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Scattering of bulk strain solitary waves in bi-layers with delamination

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Abstract

We study the scattering of longitudinal bulk strain solitary waves in delaminated bi-layers with different types of bonding. The direct numerical modelling of these problems is challenging and has natural limitations. We develop a semi-analytical approach, based on the use of several matched asymptotic multiple-scale expansions and the Integrability Theory of the Korteweg - de Vries equation by the Inverse Scattering Transform. We show that the semi-analytical approach agrees well with the direct numerical simulations and use it to study the scattering of different types of longitudinal bulk strain solitary waves in a wide range of bi-layers with delamination. In particular, we model the dynamics of a long longitudinal strain solitary wave in a symmetric perfectly bonded bi-layer with delamination. The numerical modelling confirms that delamination causes fission of an incident solitary wave and, thus, can be used to detect the defect. We then extend our approaches to the modelling of the waves in bi-layers with soft ("imperfect") bonding, described by a system of coupled Boussinesq equations and supporting radiating solitary waves. The results may help us to control the integrity of layered structures.

Keywords: Solitary waves; delamination; scattering

1. Introduction

In 2017 the world celebrates 50 years of the discovery of the Inverse Scattering Transform (IST) for the Korteweg-de Vries (KdV) equation - the method for the solution of a large class of initial-value problems on the infinite line [1]. This sophisticated and powerful technique is intrinsically linked with another fundamental discovery of the 20th century - the discovery of solitons as extremely stable localised nonlinear waves [2]. Solitons, when present, constitute the main part of the asymptotics of initial-value problems for localised initial data, and this is the reason why they are ubiquitous in the real world, across all scales.

Recently, the method has found a new application in our studies of the scattering of pure and radiating bulk strain solitons in layered bars with delamination [3–6]. The exceptional stability of bulk strain solitons makes them an attractive candidate as an additional tool for the introscopy of layered structures (see [7,8] and references therein).

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The dynamical behaviour of layered structures depends both on the properties of the bulk material and on the type of the bonding between the layers. In particular, if the material of the layers have similar properties and the bonding between the layers is sufficiently soft ("imperfect bonding"), then the bulk strain soliton is replaced with a *radiating solitary wave*, a solitary wave with a co-propagating oscillatory tail [9]. Experimental observations of the pure and radiating bulk strain solitons in layered bars have been reported in [7].

In [9] it was shown that nonlinear longitudinal bulk strain waves in a bi-layer with a sufficiently soft bonding can be modelled by a system of coupled regularised Boussinesq (cRB) equations (in the scaled, nondimensional form):

$$f_{tt} - f_{xx} = \frac{1}{2}(f^2)_{xx} + f_{ttxx} - \delta(f - g),$$
$$g_{tt} - c^2 g_{xx} = \frac{1}{2}\alpha(g^2)_{xx} + \beta g_{ttxx} + \gamma(f - g). \tag{1}$$

Here, f and g denote the longitudinal strains in the layers, while the coefficients $c, \alpha, \beta, \delta, \gamma$ are defined by the physical and geometrical parameters of the problem (see [9] for details).

In the symmetric case ($c = \alpha = \beta = 1$) system (1) admits the reduction $g = f$, where f satisfies the Boussinesq equation

$$f_{tt} - f_{xx} = \frac{1}{2}(f^2)_{xx} + f_{ttxx}, \tag{2}$$

which has particular solitary wave solutions:

$$f = 3(v^2 - 1)\operatorname{sech}^2 \frac{x - vt}{\Lambda}, \quad \Lambda = \frac{2v}{\sqrt{v^2 - 1}}, \tag{3}$$

where v is the speed of the wave. In the cRB system of equations (1), when the characteristic speeds of the linear waves in the layers are close (i.e. c is close to 1), these pure solitary wave solutions are replaced with radiating solitary waves (see [9,10]), that is a solitary wave radiating a co-propagating one-sided oscillatory tail (see, for example, [11,12]). The radiating solitary waves emerge due to a resonance between a soliton and a harmonic wave, which can be deduced from the analysis of the linear dispersion relation of the problem.

The aim of this paper is to model the scattering of pure and radiating solitary waves in delaminated areas of bonded layered structures. We use the direct and semi-analytical numerical approaches developed in [5,6]. The semi-analytical approach is based on the use of several matched asymptotic multiple-scale expansions and the IST. Our modelling confirms that the semi-analytical method produces correct results in the cases where we can use both methods. Thus, one can use the semi-analytical method to study the scattering of solitary waves in a wide range of bonded bi-layers with delamination instead of the expensive direct method.

2. Scattering of pure and radiating solitary waves in bi-layers with delamination

2.1. Perfectly bonded bi-layer: scattering of a pure solitary wave

In this section we model the scattering of a long longitudinal strain solitary wave in a perfectly bonded two-layered bar with delamination at $x > 0$ (see Figure 1). The material of the layers is assumed to be the same (symmetric bar), while the material to the left and to the right of the $x = 0$ cross-section can be different. This problem represents the propagation of a strain wave in a bar with perfect bonding between the two layers for the first boundary-value problem, and a bar with complete delamination in the second boundary-value problem, described by the following set of scaled, nondimensional equations [3]

$$u_{tt}^- - u_{xx}^- = \epsilon\left[-12u_x^- u_{xx}^- + 2u_{ttxx}^-\right], \quad x < 0,$$
$$u_{tt}^+ - c^2 u_{xx}^+ = \epsilon\left[-12\alpha u_x^+ u_{xx}^+ + 2\frac{\beta}{c^2}u_{ttxx}^+\right], \quad x > 0, \tag{4}$$

with associated continuity conditions

$$u^-|_{x=0} = u^+|_{x=0}, \qquad (5)$$

$$u_x^- + \epsilon \left[-6\left(u_x^-\right)^2 + 2u_{ttx}^- \right]\Big|_{x=0} = c^2 u_x^+ + \epsilon \left[-6\alpha\left(u_x^+\right)^2 + 2\frac{\beta}{c^2}u_{ttx}^+ \right]\Big|_{x=0}, \qquad (6)$$

and appropriate initial and boundary conditions. Here, c, α, β are constants defined by the geometrical and physical parameters of the structure, while ϵ is the small wave amplitude parameter. The functions $u^-(x,t)$ and $u^+(x,t)$ describe displacements in the bonded and delaminated areas of the structure respectively. Condition (5) is continuity of longitudinal displacement, while condition (6) is the continuity of stress. In what follows we assume that $\alpha = 1$ and $\beta = \frac{n^2+k^2}{n^2(1+k^2)}$, where n represents the number of layers in the structure and k is defined by the geometry of the waveguide (see [3]). The cross-section $x = 0$ has width $2a$ and the height of each layer is b. In terms of these values $k = b/a$ and, as there are two layers in this example, $n = 2$.

In [5] we proposed two numerical schemes for solving two boundary-value problems matched at $x = 0$. Following [3], we looked for a weakly nonlinear solution by taking a multiple-scales expansion in terms of the appropriate set of fast and slow variables. This produced three KdV equations satisfying initial-value problems, describing the leading order incident, reflected and transmitted waves. The initial values in these equations are described in terms of the leading order incident wave, with reflection and transmission coefficients being derived to describe these initial conditions. It was noted that a bar made of one and the same material will not generate a leading order reflected wave. In addition, expressions were found for the higher order corrections in terms of the leading order incident and reflected waves, and the reflection and transmission coefficients. The semi-analytical numerical approach is based on this weakly-nonlinear solution [5]. The direct method is based on the use of finite-difference techniques, which results in two tridiagonal systems and a nonlinear difference equation linking the systems [5]. These systems were solved, in terms of "ghost points" in both systems, and the result of this calculation was used to find the solution of the nonlinear equation for one of the ghost points. The solution for this ghost point is then substituted back into the implicit solution of the tridiagonal system to determine the solution at a given time value.

A typical scenario is shown in Figure 1 for an exact solitary wave initial condition, with parameters $\alpha = 1$, $\beta = 5/8$, $c = 1$, $\epsilon = 0.1$, with initial position $x = -200$ and initial speed $v = 1.04$. The results are presented for the strains $u_x^- = e^-$ and $u_x^+ = e^+$ at the initial moment of time and at $t = 1100$ respectively, when the wave is propagating in the delaminated section of the bar. The results are in good agreement, with a small phase shift for the lead soliton. Soliton fission occurs in the delaminated section of the bar, with two solitons generated from a single incident soliton [3] (see also [13,14] for the first studies of soliton fission). As the material in both sections of the bar is the same, there is no leading order reflected wave.
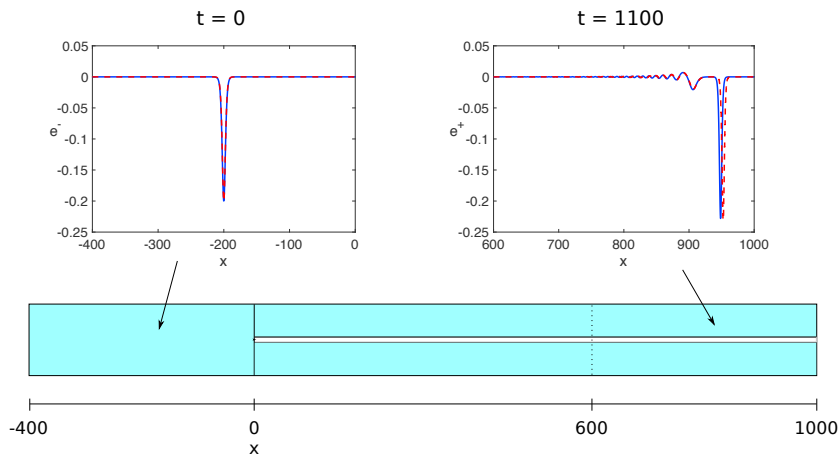

Fig. 1. The solution e^- and e^+ at the initial moment of time and at $t = 1100$, for the parameters $\alpha = 1$, $\beta = 5/8$, $c = 1$, $\epsilon = 0.1$, with initial position $x = -200$ and initial speed $v = 1.04$, for direct numerical simulations (solid line) and weakly-nonlinear solution (dashed line).

2.2. Imperfectly bonded bi-layer: scattering of a radiating solitary wave

In this section we model the generation and the scattering of a long radiating solitary wave in a two-layered imperfectly bonded bi-layer with delamination (see Figure 2 and Figure 3). Two identical homogeneous layers (the section on the left in Figure 2 and on the right in Figure 3) are "glued" to a two-layered structure with soft bonding between its layers, followed with a delaminated section in the middle, and another bonded section. The materials in the bi-layer are assumed to have close properties, leading to the generation of a radiating solitary wave in the bonded section. We model the scattering of this wave by the subsequent delaminated region, as well as the dynamics in the second bonded region.

The mathematical problem formulation consists of the following sets of scaled, nondimensional equations in the respective sections of the complex waveguide: [3,9]

$$
\begin{aligned}
u_{tt}^{(1)} - u_{xx}^{(1)} &= \epsilon\left[-12u_x^{(1)}u_{xx}^{(1)} + 2u_{ttxx}^{(1)}\right], \\
w_{tt}^{(1)} - w_{xx}^{(1)} &= \epsilon\left[-12w_x^{(1)}w_{xx}^{(1)} + 2w_{ttxx}^{(1)}\right]
\end{aligned}
\tag{7}
$$

for the section with two homogeneous layers,

$$
\begin{aligned}
u_{tt}^{(2,4)} - u_{xx}^{(2,4)} &= \epsilon\left[-12u_x^{(2,4)}u_{xx}^{(2,4)} + 2u_{ttxx}^{(2,4)} - \delta\left(u^{(2,4)} - w^{(2,4)}\right)\right], \\
w_{tt}^{(2,4)} - c^2 w_{xx}^{(2,4)} &= \epsilon\left[-12\alpha w_x^{(2,4)}w_{xx}^{(2,4)} + 2\beta w_{ttxx}^{(2,4)} + \gamma\left(u^{(2,4)} - w^{(2,4)}\right)\right]
\end{aligned}
\tag{8}
$$

for the two bonded regions, and

$$
\begin{aligned}
u_{tt}^{(3)} - u_{xx}^{(3)} &= \epsilon\left[-12u_x^{(3)}u_{xx}^{(3)} + 2u_{ttxx}^{(3)}\right], \\
w_{tt}^{(3)} - c^2 w_{xx}^{(3)} &= \epsilon\left[-12\alpha w_x^{(3)}w_{xx}^{(3)} + 2\beta w_{ttxx}^{(3)}\right]
\end{aligned}
\tag{9}
$$

for the delaminated region. Here, the functions $u^{(i)}(x,t)$ and $w^{(i)}(x,t)$ describe longitudinal displacements in the "top" and "bottom" layers of the four sections of the waveguide, respectively. The values of the constants α, β and c depend on the physical and geometrical properties of the waveguide, while the constants δ and γ depend on the properties of the soft bonding layer, and ϵ is a small amplitude parameter (see [9]). These equations are complemented with continuity conditions for the longitudinal displacements and stresses in the layers, at the interfaces between the sections similarly to (5), (6) (see [6] for details), as well as the relevant initial and boundary conditions.

Again, the direct numerical modelling of this problem is difficult and expensive because one needs to solve several boundary value problems linked to each other via matching conditions at the boundaries. Therefore, we developed an alternative semi-analytical approach based upon the use of several matched asymptotic multiple-scale expansions and averaging with respect to the fast space variable [6]. The developed direct finite-difference scheme and the scheme for the weakly nonlinear solution show good agreement in all regions of the bi-layer, with a small difference in the amplitude and minor phase shift between the results.

A typical scenario is shown in Figure 2 for an exact solitary wave initial condition, with parameters $\alpha = \beta = 1.075$, $c = 1.025$, $\delta = \gamma = 1$ and $\epsilon = 0.05$, with initial position $x = -450$ and initial speed $v = 1.03$. Two homogeneous layers, of the same material as the lower layer, are attached to the left of the bar. We see the generation of a radiating solitary wave in the bonded section of the bar, the separation of the solitary wave from its radiating tail in the delaminated section, and the re-coupling of the waves in the second bonded region.

A similar numerical experiment for the same parameters, with the homogeneous section on the right-hand side of the bi-layer, is shown in Figure 3. The results in this case are qualitatively similar, with a different length of radiating tail due to the change in the length of the relevant bonded section. We note that if the radiation wave packet in the second bonded region is closer to the leading wave when sending the waves from the right, then the delamination is closer to the left-hand side of the structure, and vice versa. We also note that the generated wave is of a larger amplitude in the case when the homogeneous layers are of the same material as the bottom layer (with a larger characteristic speed), which was the case in the numerical runs presented in this paper.
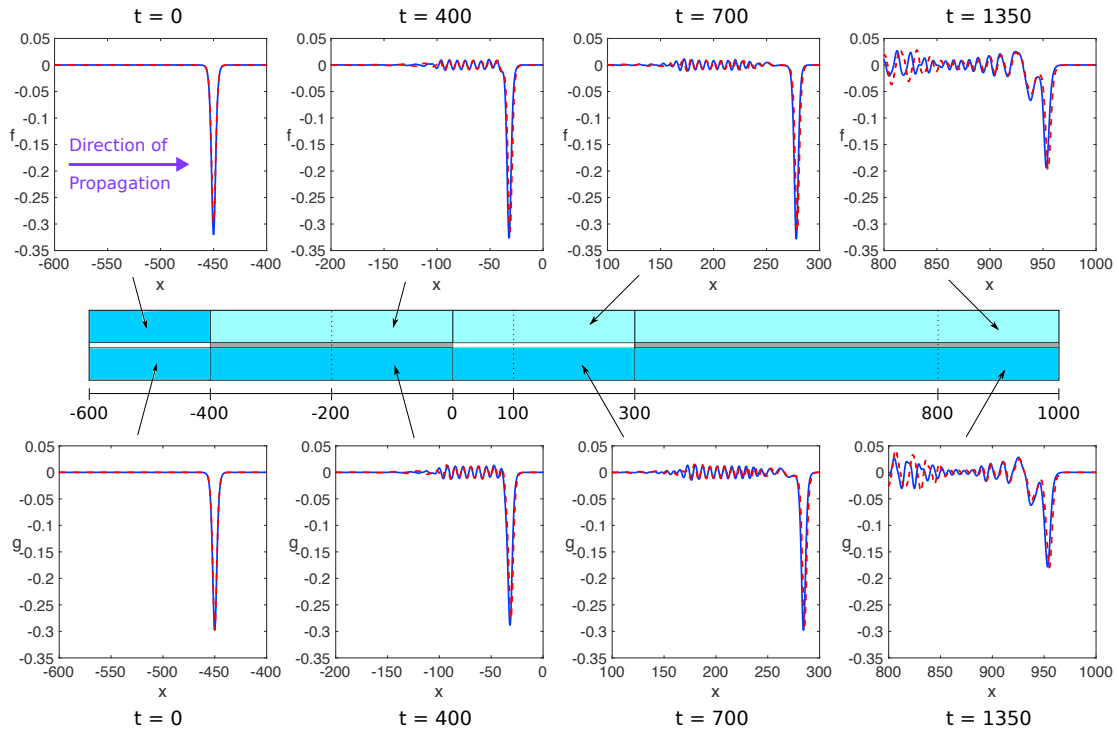
Fig. 2. The solutions f (top row) and g (bottom row) in each section of the bi-layer, for the parameters $\alpha = \beta = 1.075$, $c = 1.025$, $\delta = \gamma = 1$ and $\epsilon = 0.05$, with initial position $x = -450$ and initial speed $v = 1.03$, for direct numerical simulations (solid line) and weakly-nonlinear solution (dashed line). Two homogeneous layers, of the same material as the lower layer, are on the left and the waves propagate to the right.

3. Conclusions

In this paper we modelled the scattering of a long pure and radiating radiating bulk strain solitary waves in a delaminated bi-layer. The modelling was performed within the framework of the nondimensional Boussinesq-type equations. For a single layer, the models are asymptotically equivalent to a "doubly dispersive equation" (DDE) [15, 16], earlier derived for the long longitudinal waves in a bar of rectangular cross-section using the nonlinear elasticity approach [3]. In dimensional variables, the DDE for a bar of rectangular cross-section $\sigma = 2a \times 2b$ has the form

$$f_{tt} - c^2 f_{xx} = \frac{\beta}{2\rho}(f^2)_{xx} + \frac{Jv^2}{\sigma}(f_{tt} - c_1^2 f_{xx})_{xx}, \tag{10}$$

where $c = \sqrt{E/\rho}, c_1 = c/\sqrt{2(1+v)}, \beta = 3E+2l(1-2v)^3+4m(1+v)^2(1-2v)+6nv^2, J = 4ab(a^2+b^2)/3, \rho$ is the density, E is the Young modulus, v is the Poisson ratio, while l, m, n are the Murnaghan moduli. Nondimensionalisation, regularisation of the dispersive terms and scaling bring the equation to the form (2). Our modelling highlights key features of the behaviour of pure and radiating solitary waves in such delaminated bi-layers, which could be used for introscopy of layered structures, in addition to traditional tools. The fission of a single incident soliton into a group of solitons in the delaminated area of a perfectly bonded PMMA bi-layer has been observed in [4]. The generation of a radiating solitary wave and subsequent disappearance of the "ripples" in the delaminated area of a two-layered PMMA bar with the PCP (polychloroprene-rubber-based) adhesive has been observed in [7]. Our numerical modelling motivates further laboratory experimentation with other materials used in applications.
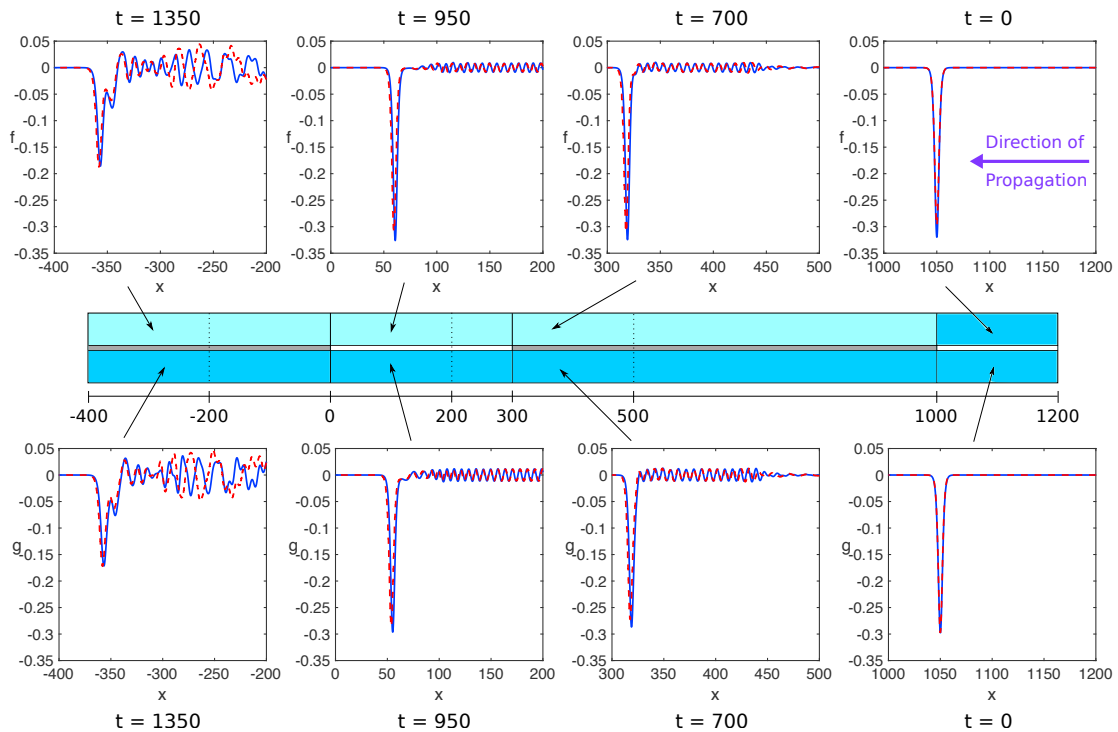
Acknowledgements

Fig. 3. The solutions f (top row) and g (bottom row) in each section of the bi-layer, for the parameters $\alpha = \beta = 1.075$, $c = 1.025$, $\delta = \gamma = 1$ and $\epsilon = 0.05$, with initial position $x = 1050$ and initial speed $v = 1.03$, for direct numerical simulations (solid line) and weakly-nonlinear solution (dashed line). Two homogeneous layers, of the same material as the lower layer, are on the right and the waves propagate to the left.

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