^n \right\} \\
< \text{tr} \left\{ C (U^T C)^n \otimes C (U^T C)^n \right\} + \text{tr} \left\{ C (U^T C)^n R_v^{(\psi)} C (U^T C)^n \right\}
\end{align*}
\]
(101)
Moreover, for the case of non-cooperative, i.e. \( L = 0 \) and \( \rho = 0 \), full diffusion case with ideal links, i.e. \( L = M, \rho = 1 \), and \( R_v = O_M \), full diffusion case with noisy links, i.e. \( L = M, \rho = 1 \) and \( R_v \neq O_M \), where \( O_M \) is and \( M \times M \) zero matrix, we have
\[
\eta_0 = \frac{1}{N} \sum_{n=0}^{\infty} \text{tr} \left\{ [(U)^n G^T (U)^n]^T \right\}
\]
\[
\eta_M = \frac{1}{N} \sum_{n=0}^{\infty} \text{tr} \left\{ C (U^T C)^n \otimes C (U^T C)^n \right\}
\]
In the first simulation, we evaluate the theoretical derivations. To this end we consider the experimental network MSD learning curves (ATC strategy) of PDLMS algorithm and theoretical results using both sequential and stochastic partial diffusion schemes under noisy links for different numbers of entries, \( L \). We use uniform weights for \( \{e_{1,k}, e_{2,k}\} \) at the combination phase at this stage. The plots are given in Fig. 3 where we can see that there is a good match between our theoretical derivations with simulation results. Similar plots...
The sequential partial-diffusion schemes outperform the
Unlike the statement above that labels a communication-
The PDLMS algorithm delivers a trade-off between com-
The ATC PDLMS strategy outperforms the adaptive CTA

for CTA strategy are given in Fig. 4.
To further examine our theoretical findings, both theoretical
and experimental steady-state network MSD of the ATC
PDLMS algorithm as a function of communicated entries, \( L \),
for different values of \( \{\sigma_{\psi,tk}^2\} \), for stochastic and sequential
schemes are plotted in Fig. 5. This figure not only supports
our analysis, but it also reveals that when channel between
agents is assumed ideal (\( \sigma_{\psi,tk}^2 = 0 \) in the figure), an increase
in communicated entries results in the network performance.
It must be noted that this is not the case when links among
the nodes are noisy and the performance of network is
deply affected by variance of channel noise. This particular
behaviour of the PDLMS algorithm in the presence of noisy
links is better understood from Fig. 6 where the steady-state
MSDs of all node \( k \) for different values of \( L \) and different
link conditions are plotted in Fig. 6.

It is also notable that in the presence of noisy links, the
PDLMS algorithm exhibits different behaviour as the step size
changes. To show this behaviour, the steady-state MSD
as a function of \( \mu \) for different values of \( \sigma_{\psi,tk}^2 \) is shown in
Fig. 7. As it is obvious from Fig. 7, for the case of ideal
links \( \sigma_{\psi,tk}^2 = 0 \) the MSD curve is a monotonically increasing
function of \( \mu \) [14], whereas, for noisy links, decreasing the
step size increases the steady-state MSD value. Also, we can
see from Fig. 7 that as \( \sigma_{\psi,tk}^2 \) increases, the effect of noisy
links increases as expected. Finally, it must be noted that
although the performance of PDLMS algorithm deteriorates
in the presence of noisy links, it is still able to provide to
deliver better performance in comparison with some similar
methods, such as a consensus based algorithm. To show this,
the MSD performance for different algorithms including non-
cooperative, consensus, full diffusion and partial-diffusion (for
\( L = 2 \) and \( L = 4 \)) under noisy links is illustrated in Fig.
8. We can observe that the DLMS algorithm exhibits better
performance than the consensus algorithm.

From the results above, we can make the following observations:

- The PDLMS algorithm delivers a trade-off between com-
  munications cost and estimation performance under ideal
  link. This statement is the main aim of [21].
- Unlike the statement above that labels a communication-
  performance trade-off to PDLMS algorithm, there is no
direct relation between MSD performance and number
of selected entries under noisy information exchange.
  In other words, the more entries are communicated at
each iteration, the more perturbed weight estimates are
interred in consultant phase that leads to worse steady-
state network MSD.
- The sequential partial-diffusion schemes outperform the
  stochastic partial-diffusion for noisy and ideal links.
- The ATC PDLMS strategy outperforms the adaptive CTA
  PDLMS strategy for both noisy and ideal cases.
Fig. 6. Theoretical and experimental steady-state MSDs at each node for different numbers of entries communicated at each iteration under ideal links (top) and noisy links (bottom). Note that solid line and dashed line represent the theoretical and experimental results respectively.

Fig. 7. The MSD of ATC-PDLMS as a function of $\mu$ for different values of $\sigma^2_{\psi,ik}$.

VI. CONCLUSION

In this work, we present a general form of PDLMS algorithms, formulate the ATC and CTA version of PDLMS under noisy links condition, and investigate the performance of partial-diffusion algorithms under several sources of noise during information exchange for both sequential and stochastic cases. We also illustrate that the PDLMS strategy can still stabilize the mean and mean-square convergence of the network with non-ideal links. We derived analytical expressions for network learning curve MSD. Furthermore, we established that there is not a direct relation between the MSD performance and the number of selected entries under imperfect information exchange. In other words, the more entries are communicated at each iteration, the more perturbed weight estimates are interred in consultant phase. The simulation results verify the theoretical findings and how well they match with theory.

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