OPTIMISING USE OF THE $\bar{X}$-CHART

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Abstract

This thesis begins with a review of the development and use of the control chart in the manufacturing industry. The review reveals scope for investigating the effects of important parameters of the production environment on control chart design parameters and the penalty incurred when suboptimal parameter values are used. The design parameters are sample size (n), sampling interval (h), and control limit coefficient (k).

A subsequent review of the economic design of the $\bar{x}$-chart reveals that objective expected cost per time unit (ECPTU) functions can be minimised in order to determine optimum values of n, h and k. In this thesis we use a special case of Lorenzen and Vance's [51] model to extend the previous literature by studying the limiting behaviour of a system of control which is based on the $\bar{x}$-chart; quantifying the response of $\bar{x}$-chart design parameters to changes in the production environment; and investigating the penalty for use of suboptimal parameter values.

ECPTU functions are minimised with respect to the $\bar{x}$-chart's design parameters. When compared with the heuristic design, the primary benefit of use of an economic design as presented in this thesis is the increased probability of detecting the OOC state. The study confirms the need to change the value of the design parameters in response to changes in the values of other process parameters such as the hourly cost of operating the process when it is out of control and the shift coefficient. Average values of the percentage changes are given.

Generally in the literature the time spent identifying and removing the assignable cause has not been regarded as a stochastic variable. In this thesis we generalise the model used to derive ECPTU functions to allow for a distribution of total time spent searching for and removing the assignable cause. Explicitly accounting for restoration time as a stochastic variate which is correlated with the period of surveillance by the $\bar{x}$-chart increases the optimum value of sampling interval. A reduction in the value of the penalty for use of suboptimal values is also observed.

The research also explores the practical issues related to production which must be addressed in order to facilitate successful implementation of the ECPTU function.

Objectives

The main objectives of the research for this thesis:

— To use the history of control charts and their economic design as a basis for continued improvement and optimisation of expected cost per time unit (ECPTU) functions for the $\bar{x}$-chart.

— To develop ECPTU functions based on different statistical distributions and assumptions about processes, thus making these mathematical formulations more robust.

— To determine which forms of the ECPTU functions provide most suitable optimal control chart parameters.

— To develop applicable mathematical formulations which may be used to quantify the effect of various factors on the cost-effectiveness of SPC techniques.

Chapter 1

The Control Chart - A Tool for Quality Management

W. Edwards Deming, Walter Shewhart, Joseph Juran and Dr. Kaoru Ishikawa are statistical process control specialists who have contributed significantly to the development and use of control charts. This chapter will highlight the contributions of these and other researchers to the development and use of control charts. In different sections of this chapter control charts will be defined; their basic uses and the different types will be outlined; and the benefits of use of statistical methods in the manufacturing process will be discussed. The relevance of probability to the function of control charts as well as the recommended values of control chart parameters will be investigated. Finally, the basis for the study of the economic design of the $\bar{x}$ chart will be discussed.

1.1 What is a control chart?

A control chart is defined as a graph or chart with statistically calculated control limit lines. It is a statistical tool which can be used for process control and process analysis. [40] The chart gives indication of the existence of special causes of variation which influence the process. Several types of control charts exist. Each has rules for calculation of the control limits [25]. Each type of control chart has three control lines — the upper control limit (UCL), the central line (CL), and the lower control limit (LCL) [40].

The British control chart is one variation of the Shewhart control chart and can have two pairs of control limits. These are the upper and lower *action lines* (UAL and LAL) — similar to those found on the Shewhart control chart — and the upper and lower *warning lines* (UWL and LWL). The warning lines are optional [7]. A cumulative sum control chart proposed in 1954 by E.S. Page, a British statistician, is used with a V-shaped mask which serves the purpose of the conventional control limit lines [10, 27].

A control chart enables the identification of the nature of changes in a process over a specified period of time. Consequently, the impact of such changes in the quality of output can be studied. The control limit lines indicate the standards for evaluation of the abnormality of the points plotted on the control chart. Each point on the graph must correctly indicate arbitrary divisions in the manufacturing process. These divisions in the data produce the sub-groups or samples [40].

1.2 A history of its use

The control chart has been effectively used to monitor variation in the state of processes and production in industry. Deming, as cited by Juran [48], states that the control chart is the tool for obtaining the most economical manufacturing methods. The main aim of correct use of the control chart is the achievement and maintenance of statistical control. According to Deming [25] a process is in a state of statistical control when there is no indication of special causes of variation so that the limits of variation are predictable. As a result, the process's behaviour in the near future is predictable. Juran in [25] states that process improvement can be effectively done when statistical control is achieved.

In the middle of the 1920s the United States of America saw the first significant application of statistical quality control (SQC) to mass production. In 1926 a team from the Bell Telephone Laboratories proposed the application of certain tools of statistical methodology to the control of the quality of manufactured telephone products. These tools included the new Shewhart control chart. Unfortunately, the use of this tool was not widely accepted during the 1920s. Work with the statistical techniques primarily involved the use of sampling inspection. [48]

There was a second surge of interest in SQC in the USA during the second world war. Again, the American economy failed to benefit as did the Japanese economy from the applications of SQC. This difference may be linked to the failure of the production sector in the

USA to make effective use of the control chart to monitor the process. Instead there was emphasis on the use of sampling inspection of output. [48]

Control charts were introduced to the Japanese manufacturing sector during the period prior to World War Two. From as early as 1931 to 1933 Yasushi Ishida had already started to use control charts to improve the quality of light-bulbs. Ishida also worked to introduce statistical methods to military production during the war. It was not until after the war, however, that there was in Japan the greatest development of the application of statistical methods to mass production.

Lectures by Deming during 1950 gave the Japanese knowledge of how to use SQC too make the cheapest, most consistent, and best quality products. The first six of the eight-day lecture series were spent teaching the Japanese how to draw control charts. The period from the mid-1950s to the 1960s saw rapid growth in the effectiveness of quality control in Japan as well as in the Japanese economy. During this period the Japanese industry also changed from sole emphasis on SQC to emphasis on managing for quality and company-wide quality control. [48]

The 3-sigma control charts developed by Dr. W.A. Shewhart are most convenient for checking unusual behaviour of a process. The success of the Japanese manufacturing sector is due, in part, to its wide use of this tool for statistical control. The most frequently used charts in Japan are the mean and range ($\bar{x} - R$) control chart, median and range ($\tilde{x} - R$) control chart,the x control chart, p control chart, pn control chart, c control chart, and u control chart. Although their use has been very beneficial to Japan, Ishikawa states that these control charts can still be improved. [41]

1.3 Types and functions of control charts

Two basic types of control charts exist: one deals with variables, and the other with attributes. Charts for variables are generally used in pairs. Examples of charts for variables are the mean and range ($\bar{x} - R$) chart; the median and range ($\tilde{x} - R$) chart; and the mean and standard deviation ($\bar{x} - s$) chart. [7] The mean or median charts monitor the process setting or location of the distribution. The range and standard deviation charts monitor variability in process output. The data handled by such charts are continuous. Examples of such data are measurements ($\frac{1}{100}mm$), volume (cc), product weight (g) and power consumption (kwh). [40]

Abnormalities will appear in the $\bar{x}$-chart in response to changes in the mean of the production process as well as changes in the dispersion. The points on this type of control chart react appreciably even to very slight changes in the process mean. When there is a change in the dispersion the points on the chart will show a greater spread and may go beyond the control limits. [40]

For some processes it is not feasible to collect measured data. However, it may be possible to collect qualitative information about product or process features. Under such circumstance the operation of the process may be monitored using attributes charts. The four principal types of attributes charts are the p chart, the np chart, the c chart and the u chart. The p chart monitors the proportion of defective units in a sample. The np chart monitors the number of defectives in a sample. The c chart monitors the number of defects in a specified area of time or space. The u chart monitors the proportion of defects in a specified area of time or space. [7]

The rules for action when the previously described control charts are used to monitor a process are generally based on the last few samples taken before the out of control — OOC — state is detected. The cumulative sum or cusum control chart adopts a rule for action that is based on all of the data. This control chart can be used to maintain current control of a process or to carry out historical analysis of a process. Particularly if the changes are small, it picks up sudden and persistent changes in the process average more rapidly than do the conventional Shewhart charts [27]. Each point plotted on a cusum chart is a cumulative sum of deviations of the sample statistic from the target value for the process. The cusum chart indicates changes in the process through changes in the slope of the points plotted. A positive, negative or zero slope indicates that the mean of the data is above, below or equal to the target value, respectively. [10]

A statistical control chart will detect the existence of a cause of variation that lies outside the process. This cause is referred to as a special cause of variation. The control chart can be used to confirm the capability of a process. It finds its basic uses as judgement and as an ongoing operation. In its function as a judgement the control chart is observed to determine whether the process is in statistical control. As an ongoing operation the control chart is used to attain and maintain statistical control during production [25]. The signals provided by control chart tell the operator when to take action to improve the uniformity of the product [24].

1.4 Benefits of Applying Statistical Methods to Mass Production

Statistical methods are very useful and helpful in quality control. The application of the control chart and other statistical methods to industry produced modern quality control in the United States during the 1930s [42]. Deming as cited by Juran [48] states that statistical methods are resources for a poor nation without natural resources to use in pursuit of international markets. Ishikawa [41] also states that Japan's advance in productivity is strongly linked to the use of statistical methods. Wherever these methods are used there is increased quality and reliability of products as well as reduced production costs [41]. The application of statistical methods to mass production creates the environment for the most efficient use of raw materials and manufacturing processes, for economies of production, and for the highest economic standards of quality for manufactured goods. [68]

Statistical techniques can be applied to a process in order to measure and analyse its variation or to measure and improve the quality of a process. The former application is known as statistical process control and the latter as statistical quality control. Control charts constitute a statistical technique commonly used to measure and analyse process variation. As a result, a state of statistical control is attained and the process can be improved. [47] Tangible economic benefits result when this state of statistical control is achieved. These include the increased predictability of the process's performance and of the costs associated with production. This is because the application of statistical methods minimises variability in process output. Regularity of output as seen in the just-in-time system of delivery is a natural consequence of a system in statistical control. In addition, productivity is maximised. A further advantage of statistical control, is that the effects of assignable causes of variation can be more quickly and accurately measured. [25]

Shewhart [67] also cites the advantages of securing a state of control through the application of statistical methods to mass production. He points out that the cost of inspecting output is reduced if the process is controlled. The comparative stability in the quality of output from a process which is in statistical control leads to reduced cost of inspection as there is less need for it. When the application of statistical methods yields controlled quality of the input for a process there is reduced rejection of output and money is saved.

A system is in a state of statistical control when all the assignable or special causes of variability have been removed. The manufacturer can then attain uniformity in the quality of output. Attainment of statistical control also yields a reduction in the tolerance limits for that particular quality even though measurement of these limits is indirect. [67]

An objective state of statistical control can be achieved. The use of statistical machinery by an engineer who makes the right kind of hypotheses will enable the establishment of criteria which indicate when the state of control has been reached. [67]

1.5 Relevance of probability theory to the function of control charts

Deming [25] states that some knowledge of the theory behind use of the control chart is a necessary requirement for its successful use. Probability theory is the foundation on which use of the control chart is based. According to Ishikawa [41], the output from all processes will have a statistical distribution. He therefore suggests that we must be guided by the concept of a statistical distribution when searching for unusual behaviour. Probability theory will also impinge on our use of statistical distributions.

The theory of probability forms the basis for the calculations which give the location of control limits on the control chart [25]. The statements which follow indicate how Shewhart [67] has used probability theory as the basis for the establishment of control limits.

The integral P gives the probability that the statistic θ lies within the limits θ_1 and θ_2. θ_1 and θ_2 can be chosen such that $P = 1$. $f_\theta(\theta, n)$ is the probability function for the statistic θ obtained from a sample sized n. Therefore,

$$P = \int_{\theta_1}^{\theta_2} f_\theta(\theta, n) d\theta$$

θ outside the limits is a positive indication that the standard of quality is not what it should be. However, the occurrence of the observed θ within the range θ_1 and θ_2 is not sufficient proof that the system has not changed. Consequently, the limits on a particular statistic must be chosen so that the associated probability P makes economic a search for trouble when any of the chosen statistics fall outside their limits. There is some economic value P and a pair of limits θ_1 and θ_2 for each quality characteristic. The economic value of P for one quality characteristic will not be the same for others. The values used in practice will be approximations. It is more economical to adopt a value which will be acceptable for nearly all quality characteristics. [67]

Equation 1.1 gives the symmetrical range for the statistic which, for each control chart, is characterised by the following limits.

$$\bar{\theta} \pm t\sigma_{\theta} \tag{1.1}$$

$\bar{\theta}$ is the expected value of θ and σ_{θ} is its standard deviation. Tchebycheff's theorem has been used to show that, so long as the quality standard is maintained, the probability P that the observed value of θ lies within these limits satisfies the inequality in Equation 1.2.

$$P > 1 - \frac{1}{t^2} \tag{1.2}$$

t could vary. However, experience has shown that $t = 3$ is an acceptable economic value. [67] For this reason the value 3 has been used as control limit coefficient for control charts.

The lines for the British variables control charts are based on probabilities given by the normal distribution. The upper and lower warning lines (UWL *and* LWL) are at $\mu \pm 1.96\sigma/\sqrt{n}$. The upper and lower actions lines (UAL *and* LAL) are at $\mu \pm 3.09\sigma/\sqrt{n}$. The position of these lines indicates that there is a 1 in 40 chance of a value occurring above the UWL and the same probability below the LWL. The probability that a value falls outside each of the action lines is 1 in 1000. For British attributes control charts the lines are also based on the probabilities of 1 in 40 and 1 in 1000. The location of the control lines is determined using the binomial or Poisson distributions. [7]

The conversion factors — d_2, d_3 — used to calculate limits for the Average Chart and the Range Chart were derived from the normal distribution. However, work done by Burr(1967) as cited by Wheeler [78] indicates that their values are not very sensitive to normality and may be used without concern for the normality or non-normality of the data.

Although the position of the control limits have their foundation in the theory of probability, Deming [25] states that it is wrong to give a figure for the probability that a statistical signal for the detection of a special cause is wrong. He further states that it would be incorrect to attach a figure to the probability that the chart could fail to send a signal when a special cause exists.

Neave [58] has stated that Deming used 3 as the value of control limit coefficient without concern for the mathematical justification for its use. Deming also used data which were not normally distributed to plot points on the control chart and often computed control limits using just a few points. This was done even though the high degree of correlation between the points and the limits would affect probabilities. Deming [25] has indicated that teaching control charts using the normal distribution assumption can be misleading and derail effective study and use of control charts.

Work done by Wheeler, Deming and Neave indicate that the value and practicality of the control chart is not dependent on probability or the normal distribution assumption. The main purpose of these charts is to provide guidance "when special causes are troublesome enough to warrant action [58]". Nevertheless, the normal distribution assumption is used in the current research to facilitate the calculation of costs associated with the operation of the control chart.

1.6 Recommended design parameters for the control chart

This study is concerned with optimising the use of the $\bar{x}$ chart with respect to its design parameters. The design parameters are sample size (n), the sampling interval (h), and the control limit coefficient(k). This section will review the recommendations of other researchers concerning the values of these parameters.

Based on Shewhart's initial work in 1925 it is established that the recommended value of the control limit coefficient is 3 [25]. This value works well in practice even though its use is not firmly rooted in probability theory [7]. (See Section 1.5.) As indicated in Section 1.5, the British control charts for variables have their control limit coefficients set at 3.09 standard deviation units for the *action limits* and 1.96 for the *warning limits*. On British control charts for attributes the values for the control limits are also based on the 1 in 40 and 1 in 1000 chance of a point falling outside either control limit.

The literature does not provide a commonly used value of sampling interval. Ishikawa [42] indicates that the sampling interval should be fixed. He states, however, that use of fixed sampling intervals generates the risk of overlooking the effect of these intervals on variation. The length of the sampling intervals should reduce the probability that assignable causes go undetected. Fixed sampling intervals give best results when relatively uniform raw materials are supplied and processed stably and machinery is infrequently adjusted. It is believed that abnormalities will occur systematically in a stable process and will be more promptly identified at the fixed times for sampling.

Ishikawa [42] further points out that, where composite samples must be taken, sample measurements may be averaged over 1 hour, 4 hours or one shift. Use of these times gives less loss of information on variation in comparison to averaging over one day. Juran [47] cites the work of Ewan(1963) who recommends an interval of $T/6$ between samples to plot a cumulative sum control chart. T is the permissible average time before a shift in the process mean is detected.

Deming [25] mentions that points should be plotted on the control charts every half-hour or every hour. This would suggest that these are lengths of the sampling intervals. Such lengths may not be feasible under some conditions of production. In selecting sampling intervals we can be guided by the fact that the length of the interval influences the estimate of variation. Subgroups taken at too large intervals will yield incorrect estimates of variation. Infrequent sampling yields too few subgroups which will give inaccurate estimates for the mean range. Consequently, there will be incorrect positioning of the control limits. [47] Since 1956 a number of researchers have investigated the selection of optimum sampling intervals for use with control charts. A review of the literature featuring selection of sampling intervals will be presented in Chapter 2.

Sample sizes ranging from 2 to 5 are recommended for use with the variables control charts and are more commonly used. However, sizes ranging from 6 to 10 have been used in special cases [40, 41]. Juran [47] gives guidelines on the subgroup size that should be used with different types of control charts. Sample sizes of four or five items are recommended for use with the $\bar{x}$−chart. He further states that the distribution of the sample mean is always normal for $n \geq 4$. This is the case even if the distribution of the observed values for the sample items do not follow the normal distribution. Larger subgroup sizes yield more sensitive $\bar{x}$ charts since the control limits will be tighter. Here it is assumed that the standard deviation units used to place the control limits are calculated from the sample. In selecting sample sizes it must be understood that smaller subgroups lead to wider control limits. This will increase the risk of missing any signals of an assignable cause of variation. Control limits become too narrow if the sample size is too large. As a result the risk of getting false out-of-control signals is increased. [47]

Betteley *et al* [7] point out that samples consisting of a single unit can be safely used. In such cases the variation is measured as a sample-to-sample difference. However, samples of size one could lead to complications in interpreting the chart if the data do not come from a normal distribution. As sample size increases the distribution of the sample means tends towards a normal distribution. For this reason, sample sizes greater than one have been used. A minimum sample size of 5 has been found to give reasonable results and is most commonly used. However, larger sample sizes are recommended for $\bar{x}$ and s charts. [7]

1.7 Basis For Investigating The Design of the $\bar{X}$ Chart

A number of researchers have continued to investigate the economic design of the $\bar{x}$ chart. This investigation may have resulted from a need to make the chart more relevant to particular production conditions or quality characteristics. The parameters of greatest influence on the use of control charts are the sample size (n), the sampling interval (h), and the control limit coefficient (k). Recommended values for these parameters have been mentioned in Section 1.6. Use of the recommended values have produced reasonable results. Nevertheless, there is still room for investigation of the specificity of n, h and k to given processes. Consequently, the research presented in this thesis represents a further study of the ways in which cost and other technical parameters associated with the process influence the design of the $\bar{x}$ chart.

In Section 1.3 it is stated that the $\bar{x}$ chart gives indication of both changes in the mean and variance of the process's output. The fact that this tool can give information about two facets of the process has singled it out for investigation. In Section 1.6 it has been pointed out that the exact values for sample size and sampling interval have not been firmly established. It is felt that the value of these parameters will be incumbent on the effect of other cost and technical parameters associated with the process. Shewhart [67] has used the inequality in Equation 1.2 to show that the value of the control limit coefficient is not necessarily 3 for all quality characteristics. (See Section 1.5). This value of 3 is heuristic. In view of the previous statements an investigation of the optimum design parameter values is aimed at selecting values specific to the process and quality characteristic under consideration.

Ishikawa [41] has stated that control charts can be improved. This study of the economic design of the $\bar{x}$ chart can help improve these charts by making them more applicable to specific types of processes. This study will further confirm the influence of certain cost and technical parameters on the optimum control chart parameters and, hence, the function of the $\bar{x}$ chart. Expected cost per time unit ($ECPTU$) functions will be derived and minimised with respect to sample size (n), the sampling interval (h), and the control limit coefficient (k). The penalty for use of the recommended design parameter values instead of optimum design parameter values will be investigated. This research will produce guidelines or general rules of thumb for the operation of the $\bar{x}$ chart under different production conditions.

The following chapter will be a review of work done by other researchers on the economic design of the $\bar{x}$ chart. Their work is used as the basis for the development of other cost per time unit equations.

Chapter 2

The Economic Design of the $\bar{X}$ chart

2.1 Criteria for optimisation

In Chapter 1 reference was made to the tangible economic benefits of using the control chart to monitor the state of a process. The control chart design parameters — sample size (n), sampling interval (h) and control limit coefficient (k) — used in the experiences cited had the heuristic values recommended by Shewhart and other specialists in the use of the charts. Recommended design parameter values for the $\bar{x}$-chart are shown in Table 2.1. The literature does not indicate these values are the economic optimum.

The onset of the study of the economic design of the $\bar{x}$-chart started investigation of optimum design parameters for the charts. Optimum parameters are selected in a manner which is dependent on the other features of the process. Hence they are specific to the process being

	n	h (hours)	k
Shewhart [67]	—	—	3
Deming [25]	—	0.5 or 1	—
Juran [47]	4, 5	—	3
Ishikawa [51, 14]	5	8	3
Feigenbaum [51, 14]	5	1	3
Burr [51, 14]	4 or 5	—	3
Dale and Oakland [23]	5	1	—

Table 2.1: $\bar{X}$-chart design parameter values proposed by various researchers.

monitored. Their use enables more rapid determination of the need to correct or improve the process. This is because sample means are taken at regular intervals and plotted on control charts whose control limits create a trade-off between minimising the probability of a false alarm — Type I error — and maximising the probability that a point falls outside the control limits when the process is in the out-of-control state — power of the test for the in-control state. Accurate use of the control chart is costly. It is therefore important that this accuracy is optimised by selecting control chart parameters which minimise the cost of using the chart.

In much of the reviewed literature, there has been simultaneous optimisation of the control chart with respect to n, h and k. The optimum h is constant for the entire period of surveillance by the control chart. Tagaras [72], however, finds the economic design by selecting k and h only. Banerjee and Rahim [4] find the economic design by selecting n, h_1 and k. They state that it is desirable to have sampling frequency increasing with the age of a process which has an increasing failure rate. Hence, h_1 is the length of the first and longest sampling interval. Sampling frequencies increase as the length of the cycle increases. The length of each sampling interval, h_j, is defined in a manner which keeps constant the probability that the process mean shifts within an interval, given no shift up to the start of this interval. In order to maintain this constant integrated hazard over each of j sampling intervals they choose sampling intervals of length $h_j = \left[j^{\frac{1}{\nu}} - (j-1)^{\frac{1}{\nu}}\right] h_1$. For this model the in-control period is assumed to follow the Weibull distribution having shape parameter ν. Chung and Lin [16] use a similar model to determine optimum values of n, h_1 and k.

Objective functions have been derived and optimised with respect to the design parameters. The functions provide an opportunity to assess the theoretical cost use of the control chart. The degree of complexity of the objective function is dependent on the assumptions which are made about the process. The most common form of the objective function has been the ratio of the expected cost to the expected time of operation of the control chart. Saniga [65] and Gupta and Sachdev [35] use an expected cost of operating the chart to determine optimum design parameters. Castillo *et al* [9] use minimum expected number of false alarms; minimum average time to detection of the OOC state; and minimum sampling cost per cycle of operation of the chart as the criteria for selection of optimum control chart design parameters. These criteria are subject to statistical constraints.

McWilliams [53] and Montgomery *et al* [56] also find the optimum values of n, h and k subject to statistical constraints. The economic-statistical design thus produced minimises the loss cost function while meeting constraints for average run length.

2.2 Control charts optimised

The $\bar{x}$-chart has been the control chart most commonly singled out for selection of an optimum design. Since the 1950's there has been increased interest in the joint economic design of the $\bar{x}$-chart and control charts for dispersion parameters.

Ho and Case [39] state that Saniga [65], in 1977, pioneered investigation of joint economically optimal design of the $\bar{x}$ and R control charts. The optimum values for the sample sizes and control limits were similar to those obtained when the design of the $\bar{x}$-chart, only, is optimised. However, optimum sampling frequency decreased as a result of joint use of the $\bar{x}$ and R charts.

Rahim [62] later produced a computer program for determining the optimal economic design for joint use of the charts. Jones and Case [46] used Duncan's work as the basis for deriving the economic design of a joint $\bar{x}$ and R chart. Unlike Saniga's [65] model Jones and Case's [46] work permitted the mean and range to be out of control at the same time. Results from the latter study suggest that the "standard" design may not be the optimal design.

Rahim *et al* [64] and Yang [80] also investigated the joint optimal design of control charts for monitoring the location and dispersion of the process output. Yang [80] used Taguchi's quadratic loss function to develop the joint economic design of $\bar{x}$ and s charts. His data analysis revealed that the optimum values of all three design parameters are significantly influenced by the cost of repair or replacement of the product. Production rate and its interaction with cost of product repair were influential to the optimum sampling interval and control limit coefficient. Rahim *et al* [64] developed a model for the joint economic design for the mean and variance charts. They found that the $\bar{x} - s^2$ chart has lower minimal cost than does the $\bar{x} - R$ chart.

Points plotted on the $\bar{x}$ control chart enable the user to detect shifts in the mean and the variance of the process. Economic design of the $\bar{x}$ chart, singly, initiates cost-effective use of the SPC techniques. However, concurrent optimisation of the R chart would ensure more rapid detection of the need to correct or improve the process in response to a shift in variation. It is outside the scope of this thesis to develop objective functions for the joint economic design of control charts used to detect shifts in the location and dispersion of the process output. However, based on a reduced model, there will be derivation of objective functions for investigating the optimised use of $\bar{x}$ charts. These functions may be modified for use in joint selection of optimum range chart parameters.

The rest of this chapter reviews the development and use of one particular form of objective function which is minimised to select optimum $\bar{x}$-chart design parameters. This form is expressed as the ratio of expected cost of a cycle of operation of the chart to the expected length of the cycle. The basis for use of this functional form will be presented. There will be a review of the assumptions related to previous use of this objective function. The different methods used for its derivation and optimisation will be highlighted. The review of the literature reveals scope for additional investigation of optimised use of the $\bar{x}$-chart. Hence, this chapter will close with an explanation of ways in which this research will facilitate this investigation.

2.3 Assumptions about the process

The statistical distribution assumed for the time to shift in the process mean is very influential on the form of objective function. Duncan [28, 29], Chung [14], Tagaras and Lee [73, 74] and Saniga [65] are among the researchers who assume that the time to shift in the process mean follows the exponential distribution. Ho and Case [39] state, however, that the exponential distribution assumption may be inappropriate for processes which deteriorate with time.

Others such as Banerjee and Rahim [4] and Chung and Lin [16] assume that the process-failure mechanism for the control chart has a non-constant hazard rate. Their cost models use the assumption that the time to shift in the process mean follows a Weibull distribution. Banerjee and Rahim [4] found that the optimal design parameters are insensitive to a moderate degree of misspecification on the Weibull parameters. In their research Surtihadi and Raghavachari [71] used the Weibull, Log-normal, Folded-normal, Folded-logistic and Gamma distribution assumptions to derive cost functions.

The models commonly assume that the process is kept in operation while there is search for the assignable cause [14, 16, 28, 29]. More recent models by Chung [15] and Tagaras and Lee [73] assume otherwise. Even when it is assumed that the process is shut down during search for the assignable cause the objective functions incorporate parameters associated with the cost and length of time for restoring the process to the in-control state.

Another common assumption is that the time to take and inspect a sample and plot the results is non-negligible [14, 15, 16, 28, 29, 73]. In their investigation of the economic design of $\bar{x}$ charts under a preventive maintenance policy Chiu and Huang [11] assume that this period is negligible. Ho and Case [39] cite work by Arnold and Collani published in 1987 which also uses the assumption of negligible sampling inspection time. There is always, however, a cost attached to the sampling and inspection of items.

The models generally assume that the production periods being monitored are infinite. Crowder [22] and Tagaras [72], however, developed SPC models for production runs of a pre-specified length. Crowder's research led to determination of the appropriate times at which the process is adjusted when the production run is finite. These optimum times for control action minimise the expected cost of operating the control chart. Tagaras's [72] model assumed a constant sample size of 1. In his model there was continuous updating the Bayesian estimate of the state of the process. This allowed for use of the dynamic programming approach to adjust values of the control limit coefficient — k — and the sampling interval — h.

The traditional statistical process control philosophy is that the process be adjusted only when the process mean is substantially off target [22]. In developing the objective functions, assumptions have been made about the point at which adjustment should occur. Adjustment is action by those involved in production to return the process mean to an in-control state after the process is declared to be out of control. In the model published by Duncan [28] in 1956, action is taken to search for and remove an assignable cause once the plotted sample mean falls outside the single pair of control limits. This feature of the model is based on the assumption that a single assignable cause shifts the process mean to the OOC state.

Extension of his work in 1956 led to a 1971 publication of his paper in which there is derivation of a cost per unit time equation for processes which are influenced by multiple assignable causes. Different levels of response were associated with the different causes. Tagaras and Lee [73, 74] and Chung [14, 15] also used this assumption in their derivation of cost equations. Tagaras and Lee [73] point out that processes subject to multiple assignable causes can be monitored, theoretically, using control charts with many sets of control limits. Administration of such charts could prove burdensome. In these cases they propose the aggregation of assignable causes and restoration activities so that charts with two or three sets of control limits and respective corrective actions are developed. Much of the reviewed work which concerns the investigation of multiple assignable causes refer to design of control charts with two pairs of control limits.

The cost and technical parameters which are included in the model vary according to the assumptions made about the process and the purpose for which the model is derived. The cost parameters which are commonly used include the costs of sampling; down-time in production because the process is shut down during search for an assignable cause; operating in the out-of-control state. Technical parameters commonly included in the functions are the shift coefficient; the time for taking a sample and plotting its mean; the expected search time for an assignable cause. The shift coefficient is the number of standard deviations by which the process mean shifts when the process goes out of control.

According to Ho and Case [39], it is commonly assumed that the cost and operating parameters for the process are known or can be precisely estimated. This information is often unavailable or difficult to obtain. They cite previous research which reveals that a cost function which accounts for this imprecision in the parameters performs better than one which does not. They further report that previous work by Krishnamoorthi presents a simple method to estimate the magnitude of the shift coefficient.

A false alarm occurs when a sample mean falls outside the control limits but the process is still in control. Some models have a cost and search time associated with false alarms.

In cost models for processes subject to multiple assignable causes, values for some parameters change according to level of the out-of-control state [14, 15, 16]. These parameters include the shift coefficient; the cost and time associated with restoring the process to the in-control state; the scale parameter for the distribution of the in-control period; and the control limit coefficient.

The model presented by Crowder [22] is one of the few for which the sampling cost is not formally considered. His model accounts for the costs associated with deviation from the target value of output and the cost of making an adjustment to the process.

2.4 Methods of derivation and optimisation

The *ECPTU* function has been derived using the length and cost of operating the chart during the in-control and the out-of-control periods. The forms of the objective functions vary according to whether it is assumed that production continues while there is search for an assignable cause and whether the values of cost or technical parameters change with the type of assignable cause influencing the process.

Duncan [28] used results from waiting time analysis to determine the average time to occurrence of an assignable cause within an interval. His method has been used by many other researchers. (See [11, 14, 64, 62]). The models assume that samples are taken at intervals of length h. $f(t)$ is the probability density function for the in-control period, t. Given the occurrence of the assignable cause in the interval between the ith and $i+1$st sample, the general expression for the average time to occurrence of an assignable cause within an interval is

$$\frac{\int_{ih}^{(i+1)h}(t-ih)f(t)dt}{\int_{ih}^{(i+1)h}f(t)dt}$$

The probability that the out-of-control state will be detected and the probability of a false alarm were found using the normal distribution assumption. Expected costs were obtained as a product of the cost parameter and the expected length of different periods or the number of events within the cycle of the chart's operation.

Lorenzen and Vance [51] presented a general method for determining the economic design of control charts. This was an attempt to unify the approach of researchers to the economic design of control charts. The expected cost per cycle was the sum of expected costs of

i) producing non-conforming items during the IC — in-control — and the OOC states;

ii) searching when there are false alarms;

iii) locating and repairing the assignable cause which exists;

iv) sampling for the duration of the cycle.

As in Duncan's model the expected costs were products of the cost parameters and the expected lengths of the states of the process or the expected number of events such as false alarms or sampling inspections. These general forms of the function can be made specific to particular types of process models. Dummy variables take different values depending on whether production ceases during search for the assignable cause and repair of the process. As originally intended, this model has been applied to different types of control chart models and generalised to statistical distributions different from the exponential distribution. (See [4, 52, 53, 71, 64]).

In his earliest work Duncan [28] found an approximation to the optimum design. He carried out Taylor series expansion of the functions for the number of samples taken during the in-control period and the average time to shift in the process mean within an interval. This produced expressions in terms of h having order two or lower. Substitution of such functions into the cost function allowed for determination of the partial derivatives. After setting to zero the partial derivatives with respect to n, h, and k explicit equations for these parameters were derived. Repeated substitution of solutions to one explicit equation into another produced the approximate optimum chart design. Tagaras and Lee [74] and Chung [15] also used Taylor series expansion of their exponential expressions to obtain explicit equations in h. The solutions to such equations were used as the starting search values for the optimum combination of design parameters.

As interest in the economic design of the control chart has increased the methods of optimising the objective functions have improved. Pattern search techniques have been more commonly used. Chung and Lin [16] used the pattern search technique to find the optimum combination of design parameters. The starting value was $n = 0$ and k and h_1 are incremented by 0.01 for each search step of the algorithm. The search ends when the combination of parameters includes a value of n which has a minimum value of 10.

Chung [14] developed an algorithm for obtaining economically optimum control chart parameters. In this algorithm n and k are treated as discrete variables. k is given unit length of 0.1 or 0.01. An explicit equation for h is solved and the solution is used as the starting value for the search procedure. The starting values for n and k are non-zero values.

A number of computer programs which calculate the optimal design for $\bar{X}$ and $\bar{X}-R$ control charts have been published. [53] Torng *et al* [75] have also produce a computer program for the economic statistical design of exponentially weighted moving average charts. The FORTRAN program developed by McWilliams [53] enables users to determine — interactively or using data files — the economic, statistical, or economic-statistical $\bar{x}$ chart design. This program is based on Lorenzen and Vance's [51] general unified model. Rahim [63] also presents a FORTRAN program which which finds the optimal economic design based on the economic model of Banerjee and Rahim [4].

Hooke and Jeeves' pattern search technique was published in 1961. Since then it has been the basis of search algorithms developed by researchers to find the optimum combination of design parameters. (See [4, 11, 15, 46, 62, 63, 64, 75].)

2.5 Basis of the objective function

For a reward renewal process the expected cumulative cost $C(t)$ per unit time tends asymptotically to the ratio of expected cost between renewals to the expected time between renewals. [3, 13, 19, 20]

The average cost of renewal per unit time up to time t is $C(t)/t$ and is expressed as follows. [3]

$$C(t)/t = \frac{\sum_{i=1}^{N(t)} E(C_i)}{t}$$

C_i denotes the cost of the ith renewal. $N(t)$ is the renewal function denoting the expected number of renewals in time t. As $t \to \infty$ the average cost per unit time tends to a limit which can be written as follows. [3]

$$\lim_{t\to\infty}\left[\frac{C(t)}{t}\right] = \frac{E(C_i)}{E(t_i)} \tag{2.1}$$

t_i represents the time to renewal on a single cycle. Derman and Sacks [26] also used this asymptotic result in their investigation of the optimal stopping rule for the replacement of periodically inspected equipment.

The operation of the $\bar{x}$-chart can be regarded as a reward renewal process. As in the model used by Derman and Sacks [26] the process under the surveillance of the control chart is checked periodically by inspection of a sample of output at specified intervals. The renewal or replacement is the detection of the OOC state when a sample mean falls outside the control limits. Work done by Ansell *et al* [3] and Christer [13] is formulated on an age replacement problem. Collani *et al* [20] investigates the specific application of this problem to the operation of the $\bar{x}$-chart. This study also investigates this application by using the limit result given in Equation 2.1.

2.6 Original work

The current research is aimed at investigating the practical application of optimal approaches to statistical control of a process. The history of control charts and their economic design will form the basis of continued improvement of the ECPTU functions for the $\bar{x}$-chart. Objective functions will be developed using different assumptions or models of the production process. They will be used to quantify the cost-effectiveness of using the $\bar{x}$-chart.

Widely accepted guidelines on the value of sampling interval do not exist [15]. On the basis of the recommendations presented in Table 2.1 the units for sampling interval in this study will be hours. Hence, the $ECPTU$ gives the hourly cost of operating the $\bar{x}$-chart.

Initially, general forms of the $ECPTU$ will be derived. They will be made specific to the exponential and Erlang distributions. Figure 2.6 shows the relationship of works by various researchers to models produced by Duncan [28] in 1956 and Lorenzen and Vance [51] in 1986.

In order to minimise the complexity of the objective function we will assume that the time for sampling inspection is negligible; the process is deemed OOC only when a sample point falls outside the control limits; the values of the cost and technical parameters do not change according to the assignable cause which affects the process. This reduced model will lessen the number of cost and technical parameters which require estimation. Work by Duncan [29], Chung [15] and Tagaras and Lee [73, 74] illustrate the complexity of objective functions which use different parameters to indicate the effect of different assignable causes.

According to Alexander *et al* [1], one of the purposes of SPC is continuous improvement. This can be achieved through adjustment of control chart design parameters over time. Hence, the objective functions derived in this study will be used to illustrate the optimised operation of the $\bar{x}$-chart. The results will give the percentages by which design parameters may be adjusted in response to changes the size of the process shift as well as particular cost ratios. The penalty for use of suboptimal combinations of design parameters will be assessed. The effect of length of the IC period on the optimal design will also be considered.

This study will also derive objective ECPTU functions by explicitly accounting for the stochastic nature of the entire period spent searching for and removing the assignable cause. This restoration period will be assumed to be correlated with the period for which the process was monitored by the $\bar{x}$-chart. These objective functions will also be simultaneously optimised with respect to all three design parameters. Benefits of incorporating the stochastic and correlated nature of restoration times to the optimised use of the control chart will be proposed.

The improved performance of computers and mathematical software will be exploited to reduce the difficulty with which the objective functions are minimised. The functions presented in this thesis will be minimised using optimisation routines in *Mathematica*, the mathematical computing package. Use of this package will improve the speed and accuracy with which the optimum values are found.

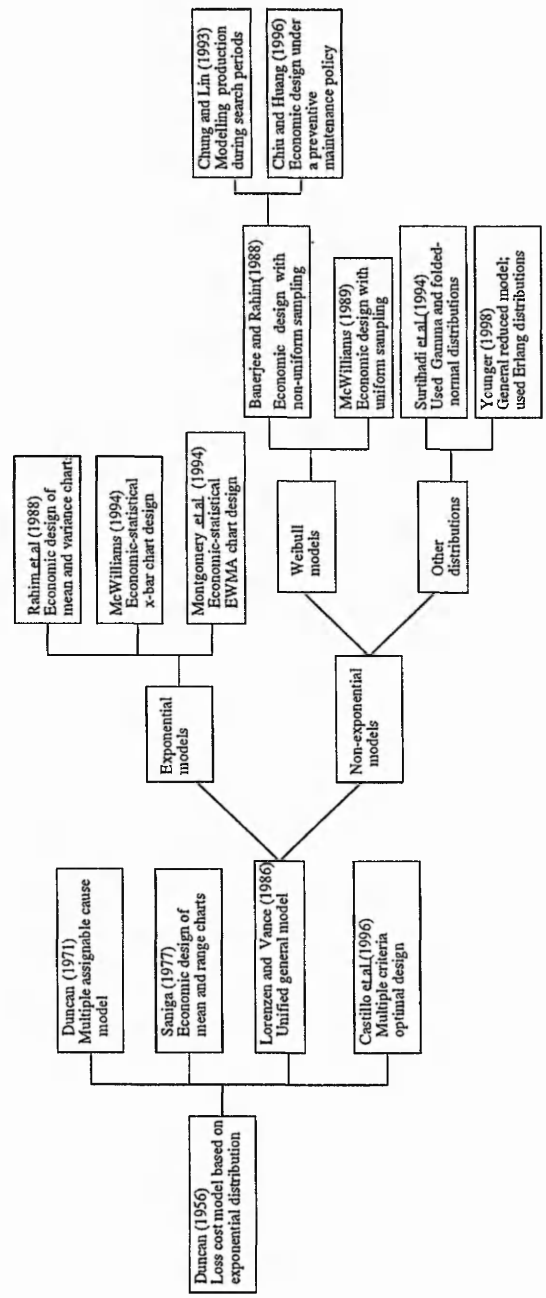


Figure 2.1: Tree diagram of various developments of Duncan's [28] model for the $\bar{x}$-chart since 1956.

Chapter 3

A Reduced Model

3.1 Introduction

Lorenzen and Vance [51] propose a unified general model which can be applied to the economic design of various control charts. Their model allows any distribution to model the time to shift in the process mean. It incorporates indicator variables which denote whether or not production continues during search for and removal of the assignable cause. In this chapter we generalise the model presented by Duncan [28] in 1956 to produce a special case of Lorenzen and Vance's [51] unified model. These models are further explained in Section 3.2. We use this special case of Lorenzen and Vance's [51] model to extend the previous literature by studying the limiting behaviour of a system of control which is based on the $\bar{x}$-chart; quantifying the response of $\bar{x}$-chart design parameters to changes in the production environment; and investigating the penalty for use of suboptimal parameter values.

As part of this research there is derivation of objective ECPTU functions based on the assumption that production continues during search for the assignable cause but not during its removal. In addition we assume that the time spent taking and inspecting samples is negligible. The time to shift in the process mean is an Erlang variate.

Different approaches are used to derive general forms of the objective functions. These are made specific to different Erlang distributions. The approaches yield probability density functions for the OOC period and for the number of samples taken while the process is under the surveillance of the $\bar{x}$-chart.

Prior to an explanation of the derivation of the $ECPTU$ functions there is an outline of the notation used in this chapter. Following this there is discussion of the assumptions made about the process.

The study reveals that the primary penalties for use of the heuristic design parameter values instead of the optimum values are reduced probabilities of detecting the OOC state and a drastic increase in the ECPTU. For the data studied, increases in the shape parameter for the Erlang distribution greatly increase optimum values of n and h.

3.2 Previous Models

3.2.1 Duncan's Model

Duncan's [28] model determines the sample size (n), the sampling interval (h), and the control limit coefficient (k) which minimise the loss cost for operating the $\bar{x}$-chart. His cost equation is

$$\frac{b + cn}{h} + \frac{B\lambda M + \frac{\alpha T}{h} + \lambda W}{1 + B\lambda}$$

where

$$B = ah + en + D$$

represents the expected length of the OOC period plus the time required to take and inspect a sample size n.

$$a = \frac{1}{P} - 0.5 + \frac{1}{12}\lambda h$$

represents the time from the shift in the process mean until the sample which detects the OOC state is taken.

M is the cost per hour for production during the OOC period.

T is the cost of searching for the assignable cause when none exists.

W is the cost of finding an assignable cause.

b is the fixed sampling cost.

c is the cost of measuring each sample item.

en is the time required to take and inspect a sample size n

D is the average time to find an assignable cause after a sample point has been found to fall outside the control limits.

The equations are based on the assumption that the in-control period is an exponential variate with mean $\frac{1}{\lambda}$. The OOC period begins when the process mean shifts from $\bar{X}''$ to $\bar{X}'' + \delta\sigma$. Action is taken to search for an assignable cause only when a single sample point falls outside the control limits. The process is assumed to continue during search for the assignable cause.

In Section 3.5.1 we explain how the model presented in this chapter generalises Duncan's [28] model.

3.2.2 Lorenzen and Vance's Model

Lorenzen and Vance [51] present a general process model which is used to derive an hourly cost function and optimise it with respect to sample size (n), sampling interval (h) and control limit coefficient (k). They define the cycle of operation of the control chart as the time between the start of successive in-control periods.

This cycle time is the sum of:

i) the time until the assignable cause occurs,

ii) the time until the next sample is taken,

iii) the time to analyze the sample and chart the result,

iv) the time until the chart gives an out-of-control signal, and

v) the time to discover the assignable cause and repair the process.

The expected cost of operating the chart is the sum of costs due to

i) production of non-conforming goods when the process is in control and out of control,

ii) searching for an assignable cause,

iii) sampling and inspection.

Their model assumes that, when the process goes out of control, the process mean shifts by a known amount. It is restored to the in-control state when the cause of this shift is removed so that the system is repaired. As does Duncan [28], Lorenzen and Vance [51] assume that the time to take and inspect a sample is non-negligible.

The model is considered a general unified one because it incorporates the indicator variables, δ_1 and δ_2, and it allows any distribution to model the time to shift in the process mean. $\delta_1 = 1$ if production continues during searches and $\delta_1 = 0$ if production ceases during searches for the assignable cause. $\delta_2 = 1$ if production continues during removal of the assignable cause and $\delta_2 = 0$ if production ceases during this period. In their derivation of the hourly cost function the researchers assume that the time to shift in the process mean is an exponential random variable with mean $\frac{1}{\lambda}$.

For an $\bar{x}$-chart this model assumes that the observations are identically and independently normally distributed with mean equal to the centre line and standard deviation σ.

Lorenzen and Vance [51] give the following expression as the expected cost per hour

$$\left(\frac{C_0}{\lambda} + C_1 \left(ARL2\,h + nE - \tau + \delta_1 T_1 + \delta_2 T_2\right)\right)$$
$$\div \left(ARL2\,h + \frac{1}{\lambda} + nE - \tau + \frac{s\,(1-\delta_1)\,T_0}{ARL1} + T_1 + T_2\right) +$$

$$\left(W + \frac{s\,Y}{ARL1} + \frac{(a+b\,n)\left(ARL2\,h + \frac{1}{\lambda} + nE - \tau + \delta_1 T_1 + \delta_2 T_2\right)}{h}\right) \div$$

$$\left(ARL2\,h + \frac{1}{\lambda} + nE - \tau + \frac{s\,(1-\delta_1)\,T_0}{ARL1} + T_1 + T_2\right)$$

Definition of notation for this expression is given in Appendix A.

In Section 3.5.1 we outline the differences between Lorenzen and Vance's [51] model and the model presented in this chapter.

3.3 Notation For This Chapter

There are random variables, design parameters, probabilities and cost and technical parameters associated with operation of the $\bar{x}$−chart. The characters used to represent these are listed below.

The random variables are denoted as follows:-

X — the measured value of the output.

T — period for which the process is in control (IC).

S — period for which the process is out of control (OOC).

J — the number of complete sampling intervals during the OOC period.

I — the number of complete sampling intervals and, hence, the number of samples taken during the IC state. $E(I) = m_1$

M — the number of samples taken during the OOC period ($M = J + 1$). $E(M) = m_2$

τ — the length of the IC period within a sampling interval before the process mean shifts within that interval.

R — the number of samples taken for the duration of the cycle.

The $\bar{x}$−chart parameters are denoted as follows:-

n — sample size.

h — sampling interval.

k — control limit coefficient.

$\{n^*, k^*, h^*\}$ — the combination of sample size, control limit coefficient and sampling interval which together minimise the ECPTU function.

The notation for cost and technical parameters is as follows:-

C — the cost of a cycle of operating the $\bar{x}$−chart.

b — the cost per sample of sampling and charting.

e — the cost of measuring each sample item.

c_2 — the hourly cost of operating in the OOC state.

c_3 — for each false alarm the cost of searching for the non-existent assignable cause.

c_4 — the average cost of searching for the existing assignable cause.

L — the length of a cycle of operating the control chart.

D — time spent searching for the assignable cause.

δ — is the number of standard deviations by which the process mean shifts when the process goes OOC.

The associated probabilities are denoted as follows:-

α_{11} — the probability that when the process is OOC a sample mean falls outside the control limits. This will also be referred to as the power of the test which determines whether the process is in control.

α_{01} — the probability of a false alarm. A false alarm occurs when there is no shift in the process mean but a sample mean falls outside the control limits.

3.4 The Assumptions

The time to shift in the process mean can be regarded as the "lifetime" of the process in the in-control state. According to Lawless [49], the exponential distribution has been widely used to model the lifetime of various processes and products. However, caution must be exercised in its use as many inferences may be sensitive to departures from the exponential model. Procedures based on the exponential distribution tend to be highly non-robust. Bendell and Edgar [6] also state that practical time to failure distributions are frequently non-exponential.

In this chapter the time to shift in the process mean is assumed to be an exponential variate. We subsequently incorporate other Erlangian distributions. The Erlang distribution can be used to model failure which occurs in multiple stages [21] as may be appropriate for the IC period. In previous literature, there has not been widespread use of the Erlang distribution assumption to model the IC period. Surtihadi and Raghavachari [71] are among the few researchers who use the assumption that the IC period is a Gamma(λ, 2) variate.

It is here assumed that the operation of the process is not halted while the samples are taken and inspected. Thus, as in work done by Chiu and Huang [11], Banerjee and Rahim [4], the time spent taking and inspecting samples is considered negligible. The objective functions produced by this study, therefore, have no parameters representing these events. This assumption facilitates the simplicity of the general theory of the $ECPTU$ function.

Duncan [28], Lorenzen and Vance [51], Rahim *et al* [64] and others [14, 16] find the economic design of the $\bar{x}$-chart using the assumption that production continues during search for an assignable cause but ceases while action is taken to remove it. For this study we assume that production is discontinued during search for and removal of the assignable cause. Nevertheless, the cost and time associated with searching for the assignable cause are input values for the objective functions. The expected length of the cycle ($E(L)$) used in our study concerns the period for which the production continues until the cause of a shift in the process mean is identified.

Removal of the assignable cause should lead to process improvement by reducing the variation in process output. It is here assumed that the $\bar{x}$-chart is being used to create opportunities for process improvement. Neave [60] indicates that, when the control chart is used in an improvement context Shewhart's criterion for declaring the process out of control — points falling outside the control limits — can be sufficient. As such, in this study, Shewhart's criterion is assumed as the signal for action. This is referred to as Detection Rule One [77].

The *ECPTU* functions take account of the fact that there will be false alarms. These are points which fall outside the control limits when there is no assignable cause of variation. According to Lorenzen and Vance [51] the expected cost of these false alarms includes the expected cost of testing and searching for the non-existent assignable cause.

Whether or not the process is in the in-control state the measurement value taken from the inspected output is assumed to be a continuous variable which follows a normal distribution. This assumption is common to all the reviewed articles on the economic design of the $\bar{x}$-chart. When the process changes from the in-control — IC — to the out-of-control — OOC — state its mean output value shifts from a value of μ_0 to $\mu_1 = \mu_0 + \delta\sigma$. We assume that δ is known.

Most naturally occurring measurements and a large number of measurements arising from industrial processes, either have a normal distribution or have distributions which are symmetrical around the mean and can be reasonably approximated by the normal distribution. [18] Work by Spedding *et al* [69] shows that the assumption of normality can be expected to increase the average run length for the lower process limit and increase the number of false alarms for the upper limit. They state that these errors are quite small, however, and would be negligible in the practical situation. The errors would, possibly, be of consequence in situations where tight control is required [69]. The findings of Spedding's *et al* [69] research indicate that it is safe to use the normal distribution assumption for the many types of data that might be presented for use on the control chart. Wheeler [78] uses previous work by Burr to demonstrate that control charts work well even if the data are not normally distributed.

The normal distribution assumption as well as the known value for δ facilitate the derivation of the probabilities associated with the operation of the control chart. These are the probability that the OOC state is detected and the probability of a false alarm, α_{11} and α_{01}, respectively.

We assume that measurement of the quality characteristic is perfect, as is assumed by Chung [15].

3.5 General Theory of the $ECPTU$ function

The $ECPTU$ function, in this study, takes the form of the ratio of two expectations, $\frac{E(C)}{E(L)}$. The basis of this formulation is discussed in Section 2.5. C is the cost of the cycle of operation of the $\bar{x}$-chart and L is the length of this cycle. In this study, a cycle begins with the operation of the process in the in-control state and ends when the special cause of variation is identified. This cycle is represented in Figure 3.1.

$E(L)$ is a summation of the expected length of the IC period, the expected length of the OOC period and the search period. $E(C)$ is a

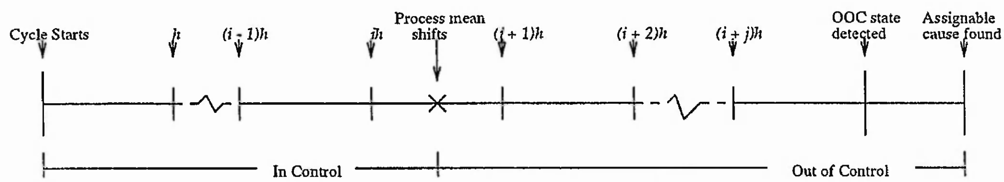


Figure 3.1: Diagram of a cycle of operating the $\bar{x}$-chart when i samples are taken during the IC period and $j+1$ during the OOC period.

summation of the associated expected costs. Some objective functions may exclude expressions for the expected cost and length of the search period. Derivation of the expectations utilises the density functions for the in-control period and the geometric random variable.

In Section 3.5.1 we present general forms for the $ECPTU$ objective function. These can be made specific to different statistical distributions used to model the IC period. In this study they are made specific to Erlang and exponential distributions. If the variate t follows the Erlang distribution with shape parameter, β, and scale parameter, λ, the probability density function can be written as

$$\frac{t^{\beta-1}\lambda^{\beta}e^{-\lambda t}}{(\beta-1)!}$$

We use this form of the density function in our derivations. $\beta = 1$ yields the probability density function for the exponential distribution. For the Erlang distributions β is always a positive integer. As β increases shape of the curve for the density function becomes similar to the normal probability density curve [45]. The proportion of the distribution in the upper right tail also increases as β increases. Plots of the generalized Gamma distribution in [12] illustrate this increase. The Erlang distributions used in this research yield closed forms of the functions before numeric values are given to the scale parameters.

3.5.1 Derivation of the ECPTU function

The methods presented here illustrate different approaches to derivation of ECPTU functions. The first approach yields a general form of the objective function which, in theory, can be made specific to any distribution assumption for the IC period. This general form which yields the objective functions $ECPTU_a$ and $ECPTU_{a1}$ represents a special case of the function presented by Lorenzen and Vance [51]. In [51] the expected cycle time is given as follows

$$ARL2\,h + \frac{1}{\lambda} + nE - \tau + \frac{s\,(1-\delta_1)\,T_0}{ARL1} + T_1 + T_2 \qquad (3.1)$$

$ARL1$ and $ARL2$ represent the average run lengths for the IC and OOC periods, respectively. $\frac{s\,(1-\delta_1)\,T_0}{ARL1} + T_1$ represents the expected time to search for and remove assignable causes. In our model this is replaced by a single parameter, D, which represents the expected search time for assignable causes. T_2, the time spent removing the assignable cause and nE, the time to analyze the sample and chart the result of the analysis, are taken as zero in our objective functions. Using Lorenzen and Vance's [51] notation, our general result for expected cycle time is as follows,

$$\frac{\beta}{\lambda} + ARL2\,h - \tau + T_1 \qquad (3.2)$$

$\frac{\beta}{\lambda}$ represents the mean IC period; $ARL2\,h - \tau$ the OOC period; and T_1, the search period.

These researchers give the expected cost per cycle as

$$\frac{C_0}{\lambda} + C_1\,(ARL2\,h + nE - \tau + \delta_1 T_1 + \delta_2 T_2) + \qquad (3.3)$$

$$W + \frac{s\,Y}{ARL1} + \frac{(a + b\,n)\,\left(ARL2\,h + \frac{1}{\lambda} + nE - \tau + \delta_1 T_1 + \delta_2 T_2\right)}{h}$$

C_0 and C_1 represent the hourly loss cost due to production of non-conforming goods during the IC and OOC periods, respectively. $a{+}b\,n$ is the per sample cost of sampling and inspection. The expected cost of false alarms and locating and repairing the true assignable cause is given by

$$W + \frac{s\,Y}{ARL1}$$

Our expected cost per cycle does not account for C_0. We assume that production ceases during search for the assignable cause and its removal, that is, $\delta_1 = \delta_2 = 0$. However, our model does incorporate the cost of searching for assignable causes. As do Lorenzen and Vance [51], we regard the expected cost of search when there is a false alarm, $\frac{s\,Y}{ARL1}$, as separate from the cost of searching for the existing assignable cause, W. Thus, our expected cost per cycle is given as a sum of the expected sampling cost; the expected cost of production while the process is out of control; and the search costs. Using notation in [51] our cost function can be written as follows

$$E(C) = C_1\,(ARL2\,h - \tau) + \tag{3.4}$$

$$W + \frac{s\,Y}{ARL1} + \frac{(a + b\,n)\,\left(ARL2\,h + \frac{\beta}{\lambda} - \tau\right)}{h}$$

The second general form of the objective function which is presented yields $ECPTU_b$ and exploits the lack of memory property of an IC period which is an exponential random variate. This general form leads to the derivation of a probability density function for, $ARL2\,h -$

τ, the OOC period. Thus the general result for the expected length and cost of a cycle can be expressed as shown in Equations 3.2 and 3.4 with $\beta = 1$.

Unlike the second general expression for the objective function, the third form can, in theory, be made specific to any distribution assumed for the IC period. Derivation of functions using this third approach exploits the discrete nature of the period of surveillance by the $\bar{x}$-chart. As a result $ECPTU_c$ is produced. For this derivation the cycle length excludes the period spent searching for the assignable cause. Here the general result for the expected length and cost of a cycle are expressed as in Equations 3.2 and 3.4 with T_1 and W set to zero. A convolution yields a density function for the number of samples taken while the process is monitored by the control chart. This final approach facilitates extension of the objective function to incorporate a stochastic restoration time which is correlated with the period for which the process is monitored by the $\bar{x}$-chart.

Our derivation of $ARL1$ uses the distribution for the IC period. We derive $ARL2$ using the distribution for a geometric random variable whose parameter is the probability that a sample mean falls outside the control limits when the process is out of control.

The derivations which are shown in the following sections generalise the objective function presented by Duncan [28]. His model is based on the assumption that the IC period is an exponential random variate with parameter, λ. In his model the production continues during search for the assignable cause. However, the cost of restoring the process to the IC state after the assignable cause is discovered is not incorporated. His loss-cost function also ignores the cost of producing non-conforming goods during the IC period. Thus for Duncan's [28] model the objective functions for the expected cost and length of a cycle are as given in Equations 3.1 and 3.3 with $\delta_1 = 1$ and $T_2 = C_0 = 0$.

3.5.1.1 Special case of previous functions

The probability density function, $f(t)$, for the IC period and the density function for a geometric random variable are used to determine the expected length of a cycle. The geometric random variable has the parameter α_{11}. The two density functions are used as separate entities to determine the expected lengths of different parts of the cycle. They are also used separately to find the expected costs incurred for the duration of the cycle. The following expressions for the expected length of the cycle, $E(L)$, and the expected cost of the cycle, $E(C)$, are a simplification of the general result given by Lorenzen and Vance [51].

$$
\begin{aligned}
E(L) &= \int_0^{\infty} tf(t)dt + h - E(\tau) + h\sum_{j=0}^{\infty} j\alpha_{11}(1-\alpha_{11})^j + E(D) \\
E(\tau) &= \sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} (t - ih) f(t)dt \\
E(C) &= E(C_1) + E(C_2) + E(C_3) + E(C_4)
\end{aligned}
$$

$E(C_1)$ is the expected cost of sampling and inspection.

$E(C_2)$ is the expected cost of operating in the OOC state.

$E(C_3)$ is the expected cost of false alarms.

$E(C_4)$ is the cost of searching for the existing assignable cause.

$$E(C_1) = (b+en)\left[\sum_{i=0}^{\infty}\int_{ih}^{(i+1)h} if(t)dt + \sum_{m=1}^{\infty} m\alpha_{11}(1-\alpha_{11})^{m-1}\right]$$

$$E(C_2) = c_2\left[h - E(\tau) + h\sum_{j=0}^{\infty} j\alpha_{11}(1-\alpha_{11})^{j}\right]$$

$$E(C_3) = c_3\,\alpha_{01}\left[\sum_{i=0}^{\infty}\int_{ih}^{(i+1)h} if(t)dt\right]$$

$$E(C_4) = c_4$$

$\int_0^{\infty} tf(t)dt$ gives the mean time to shift in the process mean.

$h - E(\tau)$ gives the expected length of time within an interval that the process operates in the OOC state.

$\sum_{j=0}^{\infty} j\alpha_{11}(1-\alpha_{11})^j$ is the expected number of complete sampling intervals which elapse during the OOC period.

$\sum_{i=0}^{\infty}\int_{ih}^{(i+1)h} if(t)dt$ is the number of samples taken while the process is in the IC state. This sum is denoted as m_1.
The derivation of m_1 using different Erlang distributions is shown in Section B.1 of Appendix B.

$\sum_{m=1}^{\infty} m\alpha_{11}(1-\alpha_{11})^{m-1}$ gives the expected number of samples taken for the duration of the OOC period. This is denoted as m_2.

While the process is in the IC state it is assumed that the mean measured value of the output follows the normal distribution with mean μ_0 and variance σ^2. This assumption enables us to calculate α_{01}, the probability that a false alarm occurs.

$$\begin{aligned}\alpha_{01} &= P\left(\bar{x} \geq \mu_0 + \frac{k\sigma}{\sqrt{n}}\right) + P\left(\bar{x} \leq \mu_0 - \frac{k\sigma}{\sqrt{n}}\right) \\ &= 2\Phi(-k)\end{aligned}$$

When the process is OOC, $\bar{x}$ is normally distributed with mean μ_1 and variance σ^2. Therefore, with $\mu_1 = \mu_0 + \delta\sigma$, the probability that the sample mean falls outside either of the control limits is

$$\begin{aligned}\alpha_{11} &= P\left(\bar{x} \geq \mu_0 + \frac{k\sigma}{\sqrt{n}}\right) + P\left(\bar{x} \leq \mu_0 - \frac{k\sigma}{\sqrt{n}}\right) \\ &= P\left(Z \geq k - \delta\sqrt{n}\right) + P\left(Z \leq -k - \delta\sqrt{n}\right) \\ &= \Phi\left(-k - \delta\sqrt{n}\right) + \Phi\left(-k + \delta\sqrt{n}\right)\end{aligned}$$

Z denotes the standard normal random variable. $\Phi(z)$ is the cumulative probability that Z is less than or equal to z.

When the time to shift in the process is an Erlang(λ, 2) variate the expressions for the expected length and cost of a cycle are as follows,

$$E(L_a) = \frac{2}{\lambda} + h - E(\tau) + h\frac{1-\alpha_{11}}{\alpha_{11}} + D$$

$$\begin{aligned}E(\tau) &= \frac{2}{\lambda} + hm_1 \\ &= \frac{2}{\lambda} - \left(h\frac{e^{-\lambda h}}{1-e^{-\lambda h}}\right)\left(\frac{1-e^{-\lambda h}+h\lambda}{1-e^{-\lambda h}}\right)\end{aligned}$$

$$\begin{aligned}E(C_a) &= (b+en)(m_1+m_2) + c_2\left(h - E(\tau) + h\frac{1-\alpha_{11}}{\alpha_{11}}\right) + \\ & c_3\alpha_{01}m_1 + c_4\end{aligned}$$

$$m_1 = \left(\frac{e^{-\lambda h}}{1-e^{-\lambda h}}\right)\left(\frac{1-e^{-\lambda h}+h\lambda}{1-e^{-\lambda h}}\right)$$

$$m_2 = \frac{1}{\alpha_{11}}$$

Another objective function is given below. This is the result of assuming that the time to shift in the process mean is an Erlang(λ, 3) variate. This is further evidence that use of the Erlang family of distributions will give closed forms of the $ECPTU$ function.

$$E(L_{a1}) = \frac{3}{\lambda} + h - E(\tau) + h\frac{1-\alpha_{11}}{\alpha_{11}} + D$$

$$E(\tau) = \frac{3}{\lambda} - hm_1$$

$$m_1 = \frac{\left(-e^{-3\lambda h} + \frac{2}{e^{2h\lambda}} + e^{-(h\lambda)}\right) h^2\lambda^2}{2\left(1-e^{-(h\lambda)}\right)^3} + \frac{1 + \frac{h\lambda}{1-e^{-(h\lambda)}} - \frac{h^2\lambda^2}{2e^{h\lambda}\left(1-e^{-(h\lambda)}\right)}}{e^{h\lambda}\left(1-e^{-(h\lambda)}\right)}$$

$$E(C_{a1}) = (b+en)(m_1+m_2) + c_2\left(h - E(\tau) + h\frac{1-\alpha_{11}}{\alpha_{11}}\right) + c_3\alpha_{01}m_1 + c_4$$

3.5.1.2 Exploiting the lack of memory property

If it is assumed that the time to shift in the process mean is an exponential variate the lack of memory property of such random variables may be exploited in order to derive an objective $ECPTU$ function.

The expected length of a cycle is the sum of the expected length of three periods.

$$E(L) = E(T) + E(S) + E(D)$$

T and S are random variables representing the IC and OOC periods, respectively. D represents the time spent searching for an assignable cause. Figure 3.2 illustrates this cycle of operation of the $\bar{x}$-chart.

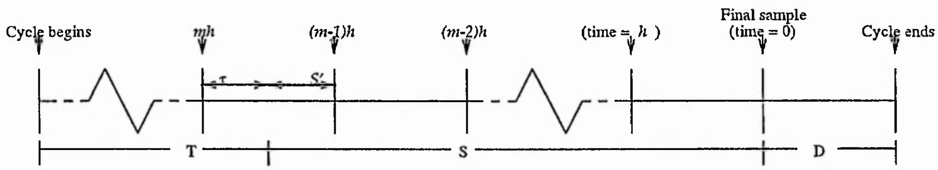


Figure 3.2: Illustration of a cycle of operation of the $\bar{x}$-chart based on a model which exploits the lack of memory property.

The expected cost of the cycle becomes the sum of the expected costs incurred in each period. This sum consists of the expected cost of sampling, the expected cost of operating in the OOC state, the expected cost of false alarms, and the expected cost of searching for an existing assignable cause.

T has an exponential distribution with scale parameter λ. $E(S)$ is derived using the lack of memory property of the variable T. $f(t)$ and $g(s)$, respectively, denote the probability density functions for the IC and OOC periods.

Since

$$f(t) = \lambda e^{-\lambda t}$$

then

$$E(T) = \frac{1}{\lambda}$$

$$E(S) = \sum_{m=1}^{\infty} \int_{s=(m-1)h}^{mh} sg(s)ds$$

In order to derive $g(s)$ we recall that $S = S' + (M-1)h$. See Figure 3.2. $S' = h - \tau$ is the random variable representing time from the shift in the process mean until the sampling inspection immediately following. τ is the random variable representing the length of the IC period within the interval in which the process mean shifts. M represents the number of samples taken during the OOC period. Therefore, $(M-1)h$ is the duration of the remaining sampling intervals in the OOC period. It is assumed that the process fails at time $\tau = t'$ from the start of the sampling interval of length h. We assume that S' is independent of $(M-1)h$.

Therefore,

$$g(s) = g(s'|t' < h)P(mh)u\left(m - \left(Int\left[\frac{s}{h}\right] + 1\right)\right)$$

for $mh \geq s \geq (m-1)h$.

$$u\left(m - \left(Int\left[\frac{s}{h}\right] + 1\right)\right) = \begin{cases} 1 & \text{if } m = Int\left[\frac{s}{h}\right] + 1 \\ 0 & \text{otherwise} \end{cases}$$

$g(s'|t' < h)$ is derived using the probability that the process remains in the IC state for a period $h - s'$ given that the process survived the previous interval of length h. $P(mh)$ is the probability that there are m sampling inspections and, therefore, $m-1$ complete sampling intervals until the shift in process mean is detected. The step function, $u(m)$, stipulates the range of $g(s)$.

The lack of memory property of exponential random variables means that whatever the present age of the process, the residual lifetime is unaffected by the past and has the same distribution as the lifetime itself.[32] For this reason τ is an exponential variate with parameter

λ since it is part of the IC period. The conditional cumulative distribution function of S' given $t' < h$ is used to derive the conditional probability density function for the variable S'.

$$\begin{aligned} G(s'|t' < h) &= P(S' \leq s'|t' < h) \\ &= 1 - P(S' > s'|t' < h) \\ &= 1 - \frac{P(S' > s', t' < h)}{P(t' < h)} \end{aligned}$$

$t' = h - S'$. It follows that, if $S' > s'$ then $t' < h - s'$. The cumulative distribution function is expressed as follows since t' is an exponential variate.

$$\begin{aligned} G(s'|t' < h) &= 1 - \frac{P(t' < h - s', t' < h)}{P(t' < h)} \\ &= 1 - \frac{1 - e^{-\lambda(h-s')}}{1 - e^{-\lambda h}} \end{aligned}$$

If X and Y are jointly continuous then the conditional cumulative distribution of Y given $X = x$ is defined as

$$F_{Y|X}(y|x) = \int_{-\infty}^{y} f_{Y|X}(z|x)dz$$

for all x such that $f_x(x) > 0$. [57] This means

$$f_{Y|X}(z|x) = \frac{dF_{Y|X}(z|x)}{dz}$$

Therefore,

$$\begin{aligned} g(s'|t' < h) &= \frac{dG(s'|t' < h)}{ds'} \\ &= \frac{\lambda e^{-\lambda(h-s')}}{1 - e^{-\lambda h}} \end{aligned}$$

$P(mh) = \alpha_{11}(1-\alpha_{11})^{m-1}$. Therefore,

$$g(s) = \frac{\lambda e^{-\lambda(h-s')}}{1-e^{-\lambda h}}\alpha_{11}(1-\alpha_{11})^{m-1}u\left(m - \left(Int\left[\frac{s}{h}\right]+1\right)\right)$$

for $mh \geq s \geq (m-1)h$.

Since $s' = s-(m-1)h$ the expression for $g(s)$ becomes

$$g(s) = \frac{\lambda e^{-\lambda(mh-s)}}{1-e^{-\lambda h}}\alpha_{11}(1-\alpha_{11})^{m-1}u\left(m - \left(Int\left[\frac{s}{h}\right]+1\right)\right)$$

Therefore, the expected period for which the process operates in the OOC state is as follows,

$$E(S) = \frac{h}{\alpha_{11}} - \frac{1}{\lambda} + \frac{he^{-\lambda h}}{1-e^{-\lambda h}}$$

Steps in the derivation of $E(S)$ are given in Section B.2 of Appendix B.

As a result the expressions for the expected length and cost of a cycle are as given below.

$$E(L_b) = \frac{h}{\alpha_{11}} + \frac{he^{-\lambda h}}{1-e^{-\lambda h}} + D$$

$$E(C) = E(C_1) + E(C_2) + E(C_3) + E(C_4)$$

$E(C_1)$ is the expected cost of sampling and inspection.

$E(C_2)$ is the expected cost of operating in the OOC state.

$E(C_3)$ is the expected cost of false alarms.

$E(C_4)$ is the cost of searching for the existing assignable cause.

$$
\begin{aligned}
E(C_1) &= (b+en)\left[\sum_{i=0}^{\infty}\int_{ih}^{(i+1)h} if(t)dt + \sum_{m=1}^{\infty}\int_{s=(m-1)h}^{mh} mg(s)ds\right] \\
&= (b+en)\left(\frac{e^{-\lambda h}}{1-e^{-\lambda h}} + \frac{1}{\alpha_{11}}\right)
\end{aligned}
$$

$$
\begin{aligned}
E(C_2) &= c_2 E(s) \\
&= c_2\left(\frac{h}{\alpha_{11}} - \frac{1}{\lambda} + \frac{he^{-\lambda h}}{1-e^{-\lambda h}}\right)
\end{aligned}
$$

$$
\begin{aligned}
E(C_3) &= c_3\alpha_{01}\left[\sum_{i=0}^{\infty}\int_{ih}^{(i+1)h} if(t)dt\right] \\
&= c_3 2\Phi(-k)\frac{e^{-\lambda h}}{1-e^{-\lambda h}}
\end{aligned}
$$

$$
E(C_4) = c_4
$$

Thus,

$$
\begin{aligned}
E(C_b) = {} & (b+en)\left(\frac{e^{-\lambda h}}{1-e^{-\lambda h}} + \frac{1}{\alpha_{11}}\right) + c_2\left(\frac{h}{\alpha_{11}} - \frac{1}{\lambda} + \frac{he^{-\lambda h}}{1-e^{-\lambda h}}\right) + \\
& c_3 2\Phi(-k)\frac{e^{-\lambda h}}{1-e^{-\lambda h}} + c_4
\end{aligned}
$$

3.5.1.3 Exploiting the discrete nature of the cycle length

In this section the expected length of a cycle is derived from a discrete probability density function. This discrete function arises because we regard the length of a cycle as an integer multiple of sampling intervals. That is, as illustrated in Figure 3.3, $L = rh$ so that

$$
E(L) = hE(R). \tag{3.5}
$$

R is the random variable representing the number of samples taken for the duration of the cycle.

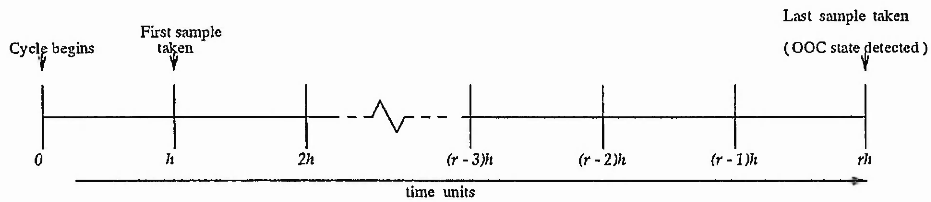


Figure 3.3: Illustration of a cycle which ends with the sampling inspection which indicates that the process is out of control.

$$E(R) = \sum_{r=1}^{\infty} rP(R=r)$$

$$P(R=r) = \sum_{i=0}^{r-1} \alpha_{11}(1-\alpha_{11})^{r-i-1} \int_{ih}^{(i+1)h} f(t)dt$$

$f(t)$ is the density function for the IC period. i represents the number of samples taken for the duration of the IC period.

The expected length of the cycle in Equation 3.5 does not account for the time spent searching for and removing the assignable cause. Hence the associated costs will not be incorporated into expected cost of the cycle. The expected costs which comprise $E(C)$ are

the expected cost of sampling and inspection, $E(C_1)$;

the expected cost of operating in the OOC state, $E(C_2)$; and

the expected cost of false alarms, $E(C_3)$.

$$E(C_1) = (b + en)E(R)$$

$$E(C_2) = c_2(hE(R) - E(T))$$

$E(T)$ is the expected length of the IC period.

$$E(C_3) = c_3\alpha_{01}\left[\sum_{i=0}^{\infty}\int_{ih}^{(i+1)h} if(t)dt\right]$$

When the time to shift in the process mean is assumed to be an Erlang(λ, 2) variate the expected length and cost of a cycle are written as follows,

$$E(L_c) = \frac{h}{\alpha_{11}} + \frac{he^{-\lambda h}}{1-e^{-\lambda h}} + \frac{\lambda h^2 e^{-\lambda h}}{(1-e^{-\lambda h})^2}$$

$$\begin{aligned} E(C_c) = {} & (b+en)\left(\frac{1}{\alpha_{11}} + \frac{e^{-\lambda h}}{1-e^{-\lambda h}} + \frac{\lambda h e^{-\lambda h}}{(1-e^{-\lambda h})^2}\right) + \\ & c_2\left(\frac{h}{\alpha_{11}} + \frac{he^{-\lambda h}}{1-e^{-\lambda h}} + \frac{\lambda h^2 e^{-\lambda h}}{(1-e^{-\lambda h})^2} - \frac{2}{\lambda}\right) + \\ & c_3 2\Phi(-k)\left(\frac{e^{-\lambda h}}{1-e^{-\lambda h}}\right)\left(\frac{1-e^{-\lambda h}+h\lambda}{1-e^{-\lambda h}}\right) \end{aligned}$$

3.5.2 Convolutions and the p.d.f. for R — the total number of samples for the duration of the cycle.

The probability function for R is derived from convolution of two discrete probability statements. Feller [32] states that if two independent random variables have densities f and g, respectively, the convolution of the two densities is denoted as $f * g$. This rule applies for discrete and continuous variables [32].

It will now be illustrated using the example of two independent random variables, X and Y, whose sum is Q. Feller [31] further states that if X and Y are two non-negative integral-valued random variables, the event $(X = j, Y = k)$ has probability $a_j b_k$. This is the case if $P\{X = j\} = a_j$, and $P\{Y = k\} = b_k$. For the new random variable $Q = X + Y$ the event $Q = q$ is the union of mutually exclusive events

$$(X = 0, Y = q), (X = 1, Y = q - 1), \ldots, (X = q, Y = 0).$$

The distribution $c_q = P\{Q = q\}$ is the sum of the probabilities of these events. This distribution is termed as the convolution of the sequences $\{a_j\}$ and $\{b_k\}$. [31]

Now, R, the random variable representing the total number of samples taken and, hence, the total number of sampling intervals for the duration of the cycle equals $I + M$. The variable I represents the number of samples taken for the duration of the IC period and $M = R - I$ the number taken during the OOC period. $\{P(I)\}$ and $\{P(R - I)\}$ are two numerical sequences. $I = 0, \ldots, R - 1$. The new sequence which defines the distribution $P\{R = r\}$ is defined by the convolution of $\{P(I)\}$ and $\{P(R - I)\}$ and can be written as follows,

$$\{P(R = r)\} = \{P(I)\}\{P(R - I)\}$$

The convolution is the sum of products of the probabilities that there are i samples before the process goes out of control and there are $m = r - i$ samples taken thereafter. The probability that there are i samples before a shift in the process mean can be written as

$$P(I) = \int_{ih}^{(i+1)h} f(t)dt$$

$$P(I) = \int_{ih}^{(i+1)h} f(t)dt$$

The probability that there are $(r-i)$ samples after the process goes out of control is written as follows,

$$P(R-I) = \alpha_{11}(1-\alpha_{11})^{r-i-1}$$

Since i can only take values ranging from zero to $r-1$ the density function for R is written as follows,

$$P(R=r) = \sum_{i=0}^{r-1} \alpha_{11}(1-\alpha_{11})^{r-i-1} \int_{ih}^{(i+1)h} f(t)dt$$

3.5.2.1 Characteristic features of $P(R=r)$

1. The sum of $P(R=r)$ over all R is 1.

   This summation will now be shown using the assumption that T is an Erlang(λ, 2) variate. Under this assumption $P(R=r)$ takes the following form,

$$P(R=r) \quad = \quad \sum_{i=0}^{r-1} \alpha_{11}(1-\alpha_{11})^{r-i-1} \int_{ih}^{(i+1)h} \frac{\lambda^2 t}{e^{\lambda t}} dt$$

   which equals

$$\sum_{i=0}^{r-1} (1-\alpha_{11})^{-1-i+r}\, \alpha_{11} \left(-\frac{h\,\lambda}{e^{h\,(1+i)\,\lambda}} + \frac{\left(1-e^{-(h\,\lambda)}\right)(1+h\,i\,\lambda)}{e^{h\,i\,\lambda}} \right)$$

   This means that

   $P(R=r) =$

$$(1-\alpha_{11})^{-1+r}\, \alpha_{11} \left(1-e^{-(h\,\lambda)}\right) \left(\sum_{i=0}^{-1+r} \frac{(1-\alpha_{11})^{-i}}{e^{h\,i\,\lambda}} \right) -$$

$$\frac{(1-\alpha_{11})^{-1+r}\,\alpha_{11}\,h\,\lambda\,\left(\sum_{i=0}^{-1+r}\frac{(1-\alpha_{11})^{-i}}{e^{h\,i\,\lambda}}\right)}{e^{h\,\lambda}} +$$

$$(1-\alpha_{11})^{-1+r}\,\alpha_{11}\,\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\left(\sum_{i=0}^{-1+r}\frac{(1-\alpha_{11})^{-i}\,i}{e^{h\,i\,\lambda}}\right)$$

$\frac{1}{(1-\alpha_{11})^{i}\,e^{h\,i\,\lambda}}$ can be written as $\left(\frac{1}{(1-\alpha_{11})\,e^{h\,\lambda}}\right)^{i}$. In order to obtain a closed form of the expression for $P(R=r)$ we assume that $\left|\frac{1}{(1-\alpha_{11})\,e^{h\,\lambda}}\right| < 1$. Application of the results for finite sums given by Prudnikov *et al* [61] yields the following result.

$P(R=r) =$

$$\frac{\left(1-e^{-(h\,\lambda)}-\frac{h\,\lambda}{e^{h\,\lambda}}\right)\left(-e^{-(h\,\lambda\,r)}+(1-\alpha_{11})^{r}\right)\alpha_{11}}{1-e^{-(h\,\lambda)}-\alpha_{11}} +$$
$$\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\alpha_{11}\,\left(-e^{-(h\,\lambda)-h\,\lambda\,r}+\frac{(1-\alpha_{11})^{r}}{e^{h\,\lambda}}\right)}{\left(1-e^{-(h\,\lambda)}-\alpha_{11}\right)^{2}} +$$
$$\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\alpha_{11}\,\left(e^{-(h\,\lambda)-h\,\lambda\,r}\,r-\frac{(1-\alpha_{11})\,r}{e^{h\,\lambda\,r}}\right)}{\left(1-e^{-(h\,\lambda)}-\alpha_{11}\right)^{2}}$$

When summed over all possible values of R, $P(R=r)$ reduces to the following expression which equals 1.

$$1-\frac{h\,\lambda}{e^{h\,\lambda}\,\left(1-e^{-(h\,\lambda)}\right)}+\frac{\left(1-\frac{\alpha_{11}}{1-e^{-(h\,\lambda)}}\right)\,h\,\lambda}{e^{h\,\lambda}\,\left(1-\alpha_{11}-e^{-(h\,\lambda)}\right)}=1$$

2. With x' as the dummy variable, the following moment generating function for R can be derived.

$$\begin{aligned} m(x') &= \sum_{r=1}^{\infty} e^{x' r} P(R=r) \\ &= \frac{e^{x'}\left(\frac{1-e^{-(h\,\lambda)}}{1-e^{-(h\,\lambda)+x'}}+\frac{\left(-1+e^{x'}\right)\,h\,\lambda}{e^{h\,\lambda}\,\left(1-e^{-(h\,\lambda)+x'}\right)^{2}}\right)\alpha_{11}}{1-e^{x'}\,(1-\alpha_{11})} \end{aligned}$$

This has been used to produce the characteristic moments about the origin and about the mean for R. We have therefore been able to derive the coefficients of skewness and kurtosis for the variate R.

3. Negative moments about the mean have also been derived. These will be used in Chapter 4.

Derivation of these features of the distribution function for R are further outlined in Appendix C.

3.6 Limiting Behaviour of ECPTU Functions

In this section we use the objective functions to investigate the limiting behaviour of a system of control which is based on use of the $\bar{x}$-chart. Plots of the functions are presented in Figures 3.5 and 3.6. Data in the first row of Table D.1 in Appendix D have been used to produce these plots.

There will be discussion of the implications of the responses of these functions to the extreme values of n, h and k — the control chart's design parameters. Examples of the ECPTU functions which will be considered are $ECPTU_a$; $ECPTU_{a1}$; $ECPTU_b$; and $ECPTU_c$. The values of the shape parameter, β, for the Erlang distribution assumption used to derive these functions are 2, 3,1 and 2, respectively. Unlike $ECPTU_c$, $ECPTU_a$ incorporates the cost and length of the period spent searching for the assignable cause. The objective functions are as follows.

$$ECPTU_a = \frac{E(C_a)}{E(L_a)} \tag{3.6}$$

$$ECPTU_{a1} = \frac{E(C_{a1})}{E(L_{a1})} \tag{3.7}$$

$$ECPTU_b = \frac{E(C_b)}{E(L_b)} \tag{3.8}$$

$$ECPTU_c = \frac{E(C_c)}{E(L_c)} \tag{3.9}$$

The functional forms of the expectations which comprise these ratios have been presented in Section 3.5.1.

k and n influence the ECPTU through their effect on, α_{11} and α_{01}, the probabilities associated with the operation of the $\bar{x}$–chart. α_{11} is the probability of detecting shift in the process mean and α_{01}, the probability of a false alarm. Figure 3.4 illustrates the relationship between n, k and α_{11} and α_{01}. n also affects the ECPTU through the cost of sampling which is dependent on this parameter. h is a more integral part of the ECPTU functions. Changes in this design parameter will therefore be more influential to the objective functions.

In order to derive general results for the limits of these functions we will assume that D and c_4, relative to other parameter values, are very small. Thus, the general forms of $E(C)$ and $E(L)$ are simplified to

$$E(L) = \sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} ihf(t)dt + h \sum_{m=1}^{\infty} m\alpha_{11}(1-\alpha_{11})^{m-1}$$

$$E(C) = (b+en)\left[\sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} if(t)dt + \sum_{m=1}^{\infty} m\alpha_{11}(1-\alpha_{11})^{m-1}\right] +$$

$$c_2\left[\sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} ihf(t)dt + h \sum_{m=1}^{\infty} m\alpha_{11}(1-\alpha_{11})^{m-1} - \int_0^{\infty} tf(t)dt\right] +$$

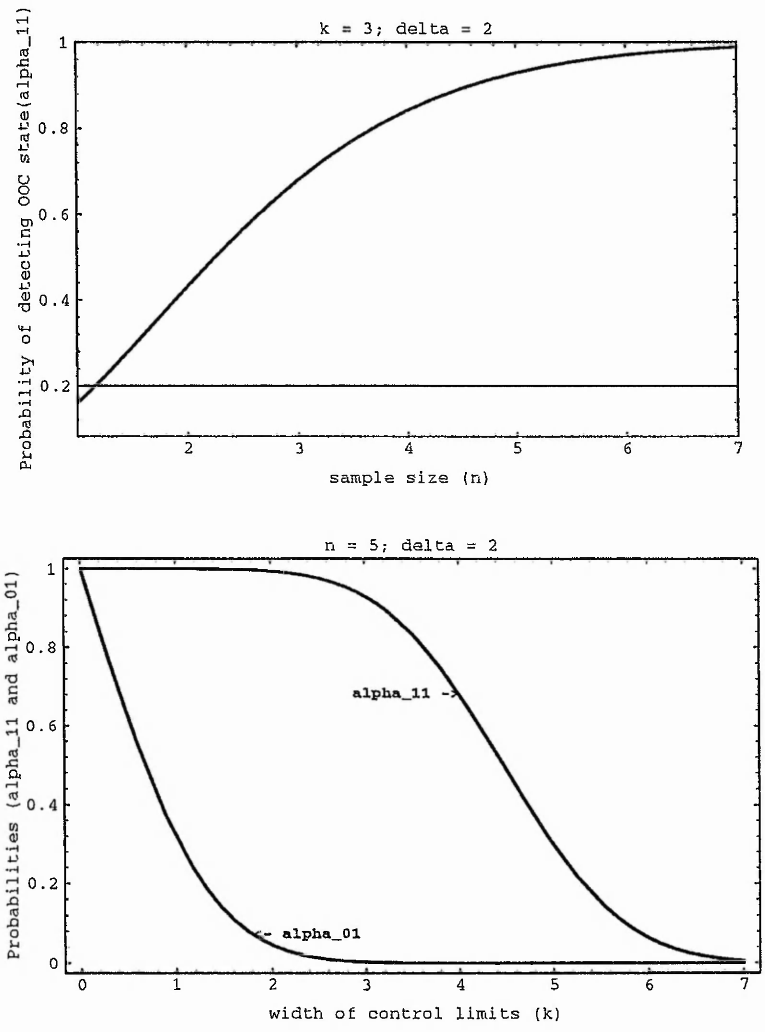


Figure 3.4: Plots of α_{11} versus n and of α_{11} and α_{01} versus k .

$$c_3 * \alpha_{01} * \left[\sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} if(t)dt\right]$$

The general results for the limits of the Equations 3.6 to 3.9 are as follows:-

As $h \longrightarrow \infty$, $ECPTU \longrightarrow c_2$.

As $h \longrightarrow 0$, $ECPTU \longrightarrow \infty$.

As $n \longrightarrow \infty$, $ECPTU \longrightarrow \infty$.

As $n \longrightarrow 1$, $\alpha_{11} \longrightarrow 0$ so that $ECPTU \longrightarrow$

$$\frac{b+e}{h} + c_2 - \frac{c_2 \int_0^{\infty} tf(t)dt}{\sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} ihf(t)dt} + \frac{c_3 * \alpha_{01}}{h} \tag{3.10}$$

As $k \longrightarrow \infty$, $\alpha_{11} \longrightarrow 0$ and $\alpha_{01} \longrightarrow 0$ so that $ECPTU \longrightarrow$

$$\frac{b+en}{h} + c_2 - \frac{c_2 \int_0^{\infty} tf(t)dt}{\sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} ihf(t)dt} \tag{3.11}$$

As $k \longrightarrow 0$, $\alpha_{11} \longrightarrow 1$ and $\alpha_{01} \longrightarrow 1$ so that $ECPTU \longrightarrow$

$$\frac{b+en}{h} + c_2 - \frac{c_2 \int_0^{\infty} tf(t)dt}{\sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} ihf(t)dt} + \frac{c_3}{h} \tag{3.12}$$

As $n \longrightarrow \infty$, the ECPTU approaches infinity. This is because, while $\alpha_{11} \longrightarrow 1$, the sampling cost approaches infinity. Thus $E(C) \longrightarrow \infty$ while $E(L)$ tends to a finite, fixed value. For D small relative to the length of the cycle this finite value is $h(m_1+1)$. This finite value is the expected length of the period of surveillance by the control chart when only one sample is taken during the OOC period. Figure 3.5 illustrates that $ECPTU_{a1}$ aproaches infinity more rapidly since it incorporates the largest of the shape parameter values.

As the $n \longrightarrow 1$, $\alpha_{11} \longrightarrow 0$. Thus, the OOC state is extended because its existence cannot be readily detected. The limit of ECPTU is predominated by the hourly costs of sampling one unit and of operating in the OOC state. See Equation 3.10. In Figure 3.5 we see that the limiting value as $n \longrightarrow 1$ is largest for $ECPTU_{a1}$.

As $k \longrightarrow 0$, $\alpha_{11} \longrightarrow 1$ and $\alpha_{01} \longrightarrow 1$. For D and c_4 small relative to the other cost and time parameters the ECPTU approaches a limit which accounts for the resulting proliferation of false alarms. This limit is also a function of the hourly costs of sampling and of operating in the OOC state. See Equation 3.12.

As $k \longrightarrow \infty$, $\alpha_{11} \longrightarrow 0$ and $\alpha_{01} \longrightarrow 0$. The length of a cycle is therefore extended because failure of the process is not readily detected. The limit of the ECPTU becomes the sum of the hourly costs of operating in the OOC state and of sampling. $c_{2'} = \$100.00$, $b = \$0.50$, $n = 5$, $h = 1$ hour and $e = \$0.10$. Therefore, in Figure 3.6 the functions all approximate to $\frac{b+en}{h} + c_2 = \$101.00$ as $k \longrightarrow \infty$.

The limit of the ECPTU, c_2, as $h \longrightarrow \infty$ indicates that a very long sampling interval could result in the process going out of control even before the end of the first sampling interval. The length of time between samples then delays the detection of the OOC state. There is even greater delay if there is a low probability that the process failure is detected. Hence, the expected length of a cycle approximates to the length of the OOC period and the expected cost becomes that of operating in the OOC state. Figure 3.6 therefore illustrates that the functions approach this limiting value of $c_2 = \$100$ per hour, the

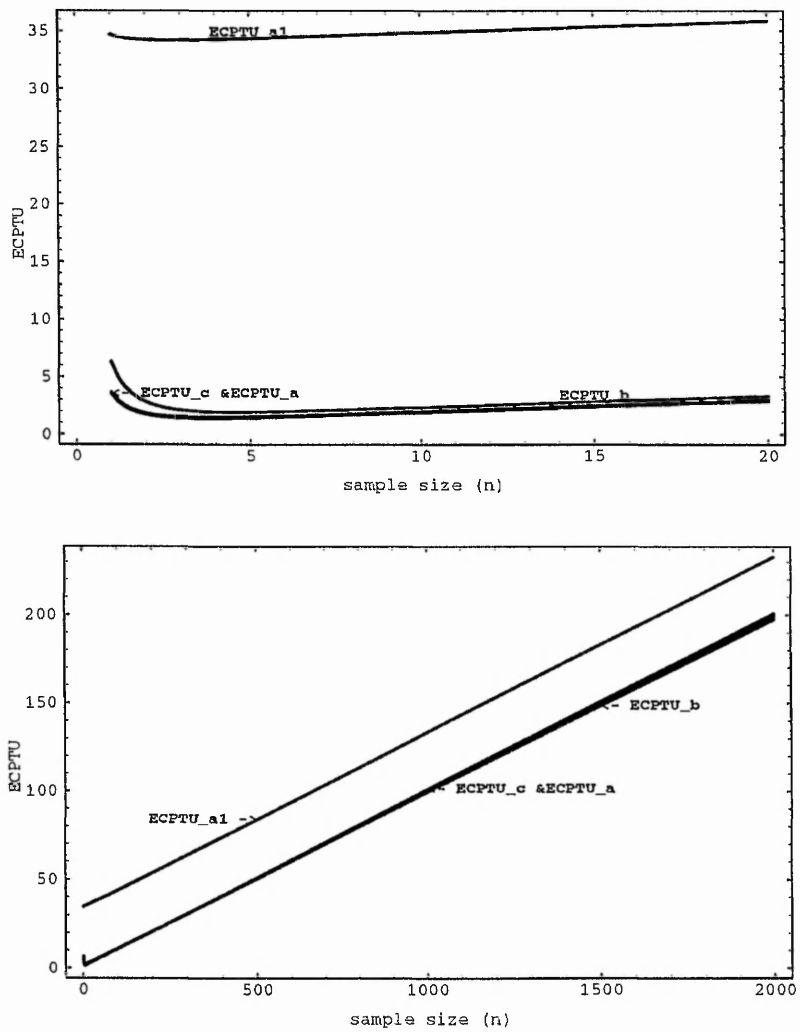


Figure 3.5: $ECPTU$ versus sample size (n) as n tends to one and to infinity.

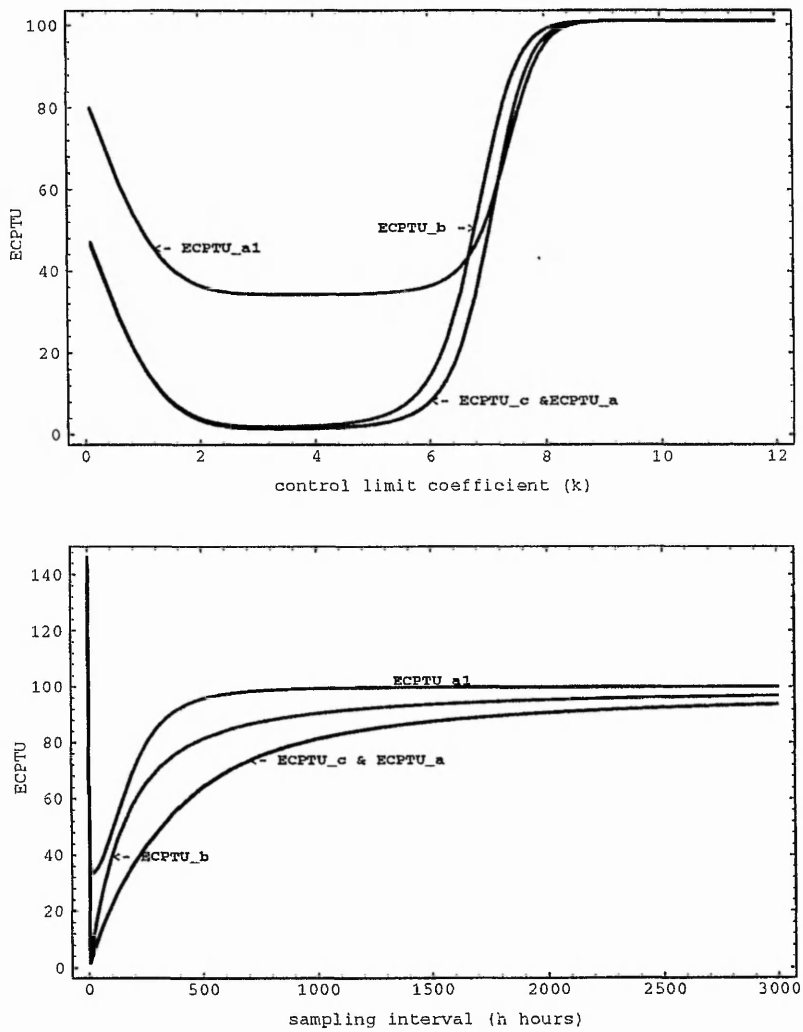


Figure 3.6: $ECPTU$ versus control limit coefficient (k) and sampling interval (h hours) as these parameters approach zero and infinity.

hourly cost of process operation during the OOC state. The limit is approached at a faster rate for larger values of β, the shape parameter. The limit as $h \longrightarrow 0$ indicates that increased sampling frequency causes a drastic increase in the ECPTU.

For the parameter values used, the plots illustrate little difference between the values of $ECPTU_a$ and $ECPTU_c$. Both these functions were derived using the Erlang distribution with shape parameter 2. The illustrations indicate, however, that changes to the value of the shape parameter for the time to shift in the process mean create differences in the response of ECPTU. These differences will be examined in the Section 3.8.

3.7 Use of the Objective Functions

In this section the objective functions are used to quantify the response of optimum design parameter values to changes in the production environment. Changes in the production environment are expressed in terms of increments in δ, r_{c23}, r_{c24} and β. δ is the shift coefficient. $r_{c23} = \frac{c_2}{c_3}$ and $r_{c24} = \frac{c_2}{c_4}$. c_2 is the hourly cost of operation during the OOC state. c_3 is the cost per occasion of searching when there is a false alarm. c_4 is the cost of searching for the existing assignable cause. β is the shape parameter for the distribution of the IC period. The penalty for use of suboptimal design parameter values will also be investigated.

In order to carry out the investigation, each objective function is, initially, minimised with respect to individual design parameters.

There is also minimisation of the ECPTU functions with respect to n, h amd k simultaneously. When there is optimisation with respect to a single design parameter the resulting combination of design parameter values may be suboptimal compared with results which could be obtained from simultaneous optimisation with respect all three design parameters. The practitioner who uses the suboptimal results must accept the penalty of increased ECPTU.

For minimisation with respect to individual parameters, there is selection of optimum value h for a given n and k; k for a given n and h; and n for a given k and h. The given design parameter values used in the numeric search are those tabulated in Section 2.1 of Chapter 2. These are $n = 4$ and 5, $k = 3$ and $h = 0.5$, 1, and 8 hours.

The minimum values of the ECPTU functions are found using the numerical minimisation function in the mathematical package *Mathematica*. This numeric minimisation procedure finds local minima using an iterative procedure. It searches for the local minimum using symbolic first partial derivatives of the function to be minimised. It works by following the path of steepest descent from the starting search value. [79] The maximum number of iterations used to produce the results shown was 1000 or more. Use of this large number of iterations was primarily aimed at increasing the probability of finding optimum values.

Data in Table D.1 of Appendix D are input values for the objective functions. Based on these optimum values the effects of changes in the production environment as well as the penalty for use of suboptimal parameter values have been quantified as percentages. Mean values for the percentages are tabulated in the subsequent sections of this chapter.

3.8 The Results

3.8.1 Effects of cost and technical parameters

There is an inverse relationship between the δ — the shift coefficient — and n^* and also between the c_2 — the hourly cost of operating the process in the OOC state — and h^*. These relationships were identified in sensitivity analyses done by Duncan [28] and Tagaras and Lee [73]. The results presented in this section illustrate that the ECPTU functions derived in this study appropriately model the expected response of each control chart parameter to changes in c_2, δ and the other design parameters.

Figure 3.7 illustrates, for all three functions, an increasing trend in n^* and h^* and a decreasing trend in the value of k^* as the example number increases. This is because the first 15 parameter sets have $\delta = 2$ while the last ten sets $\delta = 1$ or 0.5. This increasing trend in n^* leads to the decreasing trend in k^*. The lower values for c_2 among the final ten cases contribute to the increasing trend in the optimum value of h.

Table 3.1 reveals the extent to which the optimum design parameters are influenced by changes in δ. For a 50% reduction in the value of δ the optimum value of n can be expected to increase by over 100%. The

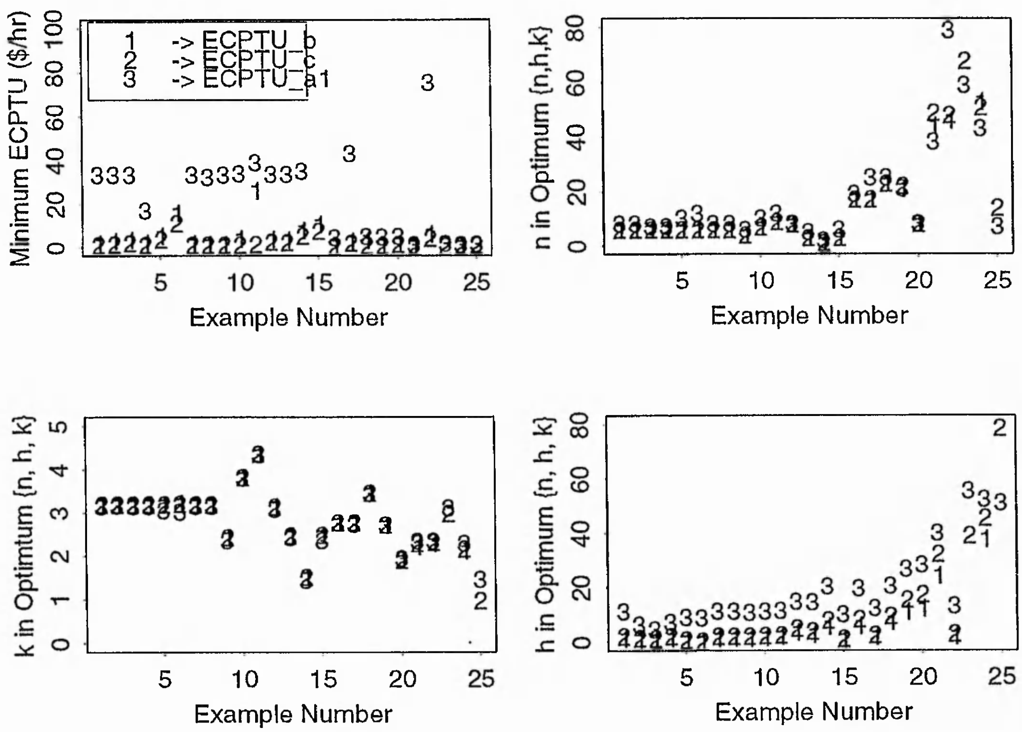


Figure 3.7: The optimum design parameters and the minimum ECPTU for the different objective functions.

optimum value of h responds similarly. For a 50% reduction in the value of δ a 10% reduction in the optimum value of k can be expected. The results in Table 3.1 correspond to the guidelines for n^* found in Duncan [28] and cited by Lorenzen and Vance [51]. They point out that the value of n^* is largely determined by δ, the magnitude of the shift. Table 3.2 gives the ranges for n^* for different ranges of δ values.

Shift Coefficient	$ECPTU_b$			$ECPTU_c$			$ECPTU_{a1}$		
	n	h	k	n	h	k	n	h	k
$\delta = 2$	6	1.7	3.05	6	2.4	3.05	8	10.7	2.98
$\delta = 1$	17	7.7	2.72	17	10.5	2.73	20	21.3	2.71
$\delta = 0.5$	48	21.4	2.21	46	39.5	2.15	45	42.4	2.3

Table 3.1: The mean values for the optimum combination of design parameters at different values of shift coefficient, δ.

	$\delta \geq 2$	$1 \leq \delta < 2$	$0.5 \leq \delta < 1$	$\delta < 0.5$
n^* [51]	2 – 10	10 – 20	20 – 40	—
n^* [28]	2 – 6	8 – 20	—	40 or more

Table 3.2: As cited by Duncan [28] and Lorenzen and Vance [51], expected ranges for n^* in response to values of δ.

Table 3.3 further illustrates the inverse response of h^* and n^* to increasing values of c_2. c_3 is the cost of searching for a non-existent assignable cause. c_4 is the expected cost of searching for the existing assignable cause. r_{c23} denotes the ratio of c_2 to c_3 and r_{c24} the ratio of c_2 to c_4. As r_{c23} or r_{c24} exceeds 1 a reduction of up to 75% in the optimum n can be expected. For the same increase in the ratios h^* can fall by as much as 66%. The results do not clearly indicate an inverse or proportional relationship between k^* and these ratios. k^* for different values of the ratio can be rounded to 3.

r_{c23}	r_{c24}	$ECPTU_b$			$ECPTU_c$			$ECPTU_{a1}$		
		n	h	k	n	h	k	n	h	k
< 1	< 1	23	12.8	2.92	27	25.0	2.75	24	31.6	2.82
$1 - 5$	$1 - 10$	10	2.1	2.85	10	3.0	2.85	15	11.3	2.81
> 5	> 10	5	0.7	2.83	5	1.0	2.83	9	9.5	2.69

Table 3.3: The mean values for the optimum combination of design parameters at different values of r_{c23}, the ratio of the c_2 to c_3 and r_{c24}, the ratio of c_2 to c_4.

Figure 3.8 illustrates that use of larger sampling intervals will require larger samples sizes for minimising the ECPTU. The percentages in Tables 3.4 and 3.5 further confirm the increment in n^* in response to a 30 minute increase in sampling interval.

r_{c23}	r_{c24}	$ECPTU_b$	$ECPTU_c$	$ECPTU_{a1}$
< 1	< 1	89	125	88
$1 - 5$	$1 - 10$	64	77	70
> 5	> 10	31	51	38

Table 3.4: For different ratios of the cost of operating in the OOC state, c_2, to the search costs, c_3 and c_4, the average percentage by which optimum n for $h = 1$ hr exceeds optimum n for $h = 0.5$ hr.

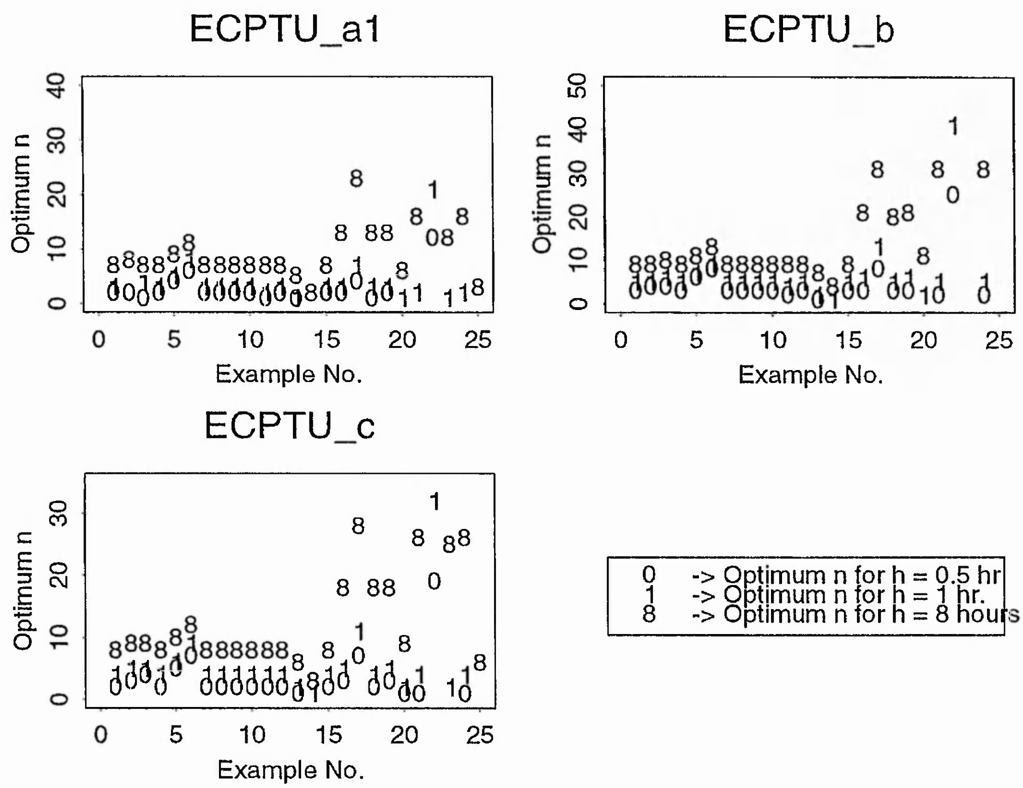


Figure 3.8: For different ECPTU functions, the optimum sample size (n) at different values of sampling interval.

Shift Coefficient	$ECPTU_b$	$ECPTU_c$	$ECPTU_{a1}$
$\delta = 2$	60	81	65
$\delta = 1$	79	78	90
$\delta = 0.5$	102	187	—

Table 3.5: For different values of the shift coefficient, δ, the average percentage by which optimum n for $h = 1$ hr exceeds optimum n for $h = 0.5$ hr.

Table 3.4 shows that the optimum value of n could increase by as much as 125% in response to a 30 minute increase in the value of h. This large difference can occur if the ratio of c_2 to c_3 or c_4 is less than 1. The value of the increment decreases as this ratio increases. It finds an average as low as 31% as the ratio exceeds 1.

The value of δ also influences the increment in the optimum value of n in response to changes in h. The increment increases as δ decreases. For example, when $\delta = 2$, a 30 minute increase in h can yield an increment of about 60% in the optimal n. This increases to over 100% for $\delta = 0.5$. See Table 3.5. These findings suggest that it is crucial that the sample size is increased as sampling interval increases if use of the $\bar{x}$-chart must be optimised. This is particularly important if the ratio of c_2 to c_3 or c_4 is less than 1 and for $\delta < 2$.

As the sample size increases for a given k, α_{11}, the probability of detecting the OOC state, increases. See Figure 3.4 in Section 3.6. With a long sampling interval the process is likely to be out of control for a longer time prior to detection. An increase in the sample size facilitates more rapid detection of the presence of an assignable cause. Thus, a larger sample size shortens the length of the OOC period. Alexander *et al* [1] state that increased sampling frequency compensates for the reduced probability that the OOC state is detected — power of the statistical test — which results from reduced n. Figure 3.8 illustrates that use of the $\bar{x}$-chart is optimised through use of larger sample sizes with larger sampling intervals .

Figure 3.9 illustrates that for a one unit increase in sample size larger k values minimise the objective function. Table 3.6 shows that, as h increases, there is an increase in the mean increment in k in response to a one unit increase in sample size. For $h \leq 1$ hour an increment of 3 to 5% can be expected. For $h = 8$ hours this increment averages 6 to 12%.

Sampling Interval	$ECPTU_b$	$ECPTU_c$	$ECPTU_{a1}$
$h = 0.5$	4	4	3
$h = 1$	5	4	4
$h = 8$	10	12	7

Table 3.6: For different values of the sampling interval, h, the average percentage by which optimum k for $n = 5$ exceeds optimum k for $n = 4$.

The response of the optimum value of k to the sample size is a trade-off between a large probability of detecting the OOC state and a small probability of a false alarm. As n increases the optimum value of k increases. While the increasing value of n increases α_{11} the associated

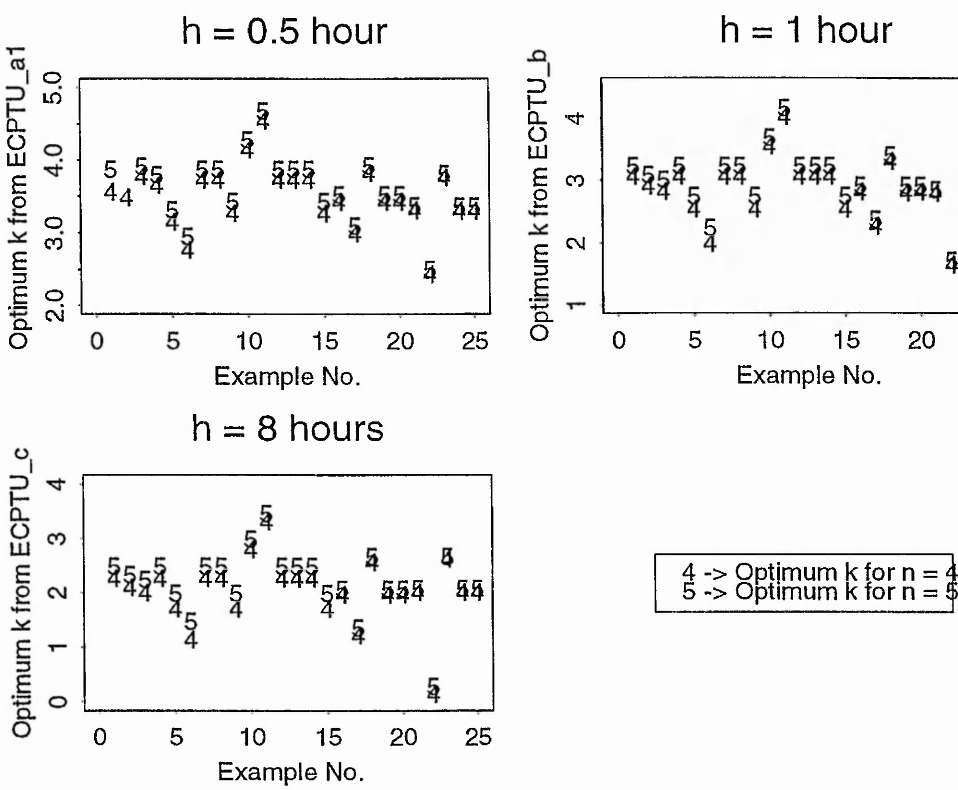


Figure 3.9: For different ECPTU functions and values of sampling interval, the optimum control limit coefficient (k) for $n = 5$ and $n = 4$.

increase in the k value which minimises the objective function will lower α_{01}, the probability of a false alarm. The corresponding lower optimum value of k for a smaller sample size illustrated in Figure 3.9 will increase the value of α_{11}. Figure 3.4 in Section 3.6 shows the relationship of k to α_{11} and α_{01}.

Figure 3.10 illustrates the inverse relationship between h and the value of k that will minimise the ECPTU. In response to a 30 minute reduction in h the increment in the optimum value of k averages 8 to 11%. This occurs whether $n = 4$ or 5. See Table 3.7. A seven hour reduction in h from 8 hours to 1 hour yields increments in excess of 30%. The findings in Table 3.8 suggest that the increment decreases as sample size increases.

Sample Size	$ECPTU_b$	$ECPTU_c$	$ECPTU_{a1}$
$n = 5$	10	9	8
$n = 4$	11	10	8

Table 3.7: For different values of the sample size (n) the average percentage by which the optimum k for $h = 0.5$ hr exceeds the optimum k for $h = 1$ hr.

Larger optimum values of k are produced when the sampling interval is smaller. Frequent checking of the process increases the possibility of false alarms. Therefore the optimum value of k increases to reduce α_{01}, the probability of a false alarm. Long sampling intervals could mean that, within an interval, the process remains in the OOC state for a longer period after it fails. Hence the optimum value of k decreases in

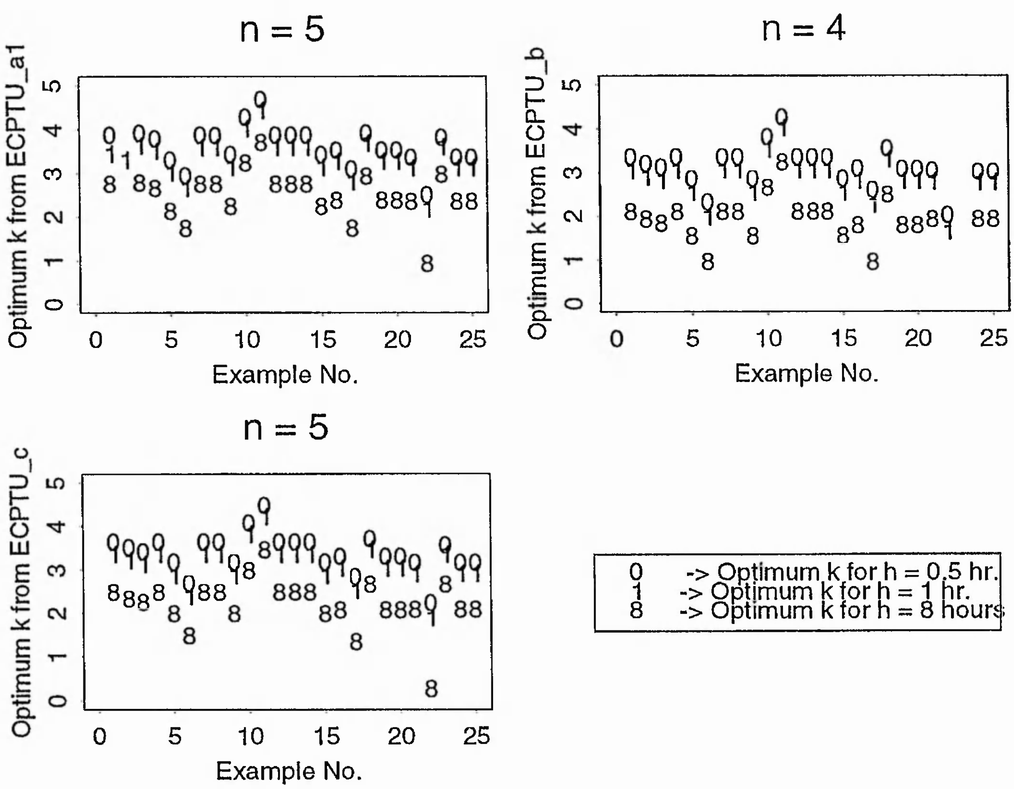


Figure 3.10: For different sample sizes and ECPTU functions, optimum control limit coefficient (k) for h =0.5 hr., 1 hr. and 8 hrs.

Sample Size	$ECPTU_b$	$ECPTU_c$	$ECPTU_{a1}$
$n = 5$	46	64	39
$n = 4$	55	90	43

Table 3.8: For different values of the sample size (n) the average percentage difference by which the optimum k for $h = 1$ hour exceeds the optimum k for $h = 8$ hours.

order to increase the probability that the OOC state is detected.

The results presented in this section indicate that, irrespective of the distribution assumed for the IC period, the objective ECPTU functions from the reduced model can illustrate the response of control chart design parameters to changes in other parameters. The results further indicate that, for a 50% reduction in δ, there should be at least a 50% increase in n and h in order to optimise use of the $\bar{x}$-chart. An attendant 10% reduction in k will also enhance optimisation. As r_{c23} or r_{c24} exceeds 1 a 60% reduction in h will facilitate optimised use of the control chart.

3.8.2 Effect of shape parameter values

β denotes the shape parameter for the distribution of the IC period. Figure 3.7 illustrates that larger values of β can yield larger optimum design parameter and $ECPTU$ values. The mean percentages given in Table 3.9 indicate that h^* is most greatly increased by a one unit change in β. The tabulated values further suggest that the increase in n^* for a one unit change in β will be more noticeable as this parameter exceeds 2.

In Figure 3.7 the optimum k obtained for all three functions appear to coincide for all twenty five sets of parameter values. This is confirmed in Table 3.9 where the mean difference between the optimum values of k is less than 1%.

The optimum values of n and h increase as the value of the shape parameter, β exceeds 1. The effect of the increase in β from 1 to 2 on n^* is, apparently, negligible since, for the data studied, the mean percentage increase in n^* is less than one percent. However, as β increases from 2 to 3 a mean 30% increase is observed. A similar level of increase in the value of h^* is observed for one unit increase in β.

The minimum $ECPTU$ decreases as β increases from 1 to 2. The percentage change in the minimum $ECPTU$ ranges from -95% to -27% in response to this one unit change in β. The negative difference follows because $ECPTU_c$, unlike $ECPTU_b$, excludes the parameters associated with search for the assignable cause. For $ECPTU_c$, $\beta = 2$ while for $ECPTU_b$, $\beta = 1$. However, as β increases beyond 2 the minimum ECPTU also increases drastically. See Table 3.9.

Difference	n	h	k	$ECPTU$
$(\beta = 3) - (\beta = 2)$	30	450	0.9	2965
$(\beta = 2) - (\beta = 1)$	0.7	38.3	0.4	-41.4

Table 3.9: The mean percentage by which the optimum parameter and function values produced by this study respond to a 1 unit increase in the shape parameter, β.

A reduction in process variation could facilitate increased length of the IC period. Thus, a larger shape parameter yielding increased mean time to failure could represent process improvement. This improvement makes a longer sampling interval and, in turn, a larger sample size more feasible. Increased n becomes feasible in response to process improvement particularly if the shift coefficient, δ, also becomes smaller.

It may not be practical to use the large n^* and h^* obtained when the time to shift in the process mean is assumed to be an Erlang$(\lambda, 3)$ variate. In Section 3.4 it was pointed out that procedures based on the exponential distribution tend to be highly non-robust and inferences may be sensitive to departures from the exponential model. Hence in order to obtain solutions of more practical value for which inferences are less sensitive to deviations from the model, the assumption that the IC period is an Erlang$(\lambda, 2)$ variate is more appropriate for use in selecting optimum design parameters.

3.8.3 The penalty for use of suboptimal values

The main penalties for use of suboptimal combinations of parameter values are reduced α_{11} — probability that the OOC state is detected; increased α_{01} — probability of false alarms; increased ECPTU for operation of the $\bar{x}$-chart. In this section the penalty for use of suboptimal parameter combinations $\{5$, 1 hour, $3\}$ and $\{5$, 8 hours, $3\}$ will be investigated.

The following expression will be used to calculate the percentages by which the heuristic probabilities, ratios or sampling intervals exceed or fall short of the optimum values.

$$\frac{heuristic - optimum}{optimum} \times 100$$

For $\delta = 2$, 1, and 0.5, the respective values for α_{11} are 0.9295, 0.2225, and 0.0299 when $n = 5$ and $k = 3$. When $k = 3$, $\alpha_{01} = 0.0027$. The n^* and k^* and the corresponding δ values have been used to calculate the probabilities α_{11} and α_{01}. Table 3.10 gives the mean values of α_{11} and α_{01} for different values of δ. The tabulated values indicate that the use of the suboptimal combination of $n = 5$ and $k = 3$ will most severely reduce the power of the test when $\delta < 2$. The results further suggest that, for $\delta = 2$, use of the n^* and k^* can shorten the average run length for the OOC period if the IC period is truly an Erlang(λ, 3) variate. The tabulated values also indicate that use of k^* instead of $k = 3$ increases α_{01}. This increase suggests that selection of k^* to minimise α_{01} may be a more suitable criterion for selecting optimum design parameters.

δ	$ECPTU_b$		$ECPTU_c$		$ECPTU_{a1}$	
	α_{11}	α_{01}	α_{11}	α_{01}	α_{11}	α_{01}
2	0.9219	0.01231	0.9216	0.01220	0.9876	0.01461
1	0.9058	0.01536	0.9057	0.01483	0.9318	0.01454
0.5	0.8867	0.02696	0.8740	0.08091	0.7627	0.04185

Table 3.10: The mean values of α_{11} and α_{01} for the optimum combination of design parameters at different values of shift coefficient, δ.

Use of $h > h^*$ can extend the OOC state. This or the increased sampling frequency for use of $h < h^*$ can lead to increased ECPTU. Table 3.11 gives the mean percentages by which $h = 1$ hour and $h = 8$ hours exceed or fall short of the h^*. The mean percentages tabulated are most negative for r_{c23} or $r_{c24} < 1$. This is the result of the inverse relationship between c_2 and h^*.

For use of $h = 8 > h^*$ the extension of the OOC state can be less than 10% for r_{c23} or $r_{c24} < 1$ when the IC period is an Erlang(λ, 2) variate. $h = 8$ falls short of h^* by less than 20% if r_{c23} or $r_{c24} \geq 1$ and the time to shift in the process mean is truly an Erlang(λ, 3) variate. Hence the penalty for use of $h = 8 < h^*$ can be at its lowest under these conditions. The tabulated values further indicate that, whatever the value of r_{c23} or r_{c24}, use of $h = 1$ will drastically increase the sampling frequency or extend the OOC period.

r_{c23}	r_{c24}	$ECPTU_b$		$ECPTU_c$		$ECPTU_{a1}$	
		1 hr	8 hrs	1 hr	8 hrs	1 hr	8 hrs
< 1	< 1	-75.4	68.7	-86.7	6.5	-95.7	-65.3
$1 - 5$	$1 - 10$	-38.2	394.5	-56.4	249.1	-89.5	-16.1
> 5	> 10	174.1	2092.4	94.8	1458.7	- 89.4	-15.2

Table 3.11: The mean percentages by which $h = 1$ hr and $h = 8$ hrs exceed or fall short of h^* at different values of r_{c23}, the ratio of c_2 to c_3 and r_{c24}, the ratio of c_2 to c_4.

Tables 3.12 and 3.13 give the mean percentages by which the $ECPTU(5, 1, 3)$ and $ECPTU(5, 8, 3)$ exceeds $ECPTU(n^*, h^*, k^*)$. The tabulated values suggest that the hourly cost penalty for use of either suboptimal combination of parameters will exceed 20% unless $\delta = 2$, r_{c23} and $r_{c24} \geq 1$, and the process is correctly described by $ECPTU_{a1}$. For $\delta = 1$ and the process described by $ECPTU_{a1}$ the mean penalty for use of the combination {5, 8 hours, 3} is also less than 20%. The very large percentages in all other cells of the tables indicate the very low efficiency and profitability with which the $\bar{x}$-chart could be used when the design parameter values are suboptimal.

δ	$ECPTU_b$		$ECPTU_c$		$ECPTU_{a1}$	
	$(5,1,3)$	$(5,8,3)$	$(5,1,3)$	$(5,8,3)$	$(5,1,3)$	$(5,8,3)$
2	431.7	8861	175.4	270.9	16.1	1.6
1	543.4	1425.8	263.0	293.0	57.3	12.3
0.5	475.9	1387.4	608.5	760.2	228.6	56.1

Table 3.12: The mean percentages by which $ECPTU(5,1,3)$ and $ECPTU(5,8,3)$ exceed $ECPTU(n^*,h^*,k^*)$ for different values of shift coefficient, δ.

r_{c23}	r_{c24}	$ECPTU_b$		$ECPTU_c$		$ECPTU_{a1}$	
		(5,1,3)	(5,8,3)	(5,1,3)	(5,8,3)	(5,1,3)	(5,8,3)
< 1	< 1	842.1	724.4	477.5	186.3	145.6	26.4
$1 - 5$	$1 - 10$	209.5	3725.5	181.8	427.5	19.0	9.2
> 5	> 10	324.7	24297.1	53.4	691.1	1.5	0.25

Table 3.13: The mean percentages by which $ECPTU(5,1,3)$ and $ECPTU(5,8,3)$ exceed $ECPTU(n^*,h^*,k^*)$ at different values of r_{c23} and r_{c24}.

From the findings presented in this section we can infer that, particularly for $\delta < 2$, an increased probability of a false alarm could result from use of k^* instead of $k = 3$. However, this increased probability is compensated by the reduced sampling frequency and the increased sample size which are optimum when $\delta < 2$. Thus the ECPTU will not be increased even though the expected number of false alarms could be increased if $k^*$is used. The penalty of reduced power of the test from use of $n = 5$ and $k = 3$ will be most costly if $\delta < 2$. The very large mean percentages given in Tables 3.12 and 3.13 imply that use of the suboptimal parameter combinations can be highly unprofitable.

3.9 Implications of the results

The probability density functions presented in this chapter can be used to study the probability distribution and other features of the OOC state and of the number of samples taken for the duration of the cycle.

Objective functions derived using a reduced model such as the one presented in this chapter effectively describe the operation of a system under the surveillance of the $\bar{x}$-chart and the response of design parameter values to changes in the production environment. This description becomes particularly useful if the changes in the production environment can be quantified using the cost and technical parameters of the model.

If the aim of use of the $\bar{x}$-chart is increased profitability, it is crucial that optimum design parameters are used for $\delta < 2$. Compared with the penalty for use of the combination {5, 1 hour, 3}, that for use of {5, 8 hours, 3} is lower when r_{c23} or $r_{c24} < 1$. If the IC period is an Erlang variate whose shape parameter is under 3 the penalty for use of {5, 1 hour, 3} is lower when r_{c23} or $r_{c24} \geq 1$.

Loss in the power of the test for the OOC state due to use of $n = 5$ and $k = 3$ is lowest when $\delta = 2$. It is essential that the n^* and k^* are used when $\delta < 2$ if the average run length for the OOC period is to be shortened with the consequent improved profitability.

Surtihadi and Raghavachari [71] state that h^* increases as the mean time to shift in the process mean increases. The results in Section 3.8.2 confirm this. The results indicate that a one unit increase in shape parameter which therefore increases the mean length of the IC period can increase h^* by over 30%. The researchers [71] state, however, that for practical purposes such large values of h^* are often of little interest. This implies that the optimum solutions produced by $ECPTU_{a1}$ which is based on the Erlang(λ, 3) variate may be of little practical value. However, such an objective function can be used to quantify effects of changes in the production environment on the operation of the control chart and on the behaviour of design parameters.

$ECPTU_c$ which is based on the Erlang(λ, 2) distribution gives solutions which have more practical value. This distribution assumption is also more appropriate for use, instead of the exponential distribution, to model failure time. In the derivation of $ECPTU_c$ the discrete nature of the cycle length is exploited. This facilitates extension of the model to incorporate stochastic restoration times which are correlated with the period, rh, for which the process has been monitored by the $\bar{x}$-chart.

In this chapter, as in the literature reviewed, we have assumed that the time and cost parameters are deterministic and independent of each other. Lorenzen and Vance [51] state that, in reality, these parameters may be stochastic. According to [51], the quantities enter the functions in a linear manner. Therefore, deterministic values can be replaced by expected values without changing the ECPTU. In the next chapter we will investigate the effect of explicitly accounting for the stochastic and correlated nature of the time spent searching for and removing the existing assignable cause.

Chapter 4

Stochastic Restoration Times

4.1 Extension of Previous Research

Generally in the literature, as in Chapter 3, the time spent identifying and removing the assignable cause has not been regarded as a stochastic variable. In the models developed by Lorenzen and Vance [51], Chung [14, 15], Duncan [28], Chung and Lin [16], Rahim *et al* [64], Banerjee and Rahim [4], Chiu and Huang [11], Tagaras and Lee [73] and Surtihadi and Raghavachari [71] the expected time to identify the assignable cause and the associated cost have fixed values. In these models, fixed values are also given to the expected time to remove the assignable cause when this is an input value for the objective ECPTU function. In Chapter 3 the time spent searching for the assignable cause also has a fixed value.

In this chapter we generalise the model to allow for a distribution of total time spent searching for and removing the assignable cause. We also allow for this time to be dependent upon the period which elapsed while the process was monitored by the $\bar{x}$-chart. It is assumed that removal of the assignable cause brings or restores the process to the IC state.

In a typical production environment the length of the period rh for which the process is monitored by the $\bar{x}$-chart can influence the length of the restoration period. For example, a shift in the process mean may have resulted from extensive tool wear [70] after a long period of surveillance. If such a shift moves the process to the OOC state, tool replacement may be required to ensure restoration to the IC state. Consequently, the restoration period may be long or short depending on the length of rh and, hence, the nature of action which must be taken to bring the process back to the IC state.

In previous research [4, 11, 51, 64, 71] on the economic design of the $\bar{x}$-chart, the term "repair" refers to activities concerned with removal of the assignable cause and is separate from the search for the assignable cause. In this chapter, as in models developed by Tagaras and Lee [73] and Chung [15], the term "repair" or "restoration" time will refer to the total period spent locating and removing the assignable cause.

4.2 The Extended Model

In previous chapters and other literature, the period spent restoring the process to the IC state has not been regarded as a random variable. In this chapter the repair periods will be regarded as variates which have increasing hazard rates. Since the Weibull and Gamma family of distributions can have increasing hazard rates it is suitable to assume that the times spent bringing the process back to the IC state can have such distributions.

For these distributions the symbols α and β will represent the scale parameter and shape parameter, respectively. Derivation of the expected restoration times will be based on the following forms of the probability density functions for Weibull and Gamma variate. For the Weibull variate

$$\beta x^{\beta-1}\alpha^{\beta}e^{-(\alpha x)^{\beta}}$$

For the Gamma variate

$$\frac{x^{\beta-1}\alpha^{\beta}e^{-\alpha x}}{(\Gamma(\beta)}$$

We assume that repair time follows the Erlang distribution which is a member of the Gamma family of distributions.

The relationship of the repair time to the period, rh, for which the process is monitored by the $\bar{x}$-chart is forged through the functions for the scale parameter. The scale parameter is assumed to be a function of rh. Here we model the relationship by $\alpha = a + vrh$ and $\alpha = a + \frac{v}{rh}$.

Four different examples of functions for the expected time to restore the process to the IC state will be presented. The symbol Z_i (i =

$1 \ldots 4$) will be used for the repair time variate in each example. The random variables for the total time spent searching for and removing the assignable cause are denoted as follows:-

Z_1 — time which follows the Erlang$(a + vrh, \beta)$ distribution.

Z_2 — time which follows the Weibull$(a + vrh, \beta)$ distribution.

Z_3 — time which follows the Erlang$(a + \frac{v}{rh}, \beta)$ distribution.

Z_4 — time which follows the Weibull$(a + \frac{v}{rh}, \beta)$ distribution.

For the distribution assumptions used, $E(Z_1)$ to $E(Z_4)$ — the expected time to restore the process to the IC state — are expressed as power series in the variable R. In the sections which follow general forms of these series are presented and the conditions for their convergence is explained. These series enable derivation of general results as well as sequential approximations for the expected repair time.

$ECPTU_c$ as developed in Chapter 3 will be extended to incorporate $E(Z_i)$, functions for the expected repair time. Four general ECPTU functions are produced when the different $E(Z_i)$ are incorporated into the basic objective function. These ECPTU functions will be denoted as $ECPTU_{Zi}$ $(i = 1 \ldots 4)$.

With the parameter values used by Duncan [28] and in Chapter 3, a numeric search procedure is carried out to find the optimum combination of sample size (n), sampling interval (h), and control limit coefficient (k). The results obtained from the $ECPTU_{Zi}$ are compared with those obtained when $ECPTU_c$ is minimised.

Explicit reckoning of repair time as a stochastic variate correlated with the period rh most appreciably increments the optimum value of h. The increments are larger when Z_i is an inverse instead of a linear function of rh. Larger increments are also obtained when Z_i is regarded as an Erlang instead of a Weibull variate. The calculated penalty for use of suboptimal values of n, h, and k is lower when the ECPTU functions incorporate the inverse relationship between Z_i and rh. These findings will be further illustrated and discussed later in this chapter.

4.2.1 Restoration time versus rh

When $\alpha = a + \frac{v}{rh}$ the rate of restoring the process to the IC state, α, decreases as the period rh increases. This relationship suggests that the mean time to restoration will be greater for a longer surveillance period. This definition of α describes the case in which extended length of the surveillance period makes restoration of the process extensive and time-consuming.

For $\alpha = a + vrh$, the rate of restoration is a direct linear function of the period for which the process is monitored by the $\bar{x}$-chart. This suggests that more restoration of the process is faster after an extended period of monitoring. For example, the process mean may shift to the OOC state when extended production leads to the breakdown of machinery. Rapid replacement of equipment could then restore the process to the IC state.

In order to illustrate the relationship between restoration time and the period spent monitoring the process using the $\bar{x}$-chart we will let $\beta = 1$. With restoration periods having distributions from the Weibull or Gamma family of distributions the mean restoration time is $\frac{1}{\alpha}$. For these examples, we assume $a > 0$ and $v > 0$.

- $\alpha = a + \frac{v}{rh}$

  Here the mean time to process restoration is $\left(a + \frac{v}{rh}\right)^{-1}$. Thus, as r and, therefore, rh increases to infinity the expected restoration rate decreases to the asymptote a. This results in expected repair time increasing to the asymptote a^{-1}. This is illustrated in Figure 4.1. As rh approaches 0 the restoration rate tends to infinity so that the expected restoration time approaches 0. This model for the scale parameter describes processes for which there is a limited period allotted for restoration of the period to the IC state. As rh increases the repair period approaches this limit.

- $\alpha = a + vrh$

  Figure 4.2 illustrates the linear relationship between restoration rate (α) and r and, therefore, rh. The resulting inverse relationship between restoration time ($\frac{1}{\alpha}$) and r and, therefore, rh is also illustrated in the picture. As rh approaches infinity the expected repair time approaches 0 as the restoration rate tends to infinity. As rh tends to 0, $\alpha \longrightarrow a$, so that the $\frac{1}{\alpha}$ increases to a^{-1}. See Figure 4.2. This model describes the process for which the repair period is longest when the process is not monitored by the $\bar{x}$-chart.

4.3 General Results For $E(Z_i)$

Distributions from the Weibull and Gamma families have mean values expressed as functions of their shape and scale parameters. The mean is inversely related to the scale parameter. With the scale and shape parameters represented by α and β, respectively, the general form of the mean for these distributions is $\frac{l(\beta)}{\alpha}$. The numerator for this ratio, $l(\beta)$, is a general function for β.

4.3.1 $\alpha = a + \frac{v}{rh}$

When $\alpha = a + \frac{v}{rh}$ the mean time to restore the process conditional on the value of R has the following general form.

$$\begin{aligned} E(Z_i|R=r) &= \frac{l(\beta)}{(a+\frac{v}{rh})} \\ &= \frac{l(\beta)}{a}\left(\frac{v}{arh}+1\right)^{-1} \end{aligned}$$

$$E\left(Z_i\right) = E_R\left(E\left(Z_i|R=r\right)\right)$$

Therefore, the expected restoration time can be expressed as follows.

$$\begin{aligned} &\sum_{r=1}^{\infty}\frac{l(\beta)}{a}\left(\frac{v}{arh}+1\right)^{-1}P(R=r) \\ = &\sum_{r=1}^{\infty}\frac{l(\beta)}{a}\left(1-\frac{v}{rha}+\left(\frac{v}{arh}\right)^2-\left(\frac{v}{arh}\right)^3+\ldots\right)P(R=r)+ \\ &\sum_{r=1}^{\infty}\frac{l(\beta)}{a}\left(\ldots+\left(-\frac{v}{arh}\right)^{q-1}+\ldots\right)P(R=r) \end{aligned}$$

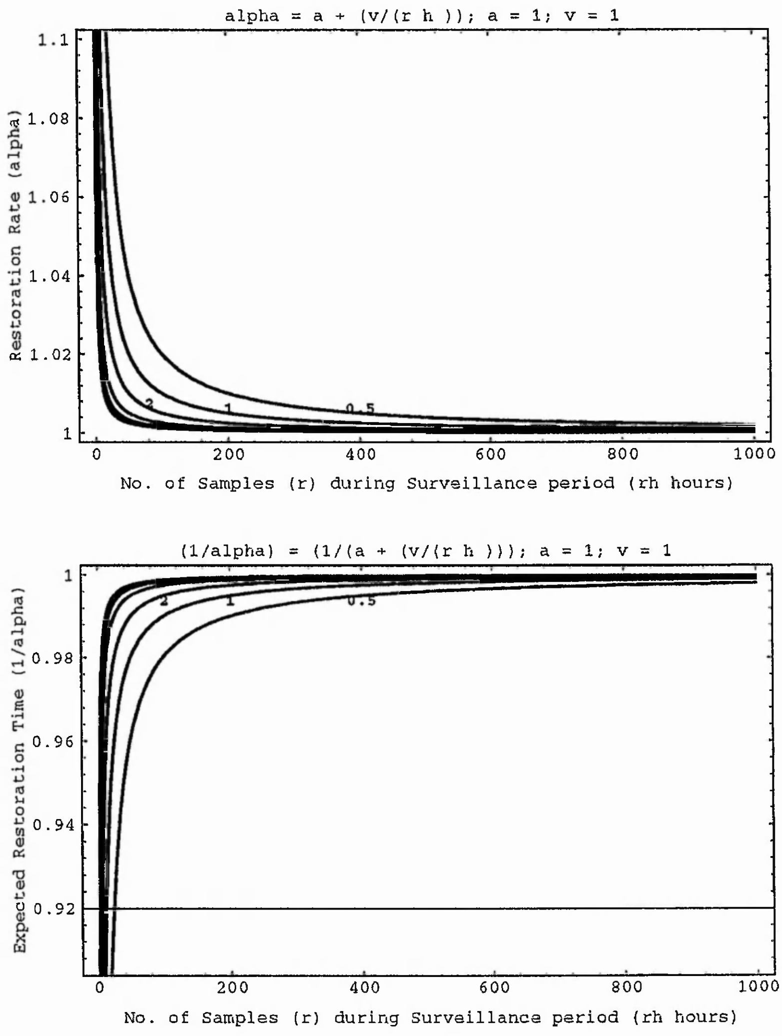


Figure 4.1: Restoration rate versus r and expected restoration time versus r when $\alpha = a + \frac{v}{rh}$ and h ranges from 0.5 to 8 hours.

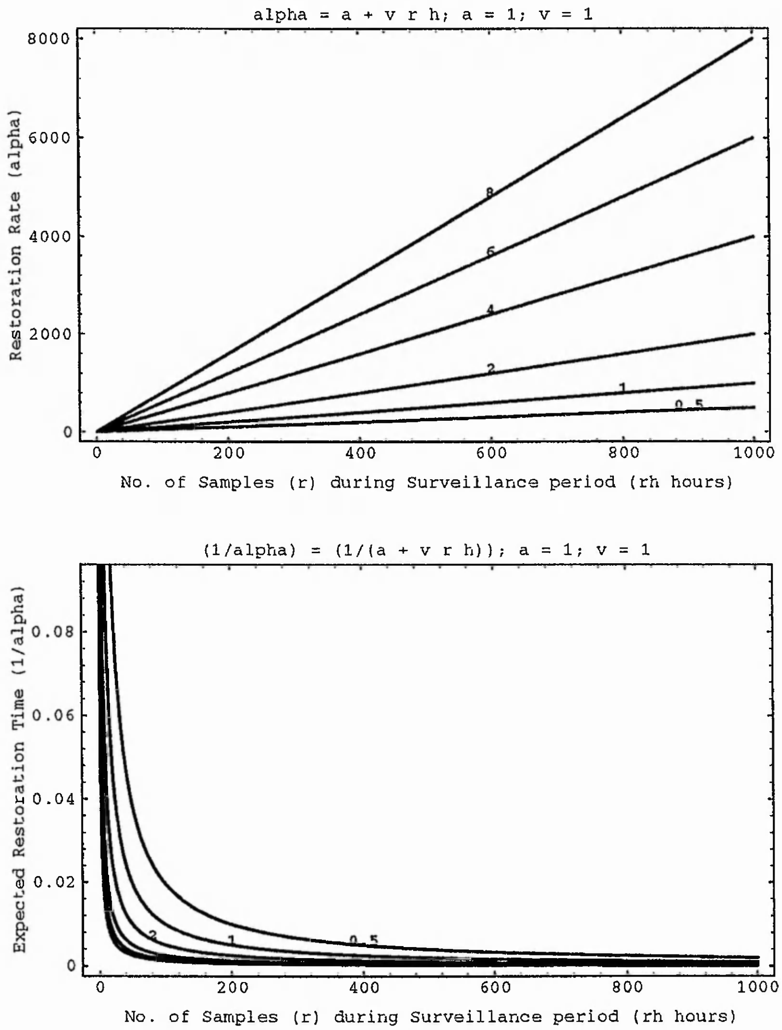


Figure 4.2: Restoration rate versus r and expected restoration time versus r when $\alpha = a + vrh$ and h ranges from 0.5 to 8 hours.

$$= \frac{l(\beta)}{a}\left(\sum_{q=0}^{\infty}\left(\frac{v}{ah}\right)^{2q} E(R^{-2q}) - \sum_{q=0}^{\infty}\left(\frac{v}{ah}\right)^{2q+1} E(R^{-(2q+1)})\right)$$

This means

$$E(Z_i) = w\left(\sum_{q=0}^{\infty}\left(\frac{v}{ah}\right)^{2q} E(R^{-2q}) - \sum_{q=0}^{\infty}\left(\frac{v}{ah}\right)^{2q+1} E(R^{-(2q+1)})\right)$$

$w = \frac{l(\beta)}{a}$ is the conditional mean of the variable Z_i when v is set to zero.

That is,

$$w = E(Z_i|R = r)_{v=0}$$

The multiplier, w , can be expressed as shown in Table 4.1 for expected restoration time following the Gamma or Weibull distributions.

Function for α	Distribution assumed	
	$\gamma : \alpha, \beta$	$W : \alpha, \beta$
$\alpha = a + \frac{v}{rh}$	$\frac{\beta}{a}$	$\frac{\Gamma(\frac{1+\beta}{\beta})}{a}$

Table 4.1: Tabulation of the expressions for the multiplier, w, when the mean restoration time is based on the Gamma or Weibull distributions.

4.3.2 $\alpha = a + vrh$

When $\alpha = a + vrh$ the mean time taken to restore the IC state conditional on the value of R has the following general form.

$$\begin{aligned} E(Z_i|R = r) &= \frac{l(\beta)}{(a + vrh)} \\ &= \frac{l(\beta)}{vrh}\left(\frac{a}{vrh} + 1\right)^{-1} \end{aligned}$$

Now

$$\begin{aligned} E(Z_i) &= E_R(E(Z_i|R=r)) \\ &= \sum_{r=1}^{\infty} \frac{l(\beta)}{vrh} \left(\frac{a}{vrh}+1\right)^{-1} P(R=r) \end{aligned}$$

Therefore the expected restoration time can be expressed as the following series expansion.

$$\sum_{r=1}^{\infty} \frac{l(\beta)}{vh} \left(\frac{1}{r} - \frac{a}{vh}\frac{1}{r^2} + \left(\frac{a}{vh}\right)^2 \frac{1}{r^3} - \left(\frac{a}{vh}\right)^3 \frac{1}{r^4} + \dots\right) P(R=r) +$$

$$\sum_{r=1}^{\infty} \frac{l(\beta)}{vh} \left(\dots + \left(-\frac{a}{vh}\right)^{q-1} \frac{1}{r^q} + \dots\right) P(R=r)$$

$$= \frac{l(\beta)}{vh} \left(\sum_{q=0}^{\infty} \left(\frac{a}{vh}\right)^{2q} E(R^{-(2q+1)}) - \sum_{q=0}^{\infty} \left(\frac{a}{vh}\right)^{2q+1} E(R^{-(2q+2)}) \right)$$

This means

$$E(Z_i) = y \left(\sum_{q=0}^{\infty} \left(\frac{a}{vh}\right)^{2q} E(R^{-(2q+1)}) - \sum_{q=0}^{\infty} \left(\frac{a}{vh}\right)^{2q+1} E(R^{-(2q+2)}) \right)$$

y is the conditional expectation of the variable Z_i when r is equal to one and a equals zero.

That is,

$$y = E(Z_i|R=r)_{a=0,r=1}$$

The multiplier, y, can be expressed as shown in Table 4.2 for expected restoration time following the Gamma or Weibull distributions.

The subsection which follows explains the conditions for convergence of the general results for the expected restoration time.

4.3.3 Explanation of convergence

Abel's test for convergence states that if the series $\sum U_n$ converges and $\{a_n\}$ is a bounded monotonic sequence then the series $\sum U_n a_n$ converges. [44]

Function for α	Distribution assumed	
	$\gamma : \alpha, \beta$	$W : \alpha, \beta$
$\alpha = a + vrh$	$\frac{\beta}{vh}$	$\frac{\Gamma(\frac{1+\beta}{\beta})}{hv}$

Table 4.2: Tabulation of the expressions for the multiplier, y, when the mean restoration time is based on the Gamma or Weibull distributions.

We recall that for $\alpha = a + \frac{v}{rh}$, $E(Z_i) =$

$$w\left(\sum_{q=0}^{\infty}\left(\frac{v}{ah}\right)^{2q} E(R^{-2q}) - \sum_{q=0}^{\infty}\left(\frac{v}{ah}\right)^{2q+1} E(R^{-(2q+1)})\right) \qquad (4.1)$$

The series $\sum_{q=0}^{\infty}\left(\frac{v}{ah}\right)^{2q}$ converges as shown below if and only if $\left|\frac{v}{ah}\right| < 1$.

$$\sum_{q=0}^{\infty}\left(\frac{v}{ah}\right)^{2q} = \left(1 - \left(\frac{v}{ah}\right)^2\right)^{-1}$$

For $\alpha = a + vrh$, $E(Z_i) =$

$$y\left(\sum_{q=0}^{\infty}\left(\frac{a}{vh}\right)^{2q} E(R^{-(2q+1)}) - \sum_{q=0}^{\infty}\left(\frac{a}{vh}\right)^{2q+1} E(R^{-(2q+2)})\right) \qquad (4.2)$$

The series $\sum_{q=0}^{\infty}\left(\frac{a}{vh}\right)^{2q}$ converges as shown below if and only if $\left|\frac{a}{vh}\right| < 1$.

$$\sum_{q=0}^{\infty}\left(\frac{a}{vh}\right)^{2q} = \left(1 - \left(\frac{a}{vh}\right)^2\right)^{-1}$$

$\left\{E\left(R^{-(2q+l)}\right)\right\}$ is the general expression for the bounded monotonic decreasing sequences in Equations 4.1 and 4.2. $l = 0, 1, 2$. This is because the variable R takes values greater than or equal to one. Consequently, $0 < E\left(R^{-(2q+l)}\right) < 1$ and each term in the sequence decreases as q increases.

$$\sum_{q=0}^{\infty} (H(a))^{2q} \left\{ E\left(R^{-(2q+l)}\right) \right\} \quad \text{and} \quad \sum_{q=0}^{\infty} (H(a))^{2q+1} \left\{ E\left(R^{-(2q+l)}\right) \right\}$$

are general expressions for the series which comprise Equations 4.1 and 4.2. $H(a)$ denotes the ratios $\frac{v}{ah}$ and $\frac{a}{vh}$. Provided $\left|\frac{v}{ah}\right| < 1$ and $\left|\frac{a}{vh}\right| < 1$, the series will converge.

According to Tranter [76], if $\sum U_n$ converges to s and $\sum V_n$ converges to t then $\sum (U_n \pm V_n)$ converges to $s \pm t$. The expectations in Equations 4.1 and 4.2 are expressed as the difference of two convergent series. This implies the series given here for the expected restoration time will also converge.

In view of this convergence the series for the expectations will be approximated as shown below and used in the ECPTU functions.

For $\alpha = a + vrh$

$$E(Z_i) = y\left(E(R^{-1}) - \left(\frac{a}{vh}\right) E(R^{-2}) + \left(\frac{a}{vh}\right)^2 E(R^{-3})\right)$$

For $\alpha = a + \frac{v}{rh}$

$$E(Z_i) = w\left(1 - \left(\frac{v}{ah}\right) E(R^{-1}) + \left(\frac{v}{ah}\right)^2 E(R^{-2})\right)$$

4.4 Illustrating $E(Z_i)$

The forms of the functions for the $E(Z_i))$ which will be incorporated into the ECPTU functions are as given in Equations 4.3 to 4.6.

$$E(Z_1) = \frac{\beta}{hv}\left(E(R^{-1}) - \left(\frac{a}{vh}\right) E(R^{-2}) + \left(\frac{a}{vh}\right)^2 E(R^{-3})\right) \quad (4.3)$$

$E(Z_2) =$

$$\frac{\Gamma(\frac{1+\beta}{\beta})}{h\,v}\left(E(R^{-1}) - \left(\frac{a}{hv}\right)E(R^{-2}) + \left(\frac{a}{hv}\right)^2 E(R^{-3})\right) \qquad (4.4)$$

$$E(Z_3) = \frac{\beta}{a}\left(1 - \left(\frac{v}{ah}\right)E(R^{-1}) + \left(\frac{v}{ah}\right)^2 E(R^{-2})\right) \qquad (4.5)$$

$$E(Z_4) = \frac{\Gamma(\frac{1+\beta}{\beta})}{a}\left(1 - \left(\frac{v}{ah}\right)E(R^{-1}) + \left(\frac{v}{ah}\right)^2 E(R^{-2})\right) \qquad (4.6)$$

$ECPTU_c$, the objective function which will be extended for incorporation of repair time, is based on the assumption that the time to shift in the process mean is an Erlang(λ, 2) variate. Consequently, we derive the negative moments and expectations given in Equations 4.3 to 4.6 using the probability function for R which is also based on the assumption that the IC period is an Erlang(λ, 2) variate. The objective function for $P(R)$ is given in Section 3.5.2.1. The expectations given below were derived using solutions to series summation given by Prudnikov *et al* [61].

$$E(R^{-1}) = \alpha_{11}\left(-\frac{h\,\lambda}{e^{h\,\lambda}\,(1 - e^{-(h\,\lambda)} - \alpha_{11})}\right) +$$

$$\alpha_{11}\left(\frac{\left(1 - e^{-(h\,\lambda)}\right)h\,\lambda\,\log(\frac{1-e^{-(h\,\lambda)}}{\alpha_{11}})}{e^{h\,\lambda}\,(1 - e^{-(h\,\lambda)} - \alpha_{11})^2} + \frac{\left(1 - e^{-(h\,\lambda)} - \frac{h\,\lambda}{e^{h\,\lambda}}\right)\log(\frac{1-e^{-(h\,\lambda)}}{\alpha_{11}})}{1 - e^{-(h\,\lambda)} - \alpha_{11}}\right)$$

$$E(R^{-2}) = \alpha_{11}\left(\frac{\left(1 - e^{-(h\,\lambda)}\right)h\,\lambda\,\log(1 - e^{-(h\,\lambda)})}{1 - e^{-(h\,\lambda)} - \alpha_{11}}\right) +$$

$$\alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\left(-\mathrm{PolyLog}(2,e^{-(h\,\lambda)})+\mathrm{PolyLog}(2,1-\alpha_{11})\right)}{e^{h\,\lambda}\,(1-e^{-(h\,\lambda)}-\alpha_{11})^2}\right)+$$

$$\alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}-\frac{h\,\lambda}{e^{h\,\lambda}}\right)\,\left(-\mathrm{PolyLog}(2,e^{-(h\,\lambda)})+\mathrm{PolyLog}(2,1-\alpha_{11})\right)}{1-e^{-(h\,\lambda)}-\alpha_{11}}\right)$$

$$E(R^{-3}) \;=\; \alpha_{11}\left(-\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\mathrm{PolyLog}(2,e^{-(h\,\lambda)})}{1-e^{-(h\,\lambda)}-\alpha_{11}}\right)+$$

$$\alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\left(-\mathrm{PolyLog}(3,e^{-(h\,\lambda)})+\mathrm{PolyLog}(3,1-\alpha_{11})\right)}{e^{h\,\lambda}\,(1-e^{-(h\,\lambda)}-\alpha_{11})^2}\right)+$$

$$\alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}-\frac{h\,\lambda}{e^{h\,\lambda}}\right)\,\left(-\mathrm{PolyLog}(3,e^{-(h\,\lambda)})+\mathrm{PolyLog}(3,1-\alpha_{11})\right)}{1-e^{-(h\,\lambda)}-\alpha_{11}}\right)$$

In addition to rh, the parameters a and v will also influence the behaviour of the expected restoration time functions. In the next section we will attempt to illustrate the response of the functions to changes in these parameters.

4.4.1 Effect of Influential parameters

Equations 4.3 to 4.6 are the general forms for the approximated functions for expected time to process restoration. The response of these functions to changes in a, v and h will be illustrated for β ranging from 1 to 11.

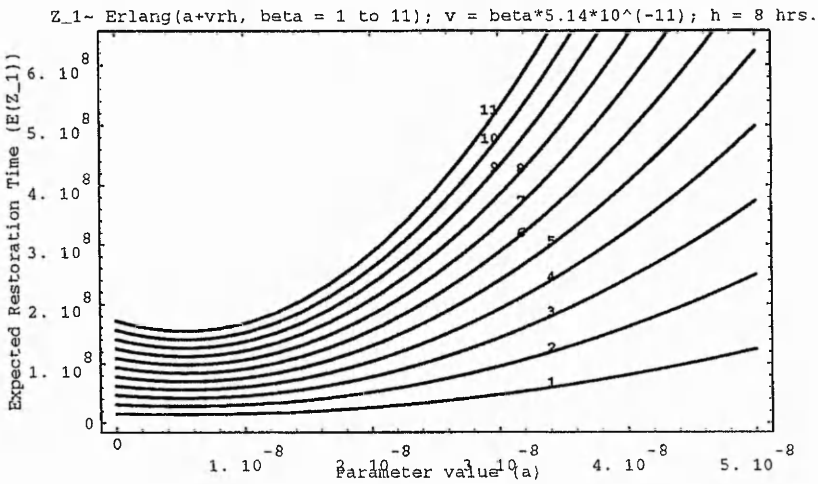
Figures 4.3 and 4.4, respectively, illustrate that, as v increases from 0.001 to 1, $E(Z_1)$ and $E(Z_2)$ tend to 0. This zero limit results as long as vh exceeds a and $l(\beta)$, the multiplier within these functions. The pictures also illustrate that, as a increases but is less than vh,

$\left(\frac{a}{vh}\right) E(R^{-2}) > \left(\frac{a}{vh}\right)^2 E(R^{-3})$, so that $E(Z_1)$ and $E(Z_2)$ decrease. However, the expectations tend to infinity as a exceeds vh so that $\left(\frac{a}{vh}\right) E(R^{-2}) < \left(\frac{a}{vh}\right)^2 E(R^{-3})$. The expectations remain decreasing functions of a only if $\left|\frac{a}{hv}\right| < 1$.

The response of $E(Z_3)$ and $E(Z_4)$ to increasing v and a is opposite to that of $E(Z_1)$ and $E(Z_2)$. Figures 4.5 and 4.6 that $E(Z_3)$ and $E(Z_4)$, respectively, approach 0 as a increases relative to the value of v. This limit is approached as long as a becomes large relative to $\frac{v}{h}$ and $l(\beta)$. Conversely, $E(Z_3)$ and $E(Z_4)$ are increasing functions of v if v exceeds ah so that $\left(\frac{v}{ah}\right) E(R^{-1}) < \left(\frac{v}{ah}\right)^2 E(R^{-2})$. This is illustrated in Figures 4.5 and 4.6. The Figures also illustrate that, as v ,increases but is less than ah, the expectations are decreasing functions of v since $\left(\frac{v}{ah}\right) E(R^{-1}) > \left(\frac{v}{ah}\right)^2 E(R^{-2})$.

The definitions for the scale parameter, α, are such that the expected restoration time should be decreasing functions of a and v. The expectations will remain decreasing functions of a and v only if, for $E(Z_3)$ and $E(Z_4)$, $\left|\frac{v}{ah}\right| < 1$ and, for $E(Z_1)$ and $E(Z_2)$, $\left|\frac{a}{hv}\right| < 1$. Under these conditions $E(Z_1)$ and $E(Z_2)$ decreases to a 0 limit as h increases while $E(Z_3)$ and $E(Z_4)$ rapidly increase to an asymptote of $\frac{l(\beta)}{a}$. See Figures 4.7 and 4.8. The values for h used in Figures 4.7 and 4.8 will not be used in practice. However, they are used here to illustrate the response of the expected restoration time to values of h which are large relative to $\left|\frac{a}{v}\right|$ in Figure 4.7 and to $\left|\frac{v}{a}\right|$ in Figure 4.8.

A one unit change in β, the shape parameter, yields a smaller increment in the mean when the Weibull distribution assumption is used.



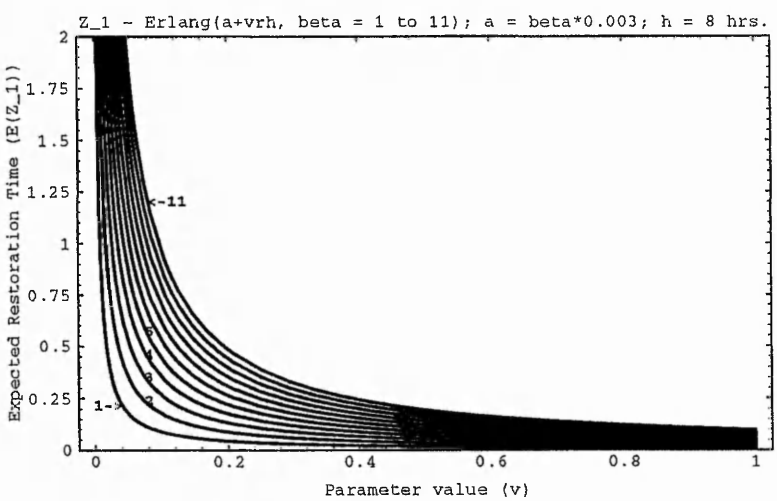
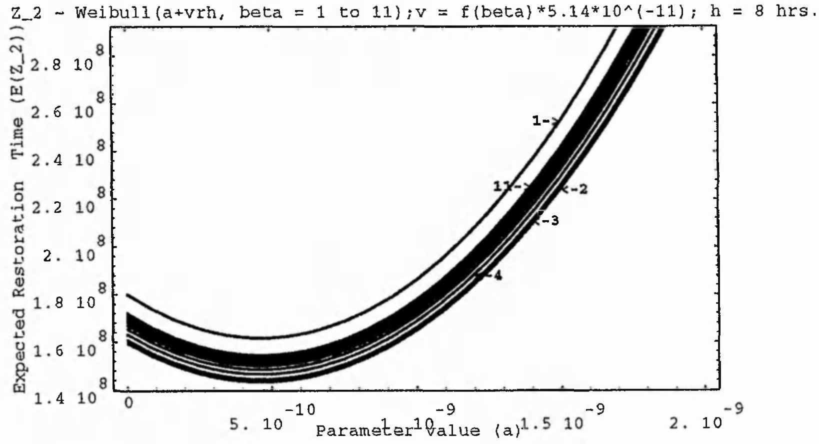


Figure 4.3: $E(Z_1)$ versus the parameters a and v for shape parameter (β) ranging from 1 to 11.



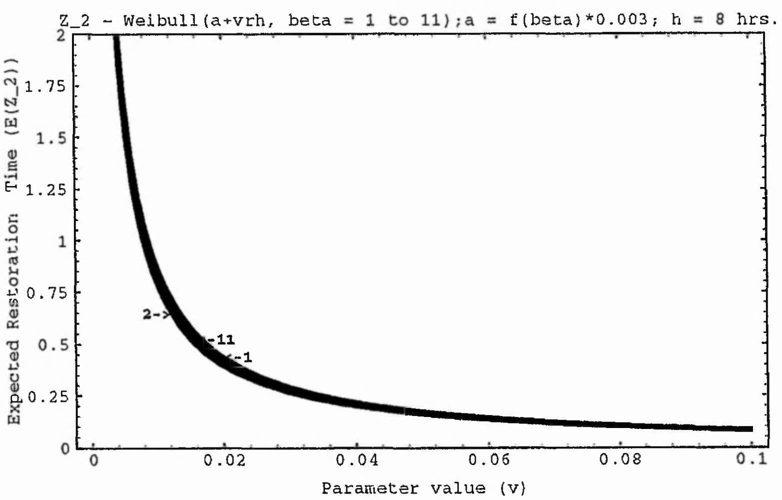


Figure 4.4: $E(Z_2)$ versus the parameters a and v for shape parameter (β) ranging from 1 to 11.

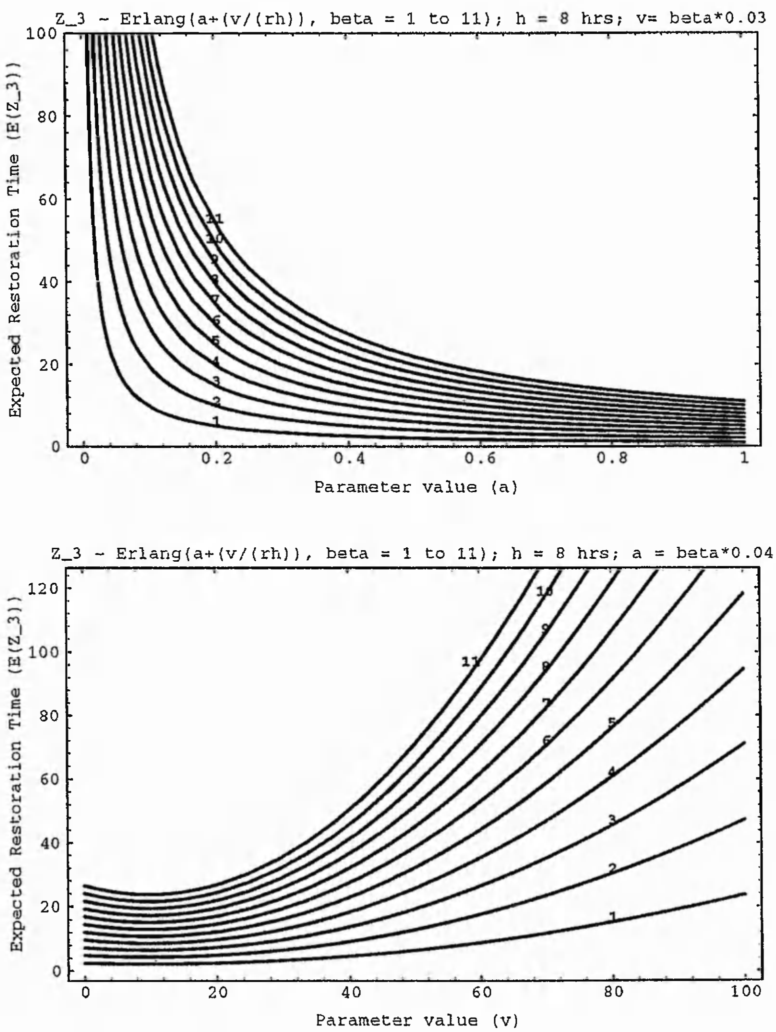


Figure 4.5: $E(Z_3)$ versus the parameters a and v for shape parameter (β) ranging from 1 to 11.

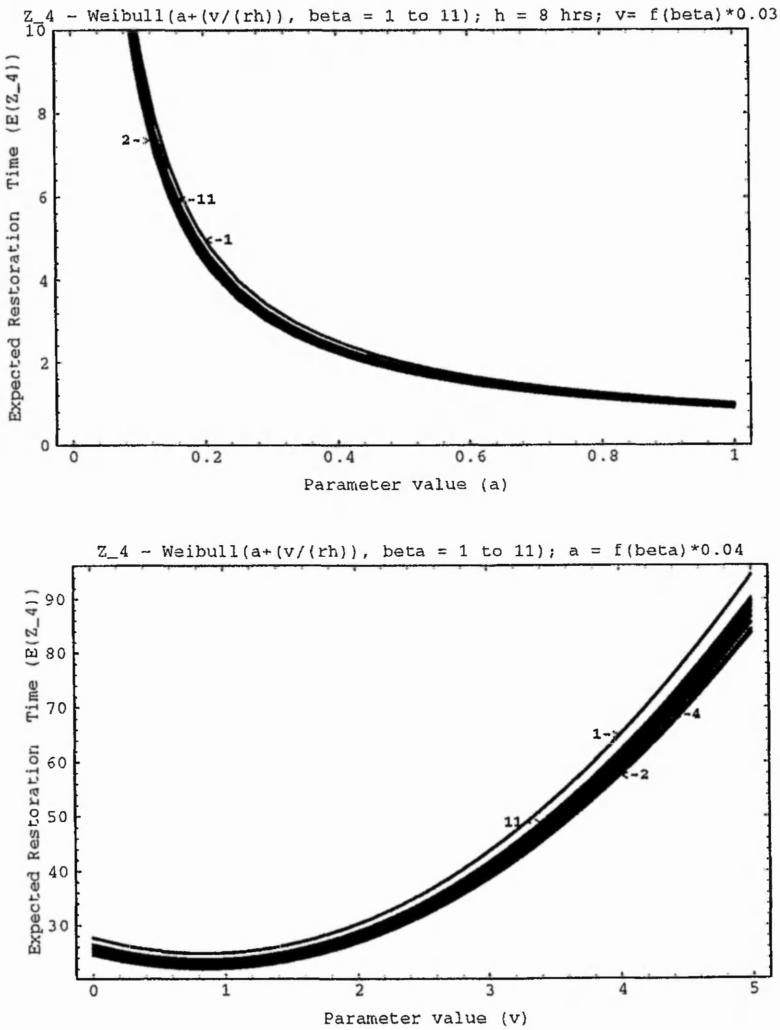


Figure 4.6: $E(Z_4)$ to the parameters a and v for shape parameter (β) ranging from 1 to 11.

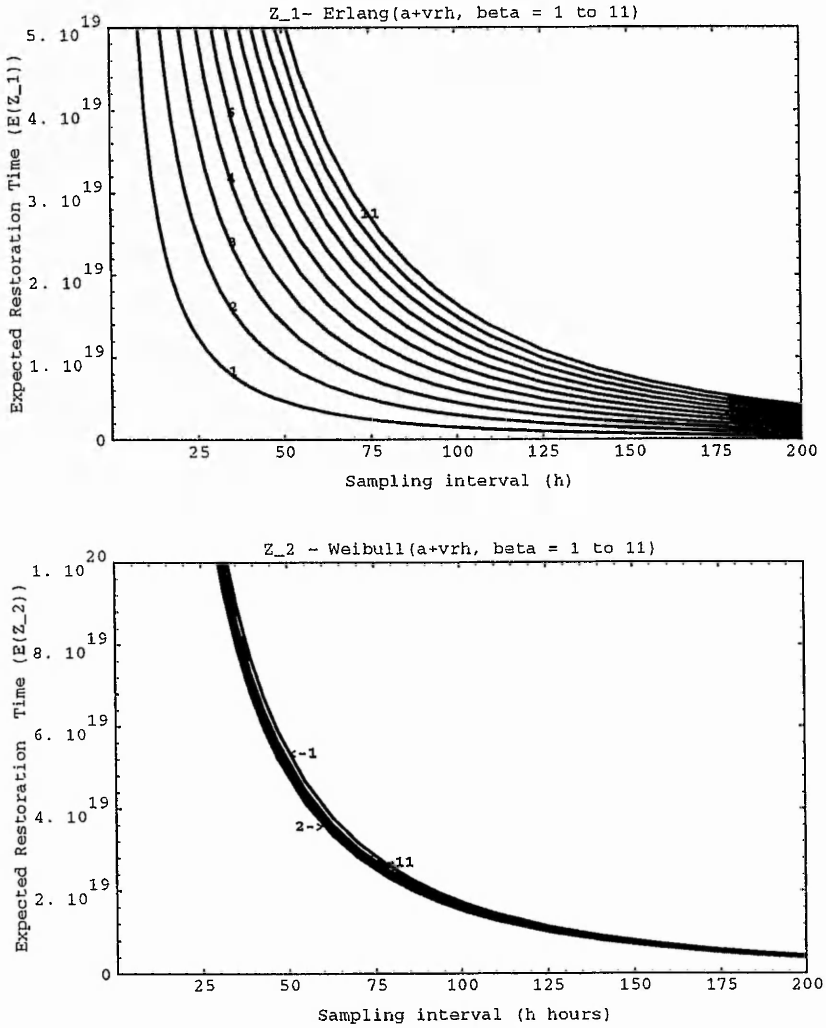


Figure 4.7: $E(Z_1)$ and $E(Z_2)$ versus sampling interval (h) for shape parameter (β) ranging from 1 to 11.

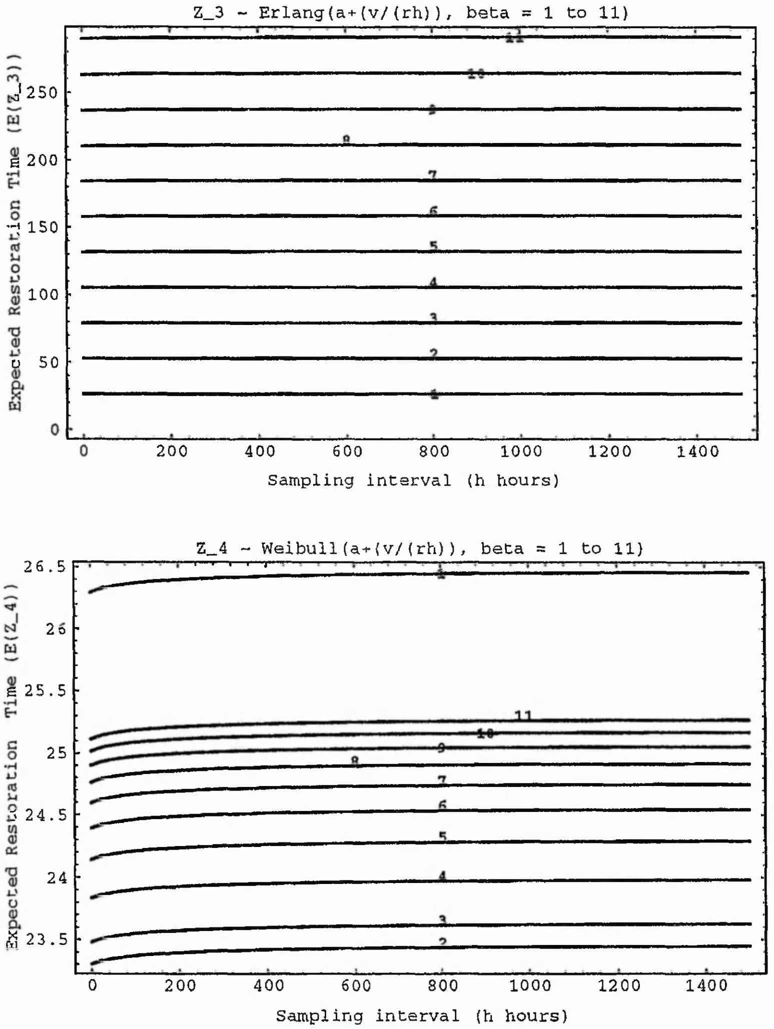


Figure 4.8: $E(Z_3)$ and $E(Z_4)$ versus sampling interval (h) for shape parameter (β) ranging from 1 to 11.

Therefore, Figures 4.3 to 4.8 illustrate the narrow range of the $E(Z_i)$ as the value of β increases. This suggests that, if the restoration time is assumed to be a Weibull variate, less precision is required in selection of the shape parameter.

4.5 $ECPTU$ functions Incorporating $E(Z_i)$

$ECPTU_c$ as presented in Chapter 3 will be extended to incorporate Equations 4.3 to 4.6 presented in Section 4.4. The resulting four ECPTU functions are expressed as the ratio of the expected costs and lengths of cycle as follows,

$$\begin{aligned} ECPTU_{Zi} &= \frac{E(C_{zi})}{E(L_{zi})} \\ E(L_{zi}) &= E(L_c) + E(Z_i) \\ E(C_{zi}) &= E(C_c) + c_4 E(Z_i) \\ &\quad \text{for } i = 1 \ldots 4 \end{aligned}$$

$E(L_{zi})$ and $E(C_{zi})$, respectively, are the expected length and cost of a cycle which ends when the process restoration period, Z_i, ends. The cycle length is illustrated in Figure 4.9. The response of the functions

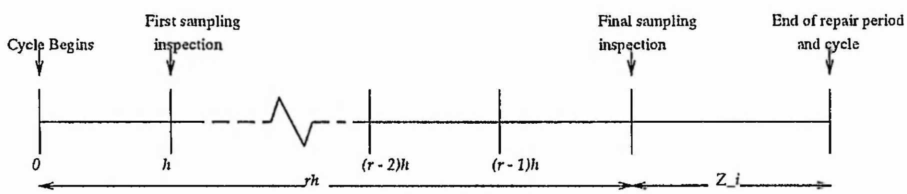


Figure 4.9: Illustration of the length of a cycle which includes the period spent searching for and removing the assignable cause.

to changes in the design parameters will be assessed through algebraic manipulation as well as observation of plots of the functions. Data from row one of Tables D.1 and D.2 in Appendix D are used to draw the plots.

Table 4.3 shows the limits approached by the expectations as the design parameters approach extreme values. $\pi(\lambda h)$ represents the limiting values of the negative moments for R as n and k approach extreme values. These limiting values are functions of λh. As $n \longrightarrow 1$, $h \longrightarrow 0$, and $k \longrightarrow \infty$ the $E(Z_i)$ tend to limits which are small relative to the limits approached by the remainder of the ECPTU functions. It therefore follows that the disparity in the ECPTU as a result of the different forms of the expected restoration time functions diminishes as the design parameters approach extreme values. Thus Figures 4.11 and 4.12 illustrate, for example, that as $h \longrightarrow \infty$ and $k \longrightarrow \infty$ the limits of the $ECPTU_{Zi}$ approximate to those given for $ECPTU_c$ in Section 3.6. These are $c_2 = \$100$ and $\frac{b+en}{h} + c_2 = \$101.00$, respectively.

Figure 4.10 illustrates that, for the data studied, $ECPTU_c$ and the $ECPTU_{Zi}$ approach infinity as n tends to infinity. As $n \longrightarrow \infty$, α_{11}(the probability of detecting the OOC state) $\longrightarrow 1$. This leads to the different functions for the asymptotes for the $E(Z_i)$. See Table 4.3. These limits influence the rate at which the ECPTU approaches infinity and maintains the difference between the the ECPTU functions over a relatively long range of values of n. See Figure 4.10.

	$E(Z_1)\&E(Z_2)$	$E(Z_3)\&E(Z_4)$
$n \longrightarrow 1$	0	$\frac{l(\beta)}{a}$
$n \longrightarrow \infty$	$\pi(\lambda h)\frac{l(\beta)}{hv}$	$\frac{l(\beta)}{a}(1-\pi(\lambda h))$
$h \longrightarrow 0$	∞	∞
$h \longrightarrow \infty$	0	$\frac{l(\beta)}{a}$
$k \longrightarrow 0$	$\pi(\lambda h)\frac{l(\beta)}{hv}$	$\frac{l(\beta)}{a}(1-\pi(\lambda h))$
$k \longrightarrow \infty$	0	$\frac{l(\beta)}{a}$

Table 4.3: Limits approached by the $E(Z_i)$ as the design parameters approach extreme values.

In Figure 4.12, as $k \longrightarrow 0$ the limits of the $ECPTU_{Zi}$ are influenced by the form of the $E(Z_i)$. They incorporate the limit of $ECPTU_c$ given in Equation 3.12 of Section 3.6 and limits of the $E(Z_i)$ given in Table 4.3. Thus, the difference between the objective ECPTU functions is maintained as $k \longrightarrow 0$.

Within the range of design parameter values which may be feasibly used in practice Figures 4.10, 4.11 and 4.12 highlight the different responses of the objective functions. For all three design parameters, $ECPTU_{Z3}$ exceeds all the other functions. For the data studied, there is no appreciable difference between $ECPTU_{Z1}$, $ECPTU_{Z2}$ and $ECPTU_c$.

Work in this section has illustrated that, as the design parameter values approach their extreme values, the $ECPTU_{Zi}$ reach limits which approximate to those reached by $ECPTU_c$. The plots illustrate that in the region of the minimum function values the functions

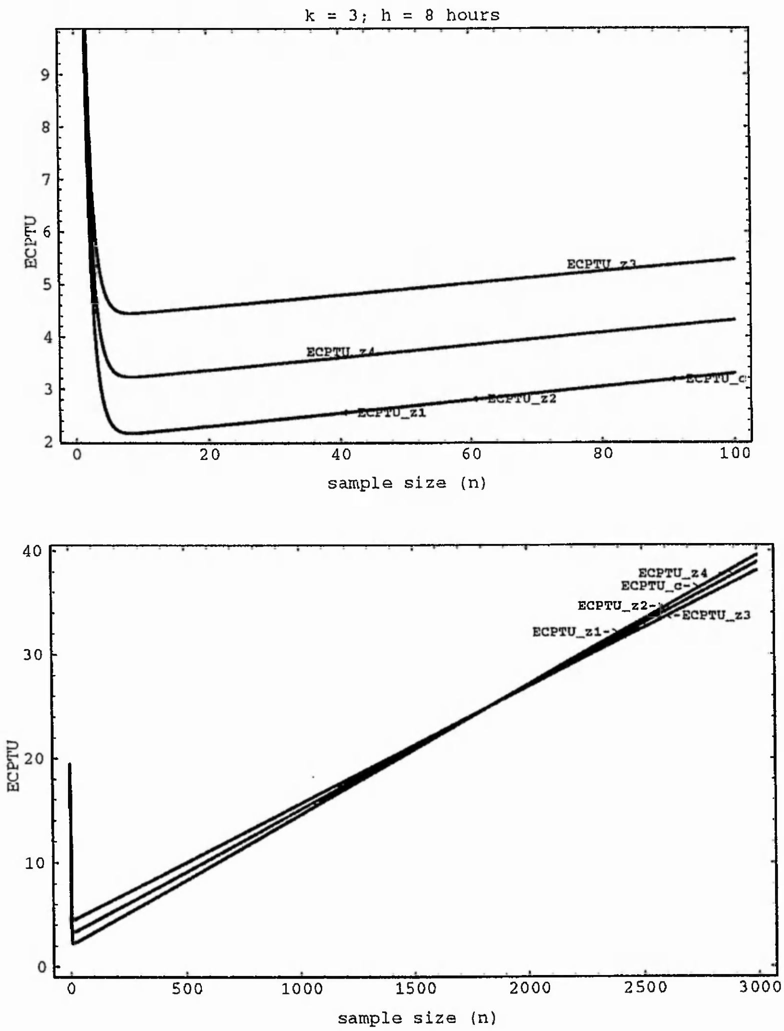


Figure 4.10: ECPTU as accounted for by $ECPTU_{Zi}$ and $ECPTU_c$ versus sample size (n) as n tends to infinity.

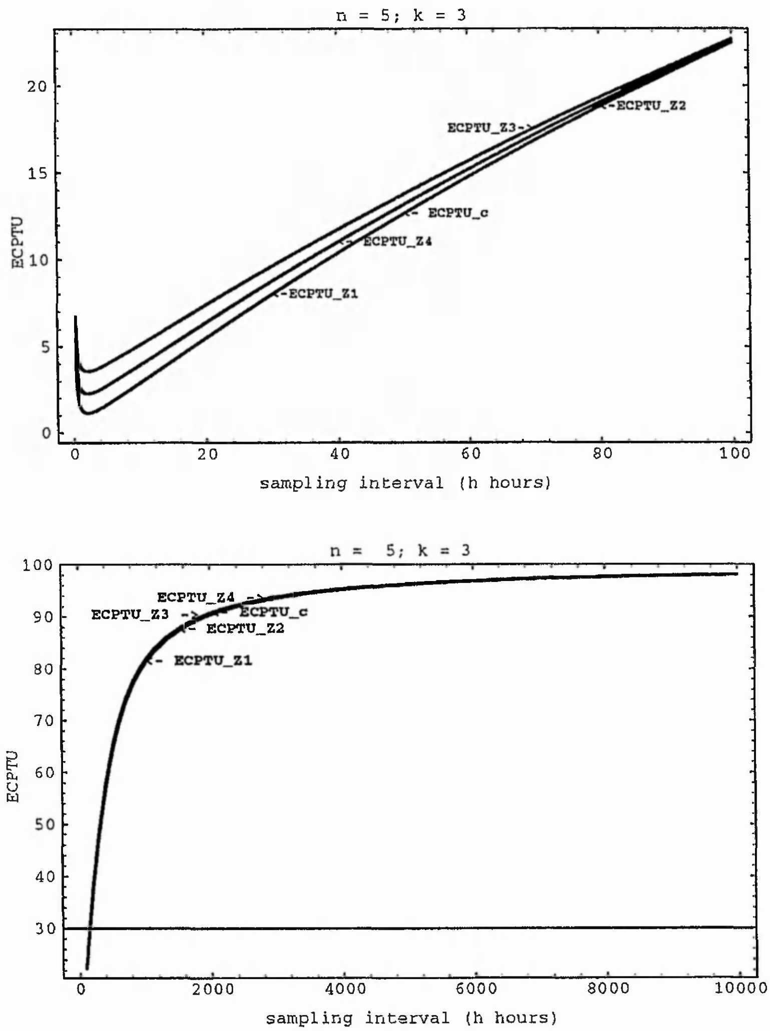


Figure 4.11: ECPTU as accounted for by $ECPTU_{Zi}$ and $ECPTU_c$ versus sampling interval (h) as h tends to infinity.

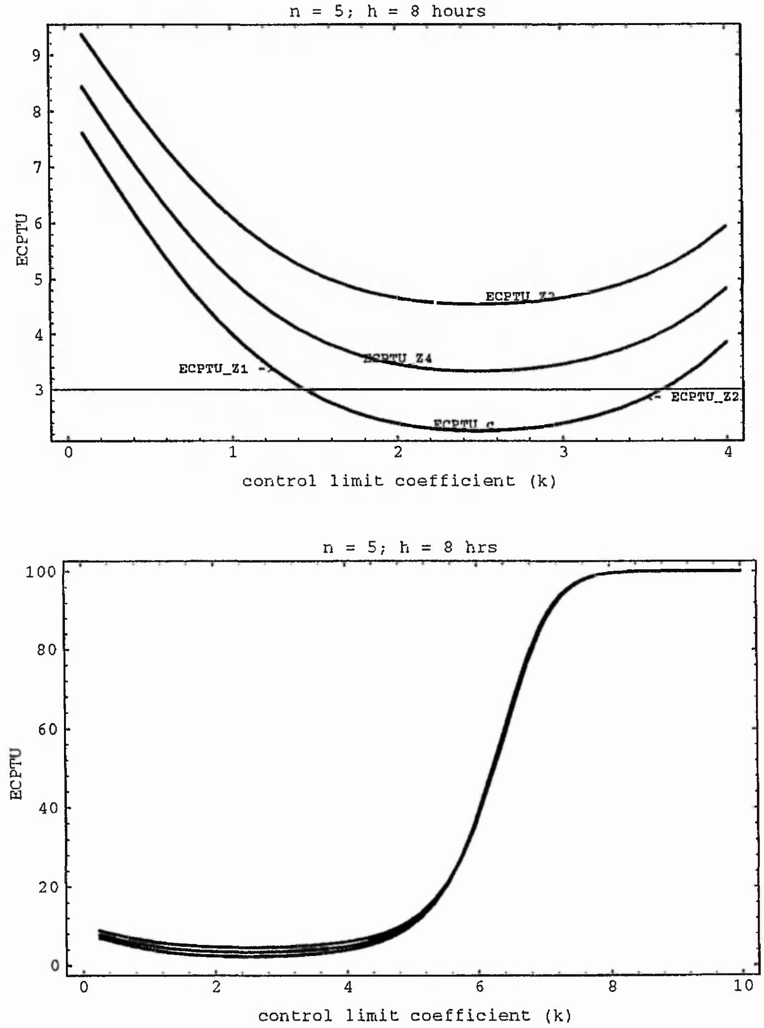


Figure 4.12: ECPTU as accounted for by $ECPTU_{Zi}$ and $ECPTU_c$ versus control limit coefficient (k) as k tends to zero and to infinity.

behave differently. The influence of the $E(Z_i)$ on the optimum combination of design parameters will be investigated in the next section. The ECPTU functions will be assessed using the data sets in Appendix D.

4.6 The Effect of Stochastic Restoration to the IC State

4.6.1 Changes to optimum design parameters

Compared with the results obtained when $ECPTU_c$ is minimised, explicit reckoning of stochastic restoration times can increase the value of h^*. Figure 4.13 illustrate that the n^* and k^* values produced by the different objective functions are approximately equal. The difference in the h^* for the functions is more detectable, particularly, for the final ten examples. We will now go on to show that low values of r_{c23}, r_{c24} and δ contribute to this increased variation in the h^* values.

Table 4.4 gives further evidence that the n^* and k^* produced by the $ECPTU_{Zi}$ are equal to those from $ECPTU_c$ for most examples. The absolute mean percentage differences for these parameters is less than 1%. The tabulated mean percentages indicate that an appreciable increment in the h^* results from minimising $ECPTU_{Z1}$, $ECPTU_{Z3}$ and $ECPTU_{Z4}$ instead of $ECPTU_c$. The increment can be as high as 20% if $ECPTU_{Z3}$ is optimised.

Values in Table 4.4 also indicate that compared with the Weibull distribution assumption, incorporation of $E(Z_i)$ based on the Erlang

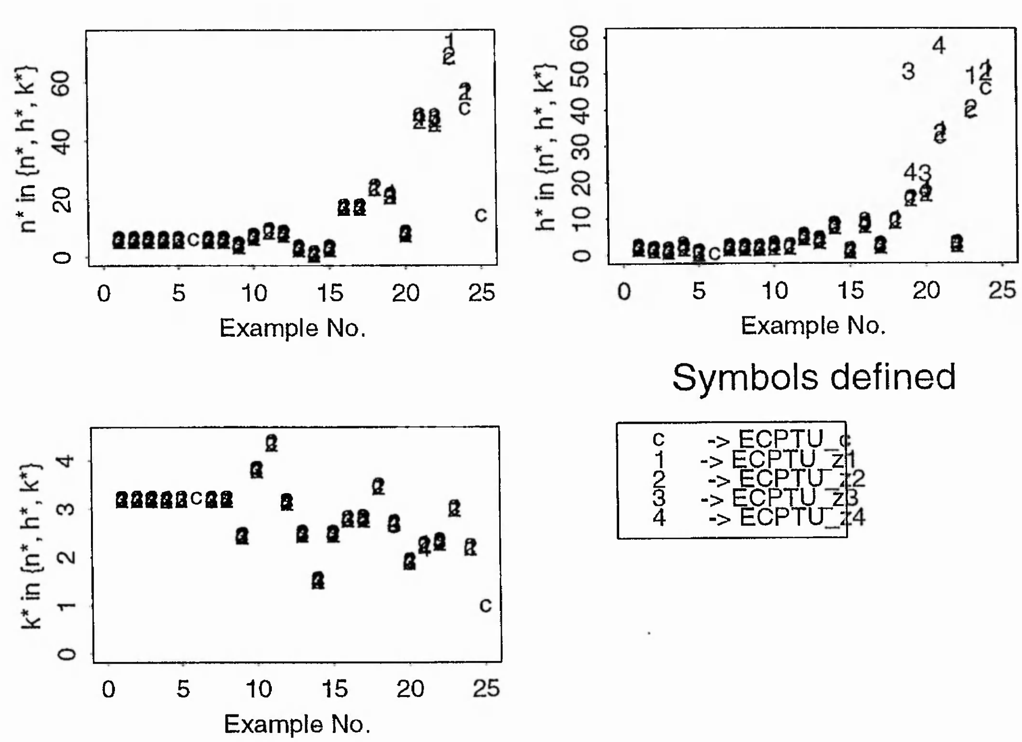


Figure 4.13: Optimum values (n^*, h^*, k^*) produced when the $ECPTU_{Zi}$ and $ECPTU_c$ are minimised simultaneously with respect to n, h and k.

ECPTU function	n	h	k
$ECPTU_{Z1}$	0.7	2.2	-0.4
$ECPTU_{Z2}$	0.2	0.7	-0.05
$ECPTU_{Z3}$	-0.38	20.3	-0.2
$ECPTU_{Z4}$	-0.48	9.7	-0.01

Table 4.4: For different $ECPTU_{Zi}$, the mean percentage by which the n^*, h^* and k^* exceed or fall short of the optimum values produced by $ECPTU_c$.

distribution will yield a higher increment in h^*. Thus the mean increments obtained for $ECPTU_{Z1}$ and for $ECPTU_{Z3}$ are higher than the mean values obtained for $ECPTU_{Z2}$ and $ECPTU_{Z4}$, respectively.

The tabulated values also give some evidence of the effect of the nature of the relationship between restoration rate, α, and the period rh. Compared with incorporation of the $E(Z_i)$ based on a linear relationship, incorporation of an inverse relationship yields a higher increment in h^*.

The mean percentages given in Table 4.5[1] indicate the influence of δ on h^* values produced by the $ECPTU_{Zi}$. The tabulated values indicate that, particularly when $\delta = 0.5$, h^* values produced by $ECPTU_{Z1}$ and $ECPTU_{Z2}$ will exceed those produced by $ECPTU_c$. From the values we can also infer that, for δ ranging from 0.5 to 2 h^* produced by minimising $ECPTU_{Z3}$ and $ECPTU_{Z4}$ will be noticeably higher than those from $ECPTU_c$.

[1]For $ECPTU_{Z3}$, optimum results obtained for only one example when $\delta = 0.5$.

ECPTU function	$\delta = 2$	$\delta = 1$	$\delta = 0.5$
$ECPTU_{Z1}$	0.1	0.6	11.1
$ECPTU_{Z2}$	0.01	-0.03	4.0
$ECPTU_{Z3}$	7.0	68.5	1.4
$ECPTU_{Z4}$	3.3	14.3	38.8

Table 4.5: For the different $ECPTU_{Zi}$ and values of δ, the mean percentage by which the h^* exceeds or falls short of h^* from $ECPTU_c$.

Table 4.6 indicates that the difference between the h^* values will be greatest for r_{c23} or $r_{c24} < 1$. Noticeable differences will also be observed for $1 \leq r_{c23} \leq 5$ and $1 \leq r_{c24} \leq 10$ when $ECPTU_{Z3}$ and $ECPTU_{Z4}$ are minimised. For $r_{c24} \leq 10$ or $r_{c23} \leq 5$ the tabulated percentages indicate that h^* from $ECPTU_{Z3}$ and $ECPTU_{Z4}$ can exceed those from $ECPTU_c$ by over 3%. The increment in h^* can be as high as 73% when $ECPTU_{Z3}$ is minimised and r_{c24} or $r_{c23} < 1$.

ECPTU function	$r_{c23} < 1$ $r_{c24} < 1$	$1 \leq r_{c23} \leq 5$ $1 \leq r_{c24} \leq 10$	$r_{c23} > 5$ $r_{c24} > 10$
$ECPTU_{Z1}$	5.5	0.02	0.01
$ECPTU_{Z2}$	2.02	-0.25	0.01
$ECPTU_{Z3}$	73.3	6.5	0.3
$ECPTU_{Z4}$	28.2	3.0	0.14

Table 4.6: For different ECPTU functions and values of the ratio of c_2 to c_3 (r_{c23}) or c_4 (r_{c24}), the mean percentage difference between h^* for $ECPTU_{Zi}$ and for $ECPTU_c$.

Table 4.7 gives further evidence that when $ECPTU_{Z3}$ and $ECPTU_{Z4}$ are minimised the h^* values will differ appreciably from those obtained when $ECPTU_c$ is minimised. Noticeable differences are observed for all values of the shape parameter for the restoration time. Noticeable differences are also observed in the h^* produced by $ECPTU_{Z1}$ and $ECPTU_{Z2}$ when $\beta = 4$. For these two functions, the values averaged to give results for $\beta = 4$ are a subset of those used to calculate mean values for $\delta = 0.5$, r_{c23} and $r_{c24} < 1$. This implies that the tabulated results could be a reflection of the influence of these parameters and not necessarily that of β.

ECPTU function	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$
$ECPTU_{Z1}$	0.2	0.7	19.2	1.0
$ECPTU_{Z2}$	-0.1	0.2	6.7	0.5
$ECPTU_{Z3}$	32.4	10.5	—	11
$ECPTU_{Z4}$	7.9	5.2	—	17.1

Table 4.7: For the different $ECPTU_{Zi}$ and values of shape parameter (β) for the restoration time distribution, the mean percentage by which the h^* exceeds or falls short of h^* from $ECPTU_c$.

Explicit reckoning of stochastic restoration times can increase significantly the value of h^*. This increase is influenced by the nature of the relationship of restoration rate to rh and the distribution of the restoration time. The observed increments are higher when the inverse relationship instead of a linear one is accounted for. Compared with increments resulting from use of the Erlang distribution, assumption for restoration time, those from the Weibull distribution are lower. In view of these differences we can expect that the penalty for use of suboptimal parameter combinations will also be influenced. This will be investigated in Section 4.6.2.

4.6.2 Changes in penalty for use of suboptimal values

In Section 4.6.1 it was established that incorporation of stochastic restoration times into the objective functions will not change, appreciably, the values of n^* and k^* compared with values obtained when $ECPTU_c$ is optimised. Thus, the penalties of reduced probability of detecting the OOC state; increased probability of a false alarm; as well as increased ECPTU due to use of $n = 5$ and $k = 3$ will be approximately equivalent to those observed for $ECPTU_c$ in Chapter 3.

The appreciable and, in some cases, statistically significant increase in h^* when the $ECPTU_{Zi}$ are optimised merits investigation of the penalty for use of suboptimal design parameter combinations. The penalty of increased ECPTU results from an extended OOC period or increased sampling frequency when $h = 8$ hours or 1 hour is used instead of h^*. In this section we will examine the variation in the penalty in response to changes in δ, r_{c24} and r_{c23}.

The inverse relationship between h^* and c_2 has been illustrated in Chapter 3. The mean percentages given in Table 4.8 illustrate this relationship. As such, the percentages by which $h = 1$ hour falls short of h^* become less as c_2 increases relative to the values of c_3 and c_4. The percentages indicate, however, that incorporation of the $E(Z_i)$

has little bearing on the penalty of increased sampling frequency for use of $h = 1$ hour instead of h^*.

ECPTU function	$r_{c23} < 1$ $r_{c24} < 1$	$1 \leq r_{c23} \leq 5$ $1 \leq r_{c24} \leq 10$	$r_{c23} > 5$ $r_{c24} > 10$
$ECPTU_{Z1}$	83	56	32
$ECPTU_{Z2}$	83	59	32
$ECPTU_{Z3}$	87	59	32
$ECPTU_{Z4}$	84	58	32
$ECPTU_c$	83	56	32

Table 4.8: For different ECPTU functions and values of the ratio of c_2 to c_3 (r_{c23}) or c_4 (r_{c24}), the mean percentage by which $h = 1$ hour falls short of optimum sampling interval (h^*).

The mean percentages in Table 4.9 illustrate that the penalty of an extended OOC state can result from use of $h = 8$ hours instead of h^*. For r_{c23} or $r_{c24} \geq 1$ this penalty is not greatly changed by incorporation of the $E(Z_i)$. However, for r_{c23} or $r_{c24} < 1$, the effect of the functional dependence of repair rate on rh becomes noticeable. Incorporation of a linear relationship — as done in $ECPTU_{Z1}$ or $ECPTU_{Z2}$ — does not change, appreciably, the percentage by which the sampling interval is extended by use of $h = 8$ hours instead of h^*. However, if an inverse relationship is the correct description for a process, the penalty — relative to that obtained from a linear relationship — can be reduced by at least 10%. See Table 4.9.

ECPTU function	$r_{c23} < 1$ $r_{c24} < 1$	$1 \leq r_{c23} \leq 5$ $1 \leq r_{c24} \leq 10$	$r_{c23} > 5$ $r_{c24} > 10$
$ECPTU_{Z1}$	40	249	446
$ECPTU_{Z2}$	40	250	446
$ECPTU_{Z3}$	7	227	444
$ECPTU_{Z4}$	29	238	445
$ECPTU_c$	40	249	445

Table 4.9: For different ECPTU functions and values of r_{c23} or r_{c24}, the mean percentage by which $h = 8$ hours exceeds h^*.

The penalties of higher ECPTU associated with use of the suboptimal parameter combinations {5, 1 hour, 3} and {5, 8 hours, 3} at different values of r_{c24} and r_{c23} are given in Tables 4.10 and 4.11. We note the appreciable reduction in the penalty of increased hourly cost due to incorporation of the inverse relationship between restoration rate and rh. If the control system based on the use of the $\bar{x}$-chart can be correctly described by $ECPTU_{Z3}$ or $ECPTU_{Z4}$ the penalty for use of the suboptimal combination {5, 1 hour , 3} can be less than 15%, particularly if $r_{c23} > 5$ or $r_{c24} > 10$. See Table 4.10. For a process described by the same objective functions the penalty for use of the combination {5, 8 hours , 3} can be less than 30% if r_{c23} or $r_{c24} < 1$. See Table 4.11.

The inverse relationship between n^* and δ contributes to the inverse relationship between the ECPTU penalty for use of the suboptimal combination {5, 1 hour, 3} or {5, 8 hours, 3} and δ. Mean percentages given in Tables 4.12 and 4.13 illustrate the inverse relationship between

ECPTU function	$r_{c23} < 1$ $r_{c24} < 1$	$1 \leq r_{c23} \leq 5$ $1 \leq r_{c24} \leq 10$	$r_{c23} > 5$ $r_{c24} > 10$
$ECPTU_{Z1}$	265	180	19
$ECPTU_{Z2}$	266	181	19
$ECPTU_{Z3}$	27	36	6
$ECPTU_{Z4}$	68	67	12
$ECPTU_c$	267	182	19

Table 4.10: For different ECPTU functions and values of r_{c23} or r_{c24}, the mean percentage by which $ECPTU(5, 1, 3)$ exceeds minimum ECPTU ($ECPTU^*$).

the penalty of increased ECPTU and δ for all the objective functions.

For restoration rate inversely related to rh and $\delta \geq 1$, explicit reckoning of a stochastic restoration time yields a more noticeable reduction in the penalty for use of either suboptimal combination of design parameters. The percentages given for $ECPTU_{Z3}$ and $ECPTU_{Z4}$ in Tables 4.12 and 4.13 indicate that an increase in hourly cost of less than 100% can be observed. The penalty can fall below 30% if $ECPTU_{Z3}$ correctly describes the process and $\delta = 2$.

We assume that, compared with $ECPTU_c$, the $ECPTU_{Zi}$ represent a truer description of a control system based on the use of the $\bar{x}$-chart. Use of the $ECPTU_{Zi}$ to select optimum design parameters gives results which lower the hourly cost penalty for use of suboptimal values values of h and, hence, for use of the suboptimal parameter combinations {5, 1 hour , 3} and {5, 8 hours , 3}. A very large

ECPTU function	$r_{c23} < 1$ $r_{c24} < 1$	$1 \leq r_{c23} \leq 5$ $1 \leq r_{c24} \leq 10$	$r_{c23} > 5$ $r_{c24} > 10$
$ECPTU_{Z1}$	137	416	168
$ECPTU_{Z2}$	138	424	168
$ECPTU_{Z3}$	11	94	68
$ECPTU_{Z4}$	30	162	121
$ECPTU_c$	139	428	168

Table 4.11: For different ECPTU functions and values of r_{c23} or r_{c24}, the mean percentage by which $ECPTU(5,8,3)$ exceeds minimum ECPTU ($ECPTU^*$).

reduction in the penalty is observed for the functions $ECPTU_{Z3}$ and $ECPTU_{Z4}$. These two functions incorporate the inverse functional dependence of repair or restoration rate on the period rh.

$ECPTU_{Z1}$ and $ECPTU_{Z2}$ incorporate the linear relationship of repair or restoration rate to the period rh. The tabulated mean percentages suggest that incorporation of this linear relationship will not change drastically the penalty for use of suboptimal values. The implications are that, where the relationship of restoration rate to rh is a linear one, explicit reckoning of this relationship will not yield results which differ appreciably from those obtained by a function which takes no account of the relationship.

The results indicate that the nature of the relationship between restoration rate and the period rh will influence the optimum results obtained when objective functions are minimised. It is, therefore, important that the nature of this relationship be correctly identified so that the optimum design parameters selected will be more specific to the system being monitored.

ECPTU function	$\delta = 2$	$\delta = 1$	$\delta = 0.5$
$ECPTU_{Z1}$	115	257	459
$ECPTU_{Z2}$	116	258	463
$ECPTU_{Z3}$	18	27	178
$ECPTU_{Z4}$	37	71	182
$ECPTU_c$	116	258	464

Table 4.12: For different ECPTU functions and values of δ, the mean percentage by which $ECPTU(5, 1, 3)$ exceeds minimum ECPTU ($ECPTU^*$).

ECPTU function	$\delta = 2$	$\delta = 1$	$\delta = 0.5$
$ECPTU_{Z1}$	114	304	1564
$ECPTU_{Z2}$	117	313	1574
$ECPTU_{Z3}$	18	82	902
$ECPTU_{Z4}$	35	80	719
$ECPTU_c$	118	316	1580

Table 4.13: For different ECPTU functions and values of δ, the mean percentage by which $ECPTU(5, 8, 3)$ exceeds minimum ECPTU ($ECPTU^*$).

4.7 Discussion of Results

In this chapter there has been an evaluation of the effect of explicitly accounting for the stochastic nature of the restoration time variate which is correlated with the period rh. The objective functions which have been produced yield optimum results which have higher h^* values than those from the function which excludes the repair time. For each function the observed differences have exceeded 25% for r_{c23} or $r_{c24} < 1$. The optimum values of n^* and k^* are not appreciably changed.

The increment in h^* due to incorporation of the $E(Z_i)$ is dependent on the nature of the relationship between restoration rate and the period rh as well as the distribution of the restoration time. Compared with the linear relationship, an inverse relationship leads to higher increments in the h^*. Use of the Erlang instead of the Weibull distribution assumption also leads to higher increments.

The results presented suggest that selection of h^* without accounting for the correlation of restoration time with the period of surveillance by the control chart could be costly. This is because use of h^* found without accounting for this correlation can increase sampling frequency.

Use of the heuristic design parameter combinations {5, 1 hour, 3} and {5, 8 hours, 3} instead of $\{n^*, h^*, k^*\}$ yield the expected increase in ECPTU. This increment is lower if the $E(Z_i)$ is based on the Erlang rather than the Weibull distribution. The penalty for their use can also be lowered if the restoration rate is an inverse function of rh. If

$ECPTU_{Z3}$ correctly describes a process, the increment from use of $\{5, 1 \text{ hour}, 3\}$ can be minimal for $r_{c23} > 5$ or $r_{c24} > 10$. For the same objective function the increment from use of $\{5, 8 \text{ hours}, 3\}$ can be minimal for r_{c23} or $r_{c24} < 1$.

If either of these suboptimal combinations must be used the loss in profitability can be reduced if the restoration time is an Erlang variate; the repair rate is inversely related to rh; and r_{c23} and r_{c24} lie within the ranges previously specified. Profitability from use of the $\bar{x}$-chart with these design parameter values can be enhanced if there is precise determination of the distribution of repair time as well as the functional dependence of this period on the period of surveillance by the control chart.

The illustrations in Section 4.4.1 indicate that the difference in the shape parameter yields wider variation in the $E(Z_i)$ based on the Erlang distribution. It is possible that differences in the shape parameter could also influence the values of n^*, k^* and h^*. This has not been identified in this study possibly because the effect of the shape parameter had been confounded with the effect of δ.

The results give additional evidence that use of a 1 hour sampling interval is least unprofitable if $r_{c23} > 5$ or $r_{c24} > 10$; an $h = 8$ hour sampling interval is most suitable, economically, if $r_{c23} < 1$ or $r_{c24} < 1$; and use of $n = 5$ and $k = 3$ will be suitable when $\delta = 2$.

Chapter 5

The Production Environment and the ECPTU

5.1 The Environment's Influence

The literature has not specifically addressed the development of the production environment to allow for selection of the optimum design parameters using the ECPTU function. It is important that there is clear understanding of the role of facets of the environment in the selection of the design parameters for the control chart.

Dale and Oakland [23] state that the design of control charts should be based on practical experience as well as statistical criteria. They further state that the frequency of sampling is subject to many practical considerations and should be determined by the economics of the process. The economics of the process relates to the stability of the process; the cost of inspection; costs incurred by the production of non-conforming output; and the frequency of changes made to the process that might affect the quality of output.

In the previous chapters we illustrated the response of design parameters n and k to changes in sampling frequency as determined by h. That the selection of sampling frequency should be determined by the economics of the process implies that the values of n and k will also be influenced by economic factors. This creates scope for use of ECPTU functions for selection of optimum control chart parameters. The practicality of selecting design parameters requires due consideration for the environment in which they will be used.

The ECPTU functions developed in Chapters 3 and 4 may be used to select optimum design parameters for processes which yield a stream of continuous data. The work environment can be prepared for utilisation of these functions through staff training; the practitioner's knowledge of the process and product; understanding the standards for quality; accurate collection and use of data.

In this chapter we will explain how each of these factors can influence the cost or technical parameters associated with operation of the the $\bar{x}$-chart. The effects on the ECPTU functions and, in turn, the optimum value of design parameters will also be discussed. We will use the objective functions, $E(L_a)$ and $E(C_a)$, found in Section 3.5.1.1 of Chapter 3 to illustrate these effects. There will also be an exposition of the practical issues which must be taken into consideration if use of the objective functions will be implemented.

5.2 Staff Training

Education in statistical thinking will equip practitioners to carry out appropriate subgrouping of data; accurate and precise measurement of process output; and correct action in response to points plotted on the control chart. This training can also equip persons to use, effectively, the seven simple and elementary quality tools. Ishikawa [41] points out that people cannot be expected to master more difficult methods of analysing data if they have not been trained to handle the simple tools. The success of the Japanese manufacturing sector since the 1960s can be attributed, in part, to the ability of workers at all levels — ranging from top management to production line workers — to use correctly the seven quality tools [41]. This competence at using the tools is indicative of an appreciation of their importance to the success of the organisation as well as an understanding of variation in data.

Education in statistical thinking and training in the use of statistical techniques should ultimately lead to improved process performance and quality of output. However, such training involves more than the study of statistical methods and the preparation of control charts. [41] Effective training in use of the quality tools — one of which is the control chart — must come against a background of training in other areas. It should be part of the quality management system and should lead to improved productivity. Areas of training which form the background for training in use of the statistical tools include [41]

1. The concept of quality.

   This involves having respect for customers; recognition of the next process as a customer; and an understanding of quality assurance.

2. Principles of implementation which concern management and improvement.

   These include quality circles and the PDCA cycle.

3. A statistical way of thinking.

   This gives an appreciation for variation in data.

Ishikawa [41] sates that improvement of human relations and self-improvement for the entire work force must be encouraged. Thus, implementation of the control chart and further optimisation of its use through ECPTU functions will be facilitated. Education encourages adaptation to change [59]. Co-operation between departments with employees experiencing "joy in work" as discussed by Neave [59] can enhance the benefits derived from training in the use of control charts and subsequent selection of design parameters using ECPTU functions.

Staff training which promotes the correct use of the control chart will ensure its cost-effective use. Any staff training that is done must be updated regularly to take account of changes in working methods and other facets of the production environment. [43] The subsequent sections indicate different forms of training which can enhance the cost-effective use of control charts and implementation of use of the ECPTU function.

5.2.1 Education In Understanding Variation

Ishikawa [41] postulates the need for staff to be educated in understanding variation. This staff includes top management. The benefits of use of the control chart will not be optimised if production-line staff are led to cheat by presenting data which has small variation. If the within-group variation is falsely presented as being very small the result could be excessive chasing after special causes of variation which are really due to common causes. This could, in turn, lead to drastic increases in the ECPTU for operating the control chart because of a proliferation of false alarms.

Education in understanding variation or any statistical training must be linked to specific organisational goals. It will be successful if it is conveyed as being critical to the achievement of the organisation's objectives. The pay-off from this knowledge and its application must be assessed. [36]

The extent to which variation in data is understood can be expected to influence the value of the cost and technical parameters used to determine the ECPTU. For example, an appreciation for variation in data will lead practitioners to seek to remove special causes of variation only and to reduce the common cause variation.

Based on the general form of the objective functions given in Section 3.5.1.1 of Chapter 3 we derived the following

$$E(L_a) \quad = \quad \frac{2}{\lambda} + h - E(\tau) + h\frac{1-\alpha_{11}}{\alpha_{11}} + D$$

$$E(\tau) = \frac{2}{\lambda} - \left(h\frac{e^{-\lambda h}}{1-e^{-\lambda h}}\right)\left(\frac{1-e^{-\lambda h}+h\lambda}{1-e^{-\lambda h}}\right)$$

$$E(C_a) = (b+en)(m_1+m_2) + c_2\left(h - E(\tau) + h\frac{1-\alpha_{11}}{\alpha_{11}}\right) + c_3\alpha_{01}m_1 + c_4$$

$$m_1 = \left(\frac{e^{-\lambda h}}{1-e^{-\lambda h}}\right)\left(\frac{1-e^{-\lambda h}+h\lambda}{1-e^{-\lambda h}}\right)$$

$$m_2 = \frac{1}{\alpha_{11}}$$

Removal of the special cause of variation can lead to a reduction in λ, the rate at which the process mean shifts to the OOC state. Thus $\lambda \longrightarrow \psi\lambda$, with $0 < \psi < 1$. Since $e^{-\lambda h} < e^{-\psi\lambda h}$, m_1, the expected number of samples taken during the IC period, and the mean IC period will increase. However, results in Section 3.8.2 indicate that longer sampling intervals and, in turn, a larger sample sizes more economically optimum. If sampling intervals are increased in response to the reduction in λ, m_1 is reduced. The consequent reduction the expected sampling costs, $(b+en)(m_1+m_2)$, and in the expected cost of false alarms, $c_3\alpha_{01}m_1$ will reduce $E(C_a)$. A reduction in $E(L_a)$ would also follow these changes in λ and h. Consequently, the ECPTU for operation of the control chart can be reduced.

A reduction in the ECPTU can also result from use larger sample sizes are used in response to a reduction in λ. This is a consequence of the increase in α_{11}, the probability that the OOC state is detected. From Section 3.5.1.1

$$\alpha_{11} = \Phi\left(-k - \delta\sqrt{n}\right) + \Phi\left(-k + \delta\sqrt{n}\right)$$

Recall that $\Phi(-k - \delta\sqrt{n})$ is the cumulative probability that the standard normal variate, Z, is less than or equal to $-k-\delta\sqrt{n}$. If n increases, then the cumulative probabilities and, therefore, α_{11} increase. Increased α_{11} will lessen $h - E(\tau) + h\frac{1-\alpha_{11}}{\alpha_{11}}$, the length of the OOC period. Consequently, the expected sampling costs and the expected cost of production during the OOC period, $c_2\left(h - E(\tau) + h\frac{1-\alpha_{11}}{\alpha_{11}}\right)$ fall. These changes can contribute to a reduction in the $E(C_a)$ and and the ECPTU.

Removal of the special cause of variation could also lead to a reduction in δ, the number of standard deviations by which the process mean shifts in response to an assignable cause. Results in Section 3.8.1 indicate that use of the control chart is optimised if sample size, n, is increased in response to a reduction in δ. As indicated in the previous paragraph, use of larger n can reduce the ECPTU.

As the process is improved through reducing common cause variation the quality of output is improved. This may reduce, c_2, the cost of poor quality output from the process during the OOC period. This will again help to reduce the ECPTU.

As well as selecting optimum design parameters the objective functions may be used to assess the effect of training in understanding variation. This could be done by comparison of ECPTU values calculated before and after the lessons learnt are applied. The benefits of training in statistical thinking will be substantial, particularly if it is given to those using SPC techniques to do their jobs [36].

5.2.2 Training in the use of SPC techniques

Hare *et al* [36] report that mass training of managers in the use of SPC techniques often meets with limited success. This results if managers do not understand why they need to possess statistical knowledge and make no effort to develop the knowledge. The managers also need to ensure that the knowledge is effectively applied. If these issues are addressed the managers can help to create a work environment which fosters positive attitudes to the use of SPC tools. Such attitudes will lead to an interest in the process so that data will be collected, recorded and used to improve the process. [23] Effective training in the use of SPC techniques can then be a reality.

Use of SPC techniques will lead the operators to understand more about statistical methods and give them more confidence to handle more advanced methods such as the design of experiments and the analysis of variance. [23] Persons who competently use SPC techniques such as the control chart can be expected to handle more confidently selection of design parameters using ECPTU functions.

Correct training in the use of techniques such as the control chart will be particularly beneficial to operators of processes which can be monitored by such tools. Training should encourage easy plotting of points on the chart. This may be facilitated by use of the range rather than the standard deviation as a measure of dispersion. It is the operators who should plot the points on the charts which should be displayed as closely as possible to the process or activity being monitored [23]. Correct training will also facilitate concurrent use of the R and $\bar{x}$-charts. More rapid detection of the OOC state can be expected as a result. The relevant efforts can then be made to reduce variation in the process output. Variation reduction should be the emphasis of process improvement.

Correct training in the use of SPC techniques can, therefore, reduce the hourly loss cost of operating the control chart by shortening the OOC period or reducing the occurrence of false alarms. There could also be an extension of the IC state in which there is a high quality of output. A more precise estimate of the ECPTU can result. This can yield more accurate estimates of the optimum design parameters.

5.2.3 Training for the measurement process

Shewhart [68] points out that there are two aspects of the operation or method of measurement. These are the physical and numerical aspects. The relevant staff need to be trained to carry out the physical aspects of measurement with speed, precision and accuracy. This can enhance the consistency of the numeric results. Speed, accuracy and precision in the physical aspects of the measurement process will make even more valid the assumption of negligible sampling inspection time. This assumption is used in the derivation of ECPTU objective functions presented in this study.

Prior to any study of the process all operators and their supervisors should fully understand the use of calibrated measurement equipment appropriate for the characteristic or parameter which is to be controlled. [23] Incorrect use of measurement equipment because of inadequate training can lead to the production of wrong readings. If the shift coefficient is calculated from these wrong readings this parameter may be estimated at $\delta = \phi\delta$ for which $0 < \phi < 1$. According to results in Section 3.8.1 use of the control chart can be optimised if smaller k, the control limit coefficient, is used with reduced δ. Recall that

$$\alpha_{01} \quad = \quad 2\Phi(-k)$$

Thus use of smaller values of k can increase the value of α_{01}, the probability of a false alarm. The increased expected cost of false alarms $c_3\alpha_{01}m_1$ can contribute to an increase in $E(C_a)$. Conversely, wrong readings can produce an inflated value of δ. Selection of a larger value of k based on this inflated value can reduce α_{11} and thus extend the OOC period as well as the cost of production during this period. Thus, an increase in the number of false alarms or the extension of the OOC state can increase the ECPTU for operation of the control chart.

5.3 Knowledge of Process and Product

Prior to implementation of use of the control chart there should be a clear knowledge of the mode of the process's operation and the different characteristics or parameters which influence the quality of products. Such knowledge will elicit appropriate reaction to different types of data and facilitate use of the control chart and ECPTU functions for providing more information about the process and product.

5.3.1 Reacting to data

It may become necessary to have different control charts monitoring different characteristics. In order to account for any correlation between these characteristics the control charts will need to be observed simultaneously. In some environments the correlated characteristics must all simultaneously conform to specifications for product acceptability. Steiner and Wesolowsky [70] have investigated the optimum design of control charts when a product has two or more correlated characteristics. They determined optimum control limits and sample sizes for each characteristic. Their study shows that the optimum sample size and control limit is dependent on the producer's and consumer's risks as well as the acceptable and rejectable mean levels associated with each characteristic. This suggests that control charts for monitoring correlated characteristics may have to be modified to ensure optimised use.

Identification of autocorrelation between process output values results if the process and product are understood. The way in which large positive autocorrelation within data is handled must be based upon a working knowledge of the process and careful interpretation of the control chart [78].

Familiarity with the desirable quality characteristics of the product is particularly crucial when qualitative measures are used to determine the state of the process. Criteria by which elimination of the assignable cause can be determined must be set up. The operator must know when the product can be declared as being in control within limits. [67]

5.3.2 Enhanced use of the control chart

Apt familiarity with the process will enable the operator to make notes at appropriate places on the $\bar{x}$-chart to indicate occurrences of technical importance. Such events include equipment and coolant adjustment, tool replacement, change of operator, new batch of material or components. These notes enable identification of sources of trouble when the chart indicates the presence of assignable causes of variation. [23] Consequently, there can be more rapid identification of the assignable cause and its removal. The associated costs and, hence, the ECPTU can then be further optimised.

Elimination of the assignable cause of variation should be followed by a recalculation of the measure of dispersion for the process. The data used should exclude the subgroups which identified the OOC state of the process. Removal of the special cause creates a "new" process. [23] A consequence of this may be changes in the process parameters such as the shift coefficient (δ), the hourly cost of operating in the OOC state (c_2), and the cost of restoring the process to the IC state (c_4). A "new" process, therefore, requires redetermination of optimum design parameters for the control chart.

5.3.3 Use of ECPTU functions

The ECPTU functions can be used to aid selection of optimum design parameters. The results of research presented in Chapter 3 indicate ways of optimising use of the $\bar{x}$-chart in response to changes in the production environment. In order to make correct use of these results the practitioner must know the expected behaviour of the process.

Output produced during the OOC period may need to be held in quarantine until results on a later sample indicating change of the process's status becomes available [23]. The loss cost due to withholding the products can be used to estimate c_2, the hourly cost of operating in the OOC state. This value of c_2 can be input to an ECPTU functions which is being used to assist or assess process performance.

The control chart cannot be guaranteed to detect, immediately, a change in the state of the process when a sample is taken. It may be necessary, therefore, to withhold output prior to detection of the OOC state. [23] The expected length of the OOC period may be calculated using functions presented in this study. Results of this calculation will give an estimate of the quantity of output which needs to be withheld.

It is imperative that the process is thoroughly understood before the ECPTU functions can be used to select their design parameters. Only a knowledge of the process obtained from a continuous study of it will enable estimation of the cost and technical parameters which are used to calculate the function values.

5.4 Ensuring Accurate Collection and Use of Data

Accurate collection and use of data requires a clear distinction between factors which affect the within-group and those that affect the between-group variation. Autocorrelated data or data made inaccurate by flaws in the measurement process can prevent clear identification of these two sources of variation. Accurate data collection and use can help lower the ECPTU for operation of the $\bar{x}$-chart. This is because special cause variation has been separated from common cause variation. Consequently, the OOC state can be more readily detected or the IC state can be extended.

Dale and Oakland [23] recommend that a minimum of 20 subgroups should be gathered and used to set up control charts. These subgroups allow for the initial study of the process so that there may be adjustments for existing autocorrelation in data. In addition, major sources of variation as well as any flaws in the measurement process can be identified. Observation of the process using these subgroups can provide estimates of cost and technical parameters associated with operation of the control chart. Once the current assignable causes have been revealed and removed and the measurement process has been suitably adjusted, the ECPTU can be more safely used to select optimum design parameters for the chart.

In the following subsections there is discussion of ways in which the accurate collection and use of data may be safeguarded.

5.4.1 Separating the types of variation

Data should be collected and used in a manner which enables measurement of the variability in a process [36] and therefore reveals the problem areas. Action based on incomplete, non-representative data yield financial losses and costs higher than those obtainable if rational subgrouping is used [36]. Wheeler [77] also emphasises the need to use the right data. He states that meaningless data yield meaningless analysis.

According to Wheeler [77], in aiming to create data which separates the types of variation accumulated data should be avoided. Accumulated data — usually in the form of summary counts and percentages — are not focussed and specific. Therefore, they do not clearly reveal the sources of variation. Each process which yields a separate data stream should be monitored by a separate control chart. [77] Ishikawa [42] further states that data on products made under similar conditions should be collected in the same subgroup. This type of subgrouping minimises the within-group variation and increases uniformity within the group. It also increases the between group variation.

A more efficient use of the $\bar{x}$-chart indicated by a lower ECPTU can result from subgrouping which minimises the within-group variation. The within-group variation is used to determine the position of the control limits. If this is minimised the limits become narrower and α_{11}, the probability that the OOC state is detected, is increased. α_{11} can be expressed as follows

$$\alpha_{11} = P\left(\bar{x} \geq \mu_0 + \frac{k\sigma}{\sqrt{n}}\right) + P\left(\bar{x} \leq \mu_0 - \frac{k\sigma}{\sqrt{n}}\right)$$

if it is assumed that $\bar{x}$ is normally distributed with mean μ_0 and variance σ^2 during the IC period. If for a given k, σ is reduced to $\gamma\sigma$ with $0 < \gamma < 1$, then $P\left(\bar{x} \geq \mu_0 + \frac{k\sigma}{\sqrt{n}}\right)$ and $P\left(\bar{x} \leq \mu_0 - \frac{k\sigma}{\sqrt{n}}\right)$ are increased. Thus, an increased α_{11} could produce a shorter OOC period, $\left(h - E(\tau) + h\frac{1-\alpha_{11}}{\alpha_{11}}\right)$ and a reduced cost of operating in this period.

Conversely, lower within group variation can be an indication of increased process stability so that the relative length of the IC period is increased. As indicated in Section 5.2.1 the lower cost of operating in the OOC period or the increased length of the IC period can both lead to reduced ECPTU for operation of the $\bar{x}$-chart.

5.4.2 Interpreting autocorrelated data

Neave [60] points out, however, that efforts to minimise within-group variation can be hazardous if the observations are naturally autocorrelated. Wheeler [78] gives guidelines on how to interpret data which may be autocorrelated.

Autocorrelation can lead to considerable misinterpretation of data displayed on the control chart. Excessive autocorrelation has a visible impact on the running record and on control limits calculated using the formula $\pm kSD(X)$[1]. Time series with a very large positive lag-one autocorrelation will have very coherent running records. As a result there may be very little need for calculation of control limits. Any common cause variation will be very small compared with the special cause variation so that the running record itself reveals the behaviour of the process. [78]

Wheeler [78] further indicates that large positive autocorrelation can make the control limits narrower than they should be. As a result, the frequency of false alarms increases. [17] [55] As such, he recommends

[1] $SD(X)$ is the estimated standard deviation for the measured output.

inflation of the natural process limits by a factor $\left(\sqrt{1-r^2}\right)^{-1}$[2], as long as the sampling frequency is suitable for the process; the full range of the output values is thought reasonable for the process; and the moving range chart does not indicate discontinuities in the process. After adjustment for autocorrelation, points falling outside the control limits will not represent a warning that the limits are too narrowly placed.

Even when data from a process are known to be serially correlated there is still scope for use of the ECPTU function to determine $\bar{x}$-chart parameters. The ECPTU functions can be used to select optimum values of n, h and k which can be used with the inflated values for the standard deviation. Increased lag size may eliminate the need for adjustment of the standard deviation for the effect of autocorrelation. Under these circumstances the ECPTU functions may be used to determine the optimum values of h and k which may be used with the $\bar{x}$-chart.

5.4.3 Adjusting the measurement process

A lack of knowledge of the appropriate measuring methods can lead to collection of wrong or useless data. [41] Inadequate calibration of the test or measuring equipment could lead to production of imprecise or inaccurate readings. Such readings can hamper the efficiency with which the control chart correctly monitors the process. This is because inaccurate readings could either delay the detection of the OOC state or increase the occurrence of false alarms. Hence, poor performance

[2] Equation (1) in [17] indicates the rationale for use of this factor.

of measurement equipment can increase the ECPTU for operation of the $\bar{x}$-chart. Measurement equipment must be properly calibrated and yield accurate measurements. Production of wrong readings can be prevented if methods are set up to check test equipment. [33] Failure of the chart to yield correct decisions about the process because of inaccurate or imprecise measurements can yield financial losses.

The $\bar{x}$-chart practitioner, as an applied scientist, has the job of getting enough data before making estimates or drawing conclusions about the process. The practitioner must also ensure that the data are both repeatable and reproducible within specified limits of tolerance. Reproducibility within tolerance limits is an indication that the measurement process is in a state of statistical control. [68] Absence of repeatability and reproducibility could be indicative of flaws in the measurement process. In order to remove such faults measurement equipment may have to be replaced.

Autocorrelated data reflects variation in the measurement process rather than common cause variation. [60] Thus, there may be need for changes to the measurement process in order to reduce the autocorrelation in the data. For example, there may be need to change the times at which measurements are taken if a running record of the data reveals definite cycles. This may become particularly necessary if the cycles prohibit correct interpretation of the data.

5.5 Understanding International Standards for Quality

According to Dale and Oakland [23] a standard is a technical or management specification which gives a precise and authoritative statement of criteria necessary to ensure that a material or procedure is fit for the purpose for which it is intended. Standardisation is activity aimed at improving efficiency by bringing consistency to products, services and processes. Standards should be a means of preventing recurrence of abnormalities in processes. Improved quality and reliability are seen as advantages of standardisation. [23] The ISO 9000 and the British Standards Institution (BSI) series are examples of documented standards for quality.

Seddon [66] has stated that ISO 9000 discourages managers from learning about the theory of variation. Understanding the theory of variation has been established as a key factor in correct interpretation of data and correct use of the control chart. Seddon indicates that following guidelines set out in the standards leads to businesses being run solely by manufacturing operations without the additional benefits gained from use of the control chart.

The role of variation and statistical tools in the operation of business must be considered when adhering to the British and international standards for quality. If this is done the profitability of certification by the ISO will be continuous. Attention is here drawn, yet again, to the success of the Japanese manufacturing industry. In Japan promotion of industrial standardisation and quality control were started at the same time [40]. Investigation of theories and experiments in sampling, division, and measurement and analysis methods for iron ore led to Japan Industrial Standards for quality of this product. These became the accepted standards for the International Standards Organisation. They helped to smooth the functioning of international trade. [41]

We will now show how adherence to the standards for quality influences use of the control chart.

5.5.1 Effects on the use of control charts

Statistical techniques have been recognised in the ISO 9000 series as well-established tools in quality assurance and quality improvement programmes. The standard requires identification of points in the process where statistical techniques should be used. Applications of the standard include establishing, controlling and verifying process capability and product characteristics. Where there are only few characteristics, application of the standards for statistical techniques may be limited. [43] Before applying the specifications of the standard there must be adequate training in the techniques which are to be standardised. There should also be a thorough knowledge of the process and the product.

There are indications, however, that the British standards currently available could hamper or limit the use of control chart for statistical process control. Harris [37] indicates that these standards reflect more of an SQC rather than an SPC stance. SQC deals more-so with measurement of product characteristics after their manufacture. The standards currently available include BS5701 — *Control charts for the number defective*; BS5703 — *Guide to data analysis and quality control using cusum techniques*; BS5700 — *Guide to control charts.* These all reflect monitoring after the event. Harris [37] regards as SPC the more up-to-date approach of identifying important process parameters which influence variation in product characteristics and monitoring them using control charts. He points out that different committees within the ISO have been set up to revise and update the standards so that they reflect more current thinking in SPC.

The BS series present the operator of the control chart with a conflict of ideas in some respects. This concerns, particularly, whether the probabilistic or heuristic control limits should be used. [37] The penalty for use of the sometimes sub-optimal heuristic values for control limits has been highlighted in Chapters 3 and 4.

Current thinking in SPC incorporates control of process parameters; requires knowledge of the relationship between the process and the product; and may require use of special techniques such as R and $\bar{x}$-charts to monitor process parameters. [37] Certification by the ISO should be a by-product of a total quality management programme which incorporates use of the relevant statistical tools. Then, certification can also become a by-product of use of ECPTU functions to optimise use of control charts.

5.6 Practical Application of ECPTU functions

Certain practical factors must be considered in order to enhance the successful application of ECPTU functions to the selection of design parameters for the $\bar{x}$-chart. Such factors include:-

1. parameter estimation;
2. availability of technology for optimisation;
3. mathematical intractability of some functions;
4. variation in the parameters omitted from the functions;
5. difficulty in obtaining a minimum value.

Ways of addressing these factors will be dealt with in turn.

5.6.1 Parameter estimation

In order to implement use of the ECPTU functions distribution parameters and cost coefficients need to be estimated. The examples of numerical values of cost and technical parameters used in this study were first found in [28]. The exact source of the data is not known but we assume that they mirror particular production conditions.

As well as describing the production environment, the numeric estimates of the parameters should have desirable statistical properties. Bard [5] states that it is desirable that estimators have sampling distributions which are concentrated in the neighbourhood of the true values for the parameters. Such estimators will be unbiased; have minimum variance; have low mean square error; and be consistent. [34]

Theoretical analysis, replication and computer simulation are approaches to the assessment of these properties. Theoretical analysis is based on the derivation of the sampling distribution for parameters and some of its relevant properties such as the mean and variance. Replication provides estimates of the mean, variance and other properties of the sampling distribution after repetition of processes which yield sample values. Computer simulation can be used to find multiple estimates of parameters. Estimates of the mean and variance of the sampling distribution can thus be obtained. In addition, the bias of the estimators can be estimated. [5]

In this study, we have derived probability density functions for S, the OOC period and Z_i, the restoration period which follows the OOC signal. A probability distribution function was also derived for R, the number of samples for the duration of the cycle. α, β and λ are distribution parameters which have been used in these derivations.

Computer simulation could be used to produce multiple estimates of S, R and Z_i for given values of α, β and λ. The mean of observed values of R and Z_i may be regarded as the functions of "true" values of the distribution parameters. Estimates of these parameters for a given process can be found through comparison of the mean of observed values of Z_i and R with those simulated for given values of α, β and λ. Distribution parameter values which will be applied to the selection of design parameters should be those which produce comparatively low estimates of the bias and variance of S, Z_i and R.

The cost coefficients enter the function in a linear manner. With all other parameters of the ECPTU function held constant, the least squares method of estimation could be used to provide estimates of these cost coefficients. If the probability density functions for these coefficients are known, maximum likelihood estimation methods could be used to find estimates.

In some organisations there are records of costs associated with identification and removal of defective items. There may also be records of costs incurred by activities to prevent production of defective items. It may be possible to obtain estimates of the cost coefficients from such records.

Montgomery [54] states that the magnitude of the shift in the process mean is easier to estimate than are the costs. Models tend to be more sensitive to inaccuracies in the shift coefficient estimates than to inaccurate cost estimates. [54] Therefore, it is important that δ is estimated with greatest precision following careful study of the nature of shift in the process mean.

5.6.2 Availability of technology for optimisation

The literature report a limited application of the results of research into the economic design of control charts. [54, 15] This limited application arises from a need for increased availability of computer programs for the economic design of the control chart and development of simplified approximate optimisation procedures suitable for manual computation [54]. If these needs are met increased practical implementation of the result would be facilitated.

The optimum design parameters presented in this study were obtained using the numerical minimisation functions of *Mathematica*, a mathematical computing package. Other researchers [1, 15] have developed their own computer programmes and optimisation routines implementable on a personal computer. Researchers such as Tagaras and Lee [74] and Duncan [28] have used manual computation of partial derivatives to begin the search for approximate optimum values.

Manual computation using partial derivatives will be feasible if the exponents which form part to the ECPTU functions are approximated to simpler expressions. Such approximations lead to polynomial expressions in terms of h. An optimum value for h can then be obtained manually.

A computing package such as *Mathematica* will minimise the requirement for advanced programming skills or for tedious manipulation of partial derivative functions in efforts to obtain the optimum parameter values. The functions as presented in the literature can be input directly into the package's minimisation subroutines. The speed with which results are obtained is thereby enhanced.

Mathematica's minimisation function has its limitations, nevertheless. As stated in Section 3.7, the package's minimisation function computes estimates using the steepest descent method. Bard [5] states that the method of steepest descent is often very inefficient. Wolfram [79] states that this method does not use infinitesimal steps in searching for a minimum value. Therefore, it is possible to "overshoot even a local minimum". Bard [5] further states that this method is not recommended for practical applications.

Appendix G gives guidelines[3] which can be used to develop an algorithm which finds n^*, h^* and k^* using partial derivatives. Unlike some algorithms presented by Chung [14, 15], Duncan [28] and Tagaras and Lee [73], the first and second partial derivatives of the objective function are used to determine the increment in the design parameter values until optimum values are found. Future work could involve development of a computer program based on the guidelines given. The efficiency of such a program could be assessed by comparison of its results with those obtained using existing algorithms.

The relative simplicity of the objective functions presented in Chapter 3 should help reduce the complexity of the optimisation procedure.

5.6.3 Mathematical intractability of some functions

In Chapter 3 the different approaches to derivation of the ECPTU functions have yielded general forms of the functions. All the approaches to their derivation involve summation over discrete variables. See Sections 3.5.1 and 4.3. Use of the Erlang distribution assumption for the IC period has produced closed forms of the objective functions. As the value of the shape parameter increases, however, the algebraic

[3] The late Dr. D.W. Wightman assisted with preparation of these notes.

expressions become more complex. Such complexity could deter their usefulness, particularly in the absence of the relevant knowledge and skills required for their effective use.

For some distribution assumptions, closed forms of the summations would not be produced. This could be a hindrance to application of general results for the functions presented in previous chapters. For example, a study of the process could reveal that the in-control or repair period is better modelled by the log-normal distribution. For the log-normal distribution, the difficulty in finding a closed form of the various summations could obstruct derivation of an objective function. If the objective function is found using a different distribution assumption the results may be somewhat sub-optimal for the process to which they are applied.

5.6.4 Variation in omitted parameters

The ECPTU functions used here neglect the parameter representing the time spent taking and inspecting a sample. For $ECPTU_c$ the time and cost associated with searching for the assignable cause are also neglected. If the values of these omitted parameters varied greatly they could lead to larger design parameter values and increased ECPTU. This uncertainty is ignored. Thus, the model is simpler and more easily minimised.

Over time, the relevant personnel should become more experienced at sampling inspection or searching for the assignable cause. Consequently, the times associated with performance of these tasks should have minimum variation. Thus, in objective functions, it becomes more feasible to have these periods relegated to a constant or omitted if they are small relative to the rest of the cycle length. Since a ratio is being used to determine optimum parameter values, omission of this time period is less likely to yield results which are sub-optimal for the process.

5.6.5 Problems with convergence

A minimum value was not obtained for all parameter sets. Some combinations of parameter values created objective functions which could not be solved simultaneously for optimum parameter values. Some minimisation routines are limited. For example, as stated in Section 5.6.2, a local minimum may not be found because of the large steps used by the steepest descent method of parameter estimation. Therefore, parameter value combinations for a particular production environment can be such that a turning point for the plane or curve is not found. This limits implementation of the ECPTU function in this environment.

The results obtained in Chapters 3 and 4 were local minimum values. At times the starting point for the numeric search had to be changed in order to obtain a result. Failure to obtain an optimum result after repeated changes to the starting point for the search led to a declaration that the function failed to converge to an optimum.

It may become necessary to resort to manual partial differentiation of objective functions in order to obtain approximate optimum solutions. If the relevant personnel are unfamiliar with use of partial derivatives to find a minimum value and the numeric search by computer fails to give a result then further use of the ECPTU function may be prevented. In such situations, the results in Chapters 3 and 4 can be used to guide selection of design parameter values which will minimise the penalty for use of sub-optimal values.

5.7 A Desirable Production Environment

In this chapter we have discussed aspects of the production environment which influence, directly and indirectly, the use of the ECPTU function for selecting optimum design parameters. Adequate and appropriate staff training; a thorough knowledge of the product and process; accurate collection and use of data; addressing practical issues relevant to implementation of the ECPTU function are desirable features of a work environment interested in applying results of this function. Certification by national and international standards organisations for quality can be a by-product of these desirable features.

The quality of the production environment is further enhanced if it makes the relevant application of the Shewhart control chart through selection of optimum control chart parameters. The ECPTU function enables selection of optimum control chart parameters. Such selection must be implemented as part of a total quality management programme if it is to be truly beneficial to production.

Inaccurate data collection; inadequate staff training; or ignorance of the process and product can produce cost and technical parameter values which do not describe the process correctly. n, h, and k can be suboptimal if they are selected using such parameter values. Use of the ECPTU function to select design parameters can be safely implemented in an environment in which there are competent operators using control charts. Their competence will be reflected in the accuracy with which data are collected due to a very clear understanding of the process and product.

A good work environment has training regularly. Here training in measurement methods; the uses of SPC techniques; understanding variation is focussed. This means that the trainer aims at meeting the specific purpose of the training. This is an important contribution to reduction in variation in performance when the lessons learnt are practised. [68]

Efforts to create an environment for use of the ECPTU functions to select design parameters will require top management's ability to identify whether or not the practitioner's performance is in a state of statistical control. Deming as cited by Neave [59] states that once performance is in a state of statistical control — that is, it becomes predictable —further training will not improve this performance. Cost-effective improvement in performance can be expected only if another worker is correctly trained and set to carry out the task. This suggests that if an individual's incorrect use of the control chart is in a state of statistical control training will not improve performance. Replacement of this worker with another who has been correctly trained in the use of control charts will yield better results. This is because training is a "one-chance opportunity". If it is done wrong initially, the damage is permanent. [59]

Research reveals that the most widely experienced benefits of ISO 9000 certification include improvements in efficiency; better management control; increased customer satisfaction; improved customer service; and improving profitability. [8] The desirable features of the production environment mentioned in this section, if they form part of a total quality management programme, will ensure that these benefits are maintained throughout the life of the organisation.

Chapter 6

Conclusions and Recommendations

6.1 Main Findings

This research has derived the probability density function for the OOC period and a probability distribution function for R, the total number of samples taken for the duration of a cycle of the control chart's operation. Derivation of the probability distribution function for R enabled us to account explicitly for the stochastic nature of the time spent restoring the process to the IC state. This stochastic variable is assumed to be correlated with the period, rh, for which the process is monitored by the $\bar{x}$-chart. Thus general results for the expected restoration time, $E(Z_i)$, have been derived.

Objective functions from the reduced model presented in Chapter 3 have been used to produce guidelines on the effects of important process parameters on n^*, h^* and k^*. The penalty for use of sub-optimal design parameter combinations has also been studied.

If $\delta = 2$ and the IC period is an Erlang variate with shape parameter equal to 2 or 1, the use of $n = 6$, $k = 3.05$ and $h = 2$ hours will be approximately optimal. A 50% reduction in δ will require an increase in n and h of up to 100% if the penalty for use of design parameter values is to be minimised. A 10% reduction in k will enhance optimality.

When the time to shift in the process mean is an Erlang variate a one unit increase in the shape parameter β increases n^* and h^* most noticeably. As β exceeds 2 the increment in n^* and h^* can exceed 30%.

Incorporation of the stochastic nature of the restoration time which is correlated with the period rh most significantly increases the value of h^*. Accounting for the restoration rate as being inversely related to rh yields higher increments. Explicit reckoning of this inverse relationship can yield mean increment of up to 20% in comparison to a mean value of less than 3% when the linear relationship is incorporated.

Penalties for use of sub-optimal values were quantified as the percentage increment in ECPTU for use of suboptimal parameter combinations $\{n, h, k\} = \{5, 1 \text{ hour}, 3\}$ and $\{5, 8 \text{ hours}, 3\}$ instead of $\{n^*, h^*, k^*\}$. Compared with values obtained for use of $\{5, 8 \text{ hours}, 3\}$, use of $\{5, 1 \text{ hour}, 3\}$ yielded relatively lower penalty when $r_{c24} > 10$ or $r_{c23} > 5$.

A mean penalty of less than 20% has been observed when the functions account for restoration time as a stochastic variate correlated with the period rh. When the stochastic and correlated nature of restoration time is not explicitly accounted for the results indicate that the penalty for use of {5, 1 hour, 3} is also lower when $\delta = 2$.

In the absence of explicit reckoning of stochastic restoration times, the penalty for use of {5, 8 hours, 3} can be under 20% if the time to shift in the process mean is truly an Erlang($\lambda, 3$) variate and δ, r_{c24} and r_{c23} are all greater than or equal to 1. Use of $\{n = 5, k = 3\}$ most appreciably reduces the probability of detecting the OOC state if $\delta < 2$.

6.2 Possible Extension of This Research

There is need for further investigation of the nature of the relationship between the restoration time and the period for which the process is monitored by the $\bar{x}$-chart. Further research is also required to determine the distribution which best models these stochastic restoration times.

Further work can be done to investigate any advantages of using distributions different from the Erlang as the basis for the probability function for the number of samples taken for the duration of the cycle. It may become necessary to provide numeric values for all the parameters before closed solutions will be obtained. Efforts can also be made to extend the use of this probability function to process improvement using other types of control chart.

General results have been given for the expected length of the OOC period based on the exponential distribution; the expected restoration time and the expected number of samples taken for the duration of the cycle. These have all been based on statistical distributions which have scale and shape parameters. There is scope for research into the use of parameter estimation techniques such as calibration or non-linear estimation to determine the values of these distribution parameters.

The results of this research have particular relevance to the manufacturing sector. However, the UK economy possibly typifies the economies of other countries in becoming increasingly service oriented. There is scope for use of statistical tools such as the $\bar{x}$-chart in the service sector. This needs further investigation in order to optimise its use in this sector. Hence, this research can be extended to deriving objective functions which could be used to select optimum control chart parameters for the service sector. This will have to follow from consideration of whether the cost and technical parameters used for the manufacturing environment are relevant to the service sector.

6.3 Conclusions

In this study relatively large values of n^* and h^* have been found optimal for $\delta < 1$. It may not be practical to make use of such values. However, use of sample sizes which are less than five is not an economic optimum if $\delta < 1$. The $\bar{x}$-chart must be concurrently used with other statistical tools such as the range or cusum chart if $\delta < 1$ and use of $n > 10$ is not feasible.

There are processes for which an inverse relationship between repair rate and the period of surveillance by the control chart holds true. In such cases, it is important that this is accounted for in ECPTU functions used to select optimum design parameters. It is particularly crucial that this is done for r_{c23} or $r_{c24} < 1$.

Values of δ, r_{c23} and r_{c24} should be used to guide selection of n, h, and k for a process. As a result, the penalty of increased hourly for use of these design parameter values can be reduced. The penalty could be less than 10% if the repair rate truly is an inverse function of the control chart's surveillance period and the repair time is an Erlang variate.

If there must be successful implementation of use of the objective ECPTU function for selection of optimum design parameters the production environment must be adequately prepared. This preparation must be based on a TQM programme which promotes self-improvement for the entire work force. Thus they will be more adaptable to the changes which become necessary for implementation of the ECPTU.

Research done by Lockyer *et al* [50] has revealed limited usage of statistical methods of quality control in the UK manufacturing industry. The major barrier to acceptance of SQC in this region is a lack of knowledge of effective steps that must be taken to improve quality [50]. Before steps can be taken to implement the economic design of the $\bar{x}$-chart this barrier will have to be surmounted.

6.4 Recommendations

Selection of optimum design parameters using ECPTU objective functions should follow a careful long term study of the process to which the values will be applied. This study should provide estimates of cost and technical parameters which will be input to the function. The study should also seek to determine any relationship between the restoration period and the surveillance period. Figure 6.1 summarises recommendations for choosing the approporiate ECPTU function used to obtain optimum design parameter values.

Figure 6.2 summarises recommendations for selecting design parameters based on values of δ, r_{c23} and r_{c24} if facilities for use of ECPTU function are unavailable. The results of this study indicate that $n = 5$ and $k = 3$ are best used when $\delta = 2$.

For a 50% reduction in δ, n should be increased by 100% and k decreased by 10%.

Use of $h = 8$ hours is best for r_{c23} or $r_{c24} < 1$. Use of $h = 1$ hour is best for $r_{c23} > 5$ and $r_{c24} > 10$. For $1 \leq r_{c23} \leq 5$ or $1 \leq r_{c24} \leq 10$, use of $h = 3$ hours will lessen the penalty for use of suboptimal values.

In the absence of facilities for minimising ECPTU functions, changes in h may be used as the basis for selecting n and k. Figure 6.3 summarises recommendations for such selection.

The cusum chart could be used in conjunction with the $\bar{x}$ and range charts, particularly when $\delta \leq 1$ and use of $n > 5$ is not feasible. Such use of the charts will facilitate a shortened OOC period.

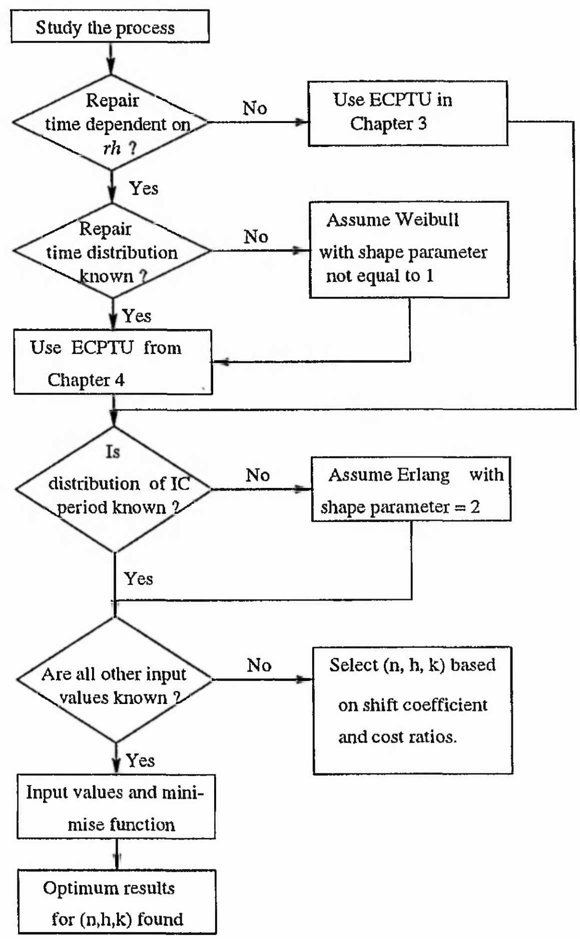


Figure 6.1: Summary of recommendations for choosing the appropriate ECPTU function.

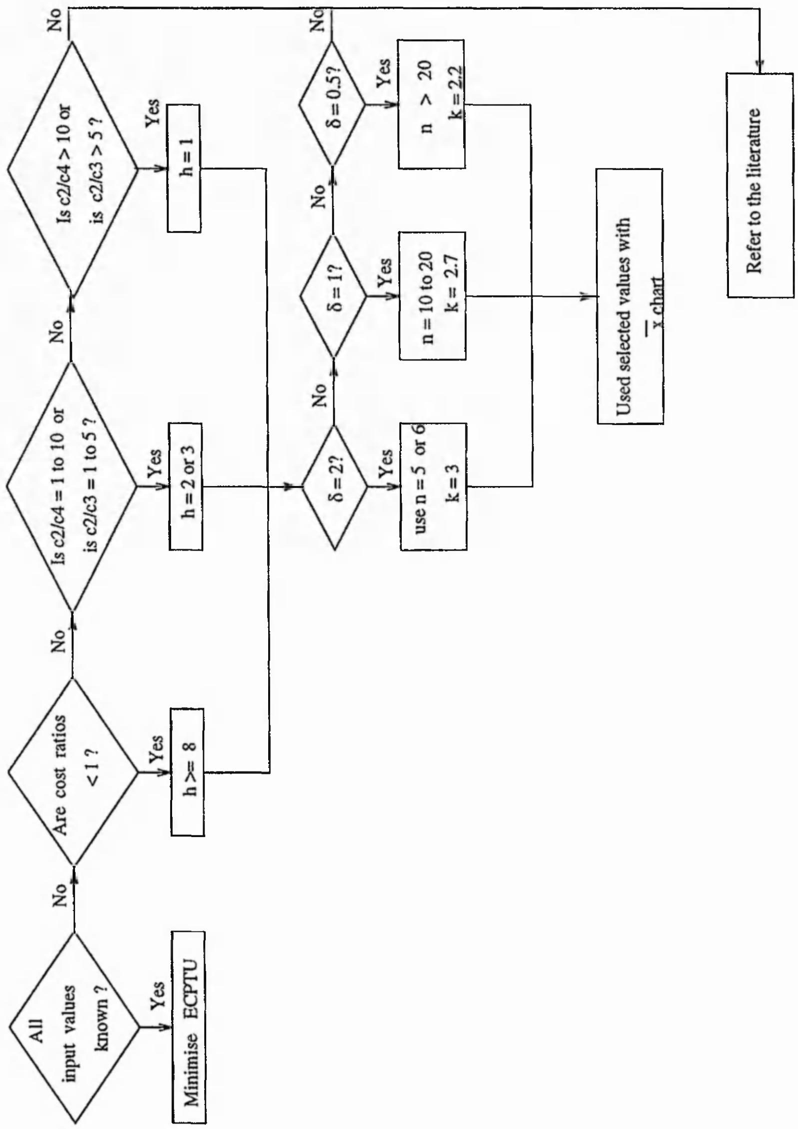


Figure 6.2: Recommendations for selecting n, h and k based on cost ratios, r_{c23} and r_{c24}, and the value of the shift coefficient, δ.

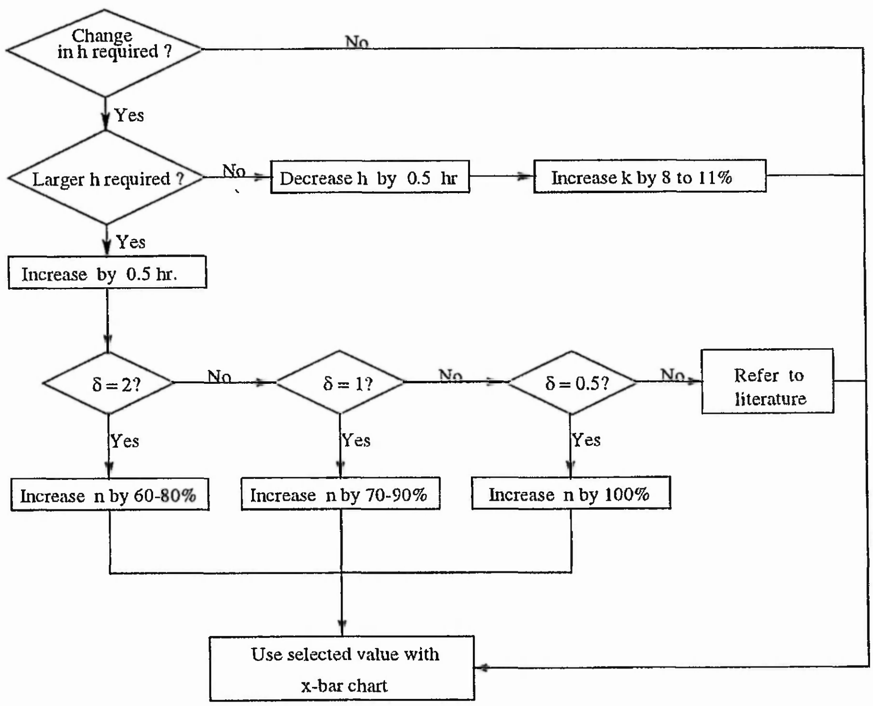


Figure 6.3: Recommendations for selecting n and k based on changes in h.

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Appendix A

Notation Used by Lorenzen and Vance [51]

A.1 Output and Calculated Variables

$$\tau = \frac{1 - \frac{1+h\,\lambda}{e^{h\,\lambda}}}{\left(1 - e^{-(h\,\lambda)}\right)\,\lambda}$$

$$s = \frac{1}{e^{h\,\lambda}\,\left(1 - e^{-(h\,\lambda)}\right)}$$

$$\begin{aligned}
\text{ARL1} &= \text{average run length while in control} \\
&= \frac{1}{\alpha} \text{ when the measured statistics are independent} \\
\alpha &= \Pr(\text{exceeding control limits—process in control}) \\
\text{ARL2} &= \text{average run length while out of control} \\
&= \frac{1}{1-\beta} \text{ when the measured statistics are independent} \\
\beta &= \Pr(\text{not exceeding control limits—process is out of control})
\end{aligned}$$

A.2 Input Variables

$$\lambda = \frac{1}{\text{mean time process is in control.}}$$

Δ = number of standard deviations slip when out of control.

E = time to sample and chart one item.

T_0 = expected search time when false alarm.

T_1 = expected time to discover the assignable cause.

T_2 = expected time to repair the process.

δ_1 = 1if production continues during searches

= 0if production ceases during searches.

δ_2 = 1if production continues during repair

= 0if production ceases during repair.

C_0 = quality cost per hour while producing in control.

C_1 = quality cost per hour while producing out of control($> C_0$).

Y = cost per false alarm.

W = cost to locate and repair the assignable cause.

a = fixed cost per sample.

b = cost per unit sampled.

Appendix B

Derivation of Objective Functions in Chapter 3

B.1 Derivation of m_1 for Erlang variates

i represents the number of samples taken for the duration of the IC period. We assume that i is an Erlang variate with scale parameter λ and shape parameter β.

$$\begin{aligned} m_1 &= \sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} i f(t) dt \\ &= \sum_{i=0}^{\infty} \left(\int_{0}^{(i+1)h} i f(t) dt - \int_{0}^{ih} i f(t) dt \right) \end{aligned}$$

We use the distribution function for an Erlang variate given in [38] to produce

$$m_1 = \sum_{i=0}^{\infty} \left(i e^{-\lambda i h} \left(\sum_{l=0}^{\beta-1} \frac{(ih\lambda)^l}{l!} \right) - i e^{-\lambda(i+1)h} \left(\sum_{l=0}^{\beta-1} \frac{((i+1)h\lambda)^l}{l!} \right) \right)$$

Thus, for $\beta = 2$,

$$\begin{aligned} m_1 &= \sum_{i=0}^{\infty} \left(ie^{-\lambda ih} \left(1 + ih\lambda\right) - ie^{-\lambda(i+1)h} \left(1 + (i+1)h\lambda\right) \right) \\ &= \sum_{i=0}^{\infty} \left(ie^{-\lambda ih} \left(1 + ih\lambda\right) \left(1 - e^{-\lambda h}\right) - \lambda h e^{-\lambda h} ie^{-\lambda ih} \right) \end{aligned}$$

For $\beta = 3$,

$$\begin{aligned} m_1 &= \sum_{i=0}^{\infty} \left(ie^{-\lambda ih} \left(1 + ih\lambda + \frac{(ih\lambda)^2}{2}\right) \right) - \\ & \sum_{i=0}^{\infty} \left(ie^{-\lambda(i+1)h} \left(1 + (i+1)h\lambda + \frac{((i+1)h\lambda)^2}{2}\right) \right) \\ &= \sum_{i=0}^{\infty} \left(ie^{-\lambda ih} \left(1 + ih\lambda + \frac{(ih\lambda)^2}{2}\right) \left(1 - e^{-\lambda h}\right) \right) - \\ & \sum_{i=0}^{\infty} \left(e^{-\lambda h} ie^{-\lambda ih} (\lambda h + i(\lambda h)^2 + \frac{(\lambda h)^2}{2}) \right) \end{aligned}$$

$e^{-\lambda ih}(1 - e^{-\lambda h})$ can be regarded as the probability function for a geometric variate which has parameter $(1 - e^{-\lambda h})$. With x' as the dummy variable the moment generating function — denoted as $m(x')$— for this probability function is as follows

$$m(x') = \frac{e^{x} \left(1 - e^{-(h\,\lambda)}\right)}{1 - e^{-(h\,\lambda)+x}} \tag{B.1}$$

This m.g.f. was differentiated repeatedly to produce moments about the origin. Differentiation routines in the package *Mathematica* were used.

The moments about the origin produced by Equation B.1 are as follows

$$\begin{aligned} ie^{-\lambda ih}\left(1 - e^{-\lambda h}\right) &= \frac{e^{-\lambda h}}{(1 - e^{-\lambda h})} \\ i^2 e^{-\lambda ih}\left(1 - e^{-\lambda h}\right) &= \frac{e^{-\lambda h}(1 + e^{-\lambda h})}{(1 - e^{-\lambda h})^2} \\ i^3 e^{-\lambda ih}\left(1 - e^{-\lambda h}\right) &= \frac{e^{-\lambda h}(1 + 4e^{-\lambda h} + e^{-2\lambda h})}{(1 - e^{-\lambda h})^3} \end{aligned}$$

Substitution of the respective moments about the origin yield the following expressions.

For $\beta = 2$

$$\begin{aligned} m_1 &= \frac{e^{-\lambda h}}{(1-e^{-\lambda h})} + \lambda h \frac{e^{-\lambda h}(1+e^{-\lambda h})}{(1-e^{-\lambda h})^2} - \lambda h \frac{e^{-2\lambda h}}{(1-e^{-\lambda h})^2} \\ &= \frac{e^{-\lambda h}}{(1-e^{-\lambda h})}\left(1+\frac{\lambda h}{(1-e^{-\lambda h})}\right) \end{aligned}$$

For $\beta = 3$,

$m_1 =$

$$\frac{e^{-\lambda h}}{(1-e^{-\lambda h})} + \lambda h \frac{e^{-\lambda h}(1+e^{-\lambda h})}{(1-e^{-\lambda h})^2} + \frac{(\lambda h)^2 e^{-\lambda h}(1+4e^{-\lambda h}+e^{-2\lambda h})}{2(1-e^{-\lambda h})^3} -$$

$$\left(\lambda h\left(\frac{e^{-\lambda h}}{(1-e^{-\lambda h})}\right)^2 + (\lambda h)^2 \frac{e^{-2\lambda h}(1+e^{-\lambda h})}{(1-e^{-\lambda h})^3} - \frac{(\lambda h)^2}{2}\left(\frac{e^{-\lambda h}}{(1-e^{-\lambda h})}\right)^2\right)$$

$$= \frac{\left(-e^{-3\lambda h} + \frac{2}{e^{2h\lambda}} + e^{-(h\lambda)}\right) h^2\lambda^2}{2\left(1-e^{-(h\lambda)}\right)^3} + \frac{1+\frac{h\lambda}{1-e^{-(h\lambda)}} - \frac{h^2\lambda^2}{2e^{h\lambda}\left(1-e^{-(h\lambda)}\right)}}{e^{h\lambda}\left(1-e^{-(h\lambda)}\right)}$$

B.2 Derivation of $E(S)$

S is the random variable representing the period for which the process is out of control. The derived density function for this variable is shown below.

$$g(s) = \frac{\lambda e^{-\lambda(mh-s)}}{1-e^{-\lambda h}} \alpha_{11}(1-\alpha_{11})^{m-1} u\left(m - \left(Int\left[\frac{s}{h}\right]+1\right)\right)$$

for $mh \geq s \geq (m-1)h$. m is the number of samples taken for the duration of the OOC period.

Now $E(S) = \int_{(m-1)h}^{mh} sg(s)ds$.

$$E(S) = \sum_{m=1}^{\infty} \frac{\lambda e^{-\lambda mh}\alpha_{11}(1-\alpha_{11})^{m-1}}{1-e^{-\lambda h}} \int_{(m-1)h}^{mh} se^{\lambda s}ds$$

$$\int_{(m-1)h}^{mh} se^{\lambda s} ds = \frac{\lambda e^{\lambda mh}}{\lambda}\left(mh(1-e^{-\lambda h}) - \frac{1}{\lambda}(1-e^{-\lambda h}) + he^{-\lambda h}\right)$$

Therefore, the expected length of the OOC period when time to shift in the process mean is an Exponential variate is

$$\begin{aligned}
E(S) &= \sum_{m=1}^{\infty} \frac{\lambda e^{-\lambda(mh}\alpha_{11}(1-\alpha_{11})^{m-1}}{1-e^{-\lambda h}} \frac{\lambda e^{\lambda mh}}{\lambda} \times \\
&\quad \left(mh(1-e^{-\lambda h}) - \frac{1}{\lambda}(1-e^{-\lambda h}) + he^{-\lambda h}\right) \\
&= \sum_{m=1}^{\infty} \frac{\alpha_{11}(1-\alpha_{11})^{m-1}}{1-e^{-\lambda h}} \left(mh(1-e^{-\lambda h}) - \frac{1}{\lambda}(1-e^{-\lambda h}) + he^{-\lambda h}\right) \\
&= \frac{h}{\alpha_{11}} - \frac{1}{\lambda} + \frac{he^{-\lambda h}}{1-e^{-\lambda h}}
\end{aligned}$$

Appendix C

Derivation of Characteristic Features of R

C.1 Summation of $P(R=r)$ over all R

R is the random variable representing the number of samples taken for the duration of the cycle. We recall that when the IC period is assumed to be an Erlang(λ, 2) variate the expression for $P(R=r)$ is as follows

$$P(R=r) \quad = \quad \sum_{i=0}^{r-1} \alpha_{11}(1-\alpha_{11})^{r-i-1} \int_{ih}^{(i+1)h} \frac{\lambda^2 t}{e^{\lambda t}} dt$$

This equals

$$\sum_{i=0}^{r-1} (1-\alpha_{11})^{-1-i+r} \alpha_{11} \left(-\frac{h\lambda}{e^{h(1+i)\lambda}} + \frac{\left(1-e^{-(h\lambda)}\right)(1+hi\lambda)}{e^{hi\lambda}} \right)$$

Thus $P(R=r)$

$$= \quad (1-\alpha_{11})^{-1+r} \alpha_{11} \left(1-e^{-(h\lambda)}\right) \left(\sum_{i=0}^{-1+r} \frac{(1-\alpha_{11})^{-i}}{e^{hi\lambda}} \right) -$$

$$\frac{(1-\alpha_{11})^{-1+r}\alpha_{11}\,h\,\lambda\left(\sum_{i=0}^{-1+r}\frac{(1-\alpha_{11})^{-i}}{e^{h\,i\,\lambda}}\right)}{e^{h\,\lambda}}+$$
$$(1-\alpha_{11})^{-1+r}\alpha_{11}\left(1-e^{-(h\,\lambda)}\right)h\,\lambda\left(\sum_{i=0}^{-1+r}\frac{(1-\alpha_{11})^{-i}\,i}{e^{h\,i\,\lambda}}\right)$$

We assume $\left|\frac{1}{(1-\alpha_{11})\,e^{h\,\lambda}}\right| < 1$. Application of the results for finite sums given by Prudnikov *et al* [61] yields

$$\sum_{i=0}^{r-1}\left(\frac{e^{-(\lambda h)}}{1-\alpha_{11}}\right)^{i} = \frac{1-\left(\frac{e^{(-\lambda\,h)}}{1-\alpha_{11}}\right)^{r}}{1-\left(\frac{e^{(-\lambda\,h)}}{1-\alpha_{11}}\right)}$$
$$\sum_{i=0}^{r-1}i\left(\frac{e^{-(\lambda h)}}{1-\alpha_{11}}\right)^{i} = \frac{\left(\frac{e^{(-\lambda\,h)}}{1-\alpha_{11}}\right)+\left((r-1)\left(\frac{e^{(-\lambda\,h)}}{1-\alpha_{11}}\right)-r\right)\left(\frac{e^{(-\lambda\,h)}}{1-\alpha_{11}}\right)^{r}}{\left(1-\left(\frac{e^{(-\lambda\,h)}}{1-\alpha_{11}}\right)\right)^{2}}$$

Thus,

$$P(R=r) = \frac{\left(1-e^{-(h\,\lambda)}-\frac{h\,\lambda}{e^{h\,\lambda}}\right)\left(-e^{-(h\,\lambda\,r)}+(1-\alpha_{11})^{r}\right)\alpha_{11}}{1-e^{-(h\,\lambda)}-\alpha_{11}}+$$
$$\frac{\left(1-e^{-(h\,\lambda)}\right)h\,\lambda\,\alpha_{11}\left(-e^{-(h\,\lambda)-h\,\lambda\,r}+\frac{(1-\alpha_{11})^{r}}{e^{h\,\lambda}}\right)}{\left(1-e^{-(h\,\lambda)}-\alpha_{11}\right)^{2}}+$$
$$\frac{\left(1-e^{-(h\,\lambda)}\right)h\,\lambda\,\alpha_{11}\left(e^{-(h\,\lambda)-h\,\lambda\,r}\,r-\frac{(1-\alpha_{11})\,r}{e^{h\,\lambda\,r}}\right)}{\left(1-e^{-(h\,\lambda)}-\alpha_{11}\right)^{2}}$$

Hence we have $P(R=r)$

$$=\frac{\left(1-e^{-(h\,\lambda)}-\frac{h\,\lambda}{e^{h\,\lambda}}\right)\left(-e^{-(h\,\lambda\,r)}+(1-\alpha_{11})^{r}\right)\alpha_{11}}{1-e^{-(h\,\lambda)}-\alpha_{11}}+$$
$$\frac{\alpha\left(1-e^{-(h\,\lambda)}\right)h\,\lambda}{\left(1-\alpha_{11}-e^{-(h\,\lambda)}\right)^{2}}\times$$
$$\left(\frac{(1-\alpha_{11})^{r}}{e^{h\,\lambda}}-e^{-(h\,\lambda)-h\,\lambda\,r}-\frac{(1-\alpha_{11})\;r}{e^{h\,\lambda\,r}}+e^{-(h\,\lambda)-h\,\lambda\,r}\,r\right)$$

In order to sum $P(R=r)$ over all R we use the following results.

$$\sum_{r=1}^{\infty}(1-\alpha_{11})^{r} = \frac{1-\alpha_{11}}{\alpha_{11}}$$

$$\sum_{r=1}^{\infty} e^{(-h\,\lambda\,r)} = \frac{e^{-(h\,\lambda)}}{1-e^{-(h\,\lambda)}}$$
$$\sum_{r=1}^{\infty} r e^{(-h\,\lambda\,r)} = \frac{e^{-(h\,\lambda)}}{\left(1-e^{-(h\,\lambda)}\right)^2}$$

Substitution of the results yields

$\sum_{r=1}^{\infty} P(R=r)$

$$= \alpha_{11}\frac{1-e^{(-\lambda\,h)}-h\lambda e^{(-\lambda\,h)}}{1-e^{(-\lambda\,h)}-\alpha_{11}}\left(\frac{1-\alpha_{11}}{\alpha_{11}}-\frac{e^{(-\lambda\,h)}}{1-e^{(-\lambda\,h)}}\right)+$$
$$\frac{\alpha_{11}\left(1-e^{-(h\,\lambda)}\right)h\,\lambda}{\left(1-\alpha_{11}-e^{-(h\,\lambda)}\right)^2}\times$$

$$\left(\frac{e^{-\lambda\,h}(1-\alpha_{11})}{\alpha_{11}}+\left(\frac{e^{-\lambda\,h}}{(1-e^{-\lambda\,h})}\right)^2-\frac{e^{-2\,\lambda\,h}}{(1-e^{-\lambda\,h})}-\frac{(1-\alpha_{11})e^{-\lambda\,h}}{(1-e^{-\lambda\,h})^2}\right)$$

$$= \alpha_{11}\left(\frac{1-e^{(-\lambda\,h)}-h\lambda e^{(-\lambda\,h)}}{1-e^{(-\lambda\,h)}-\alpha_{11}}\right)\left(\frac{1-e^{(-\lambda\,h)}-\alpha_{11}}{\alpha\left(1-e^{(-\lambda\,h)}\right)}\right)+$$
$$\frac{\alpha\left(1-e^{-(h\,\lambda)}\right)h\,\lambda}{\left(1-\alpha_{11}-e^{-(h\,\lambda)}\right)^2}\times$$
$$\left(\frac{e^{(-\lambda\,h)}\left(1-e^{(-\lambda\,h)}\right)^2(1-\alpha_{11})-e^{(-2\,\lambda\,h)}\alpha_{11}\left(1-e^{(-\lambda\,h)}\right)}{\alpha_{11}\left(1-e^{(-\lambda\,h)}\right)^2}\right)+$$
$$\frac{\alpha\left(1-e^{-(h\,\lambda)}\right)h\,\lambda}{\left(1-\alpha_{11}-e^{-(h\,\lambda)}\right)^2}\left(\frac{\alpha_{11}\,e^{(-2\,\lambda\,h)}-\alpha_{11}\,e^{(-\lambda\,h)}+\alpha_{11}^2 e^{(-\lambda\,h)}}{\alpha_{11}\left(1-e^{(-\lambda\,h)}\right)^2}\right)$$

Thus,

$\sum_{r=1}^{\infty} P(R=r) =$

$$1-\frac{\lambda\,h e^{(-\lambda\,h)}}{1-e^{(-\lambda\,h)}}+$$
$$\frac{h\lambda e^{(-\lambda\,h)}\left(1-\alpha_{11}-2e^{(-\lambda\,h)}+2\alpha_{11}\,e^{(-\lambda\,h)}+e^{(-2\,\lambda\,h)}\right)}{\left(1-e^{(-\lambda\,h)}\right)\left(1-\alpha_{11}-e^{(-\lambda\,h)}\right)^2}+$$
$$\frac{h\lambda e^{(-\lambda\,h)}\left(-\alpha_{11}\,e^{(-2\,\lambda\,h)}-\alpha_{11}\,e^{(-\lambda\,h)}+\alpha_{11}\,e^{(-2\,\lambda\,h)}\right)}{\left(1-e^{(-\lambda\,h)}\right)\left(1-\alpha_{11}-e^{(-\lambda\,h)}\right)^2}+$$

$$\frac{h\lambda e^{(-\lambda\,h)}\left(\alpha_{11}\,e^{(-\lambda\,h)}-\alpha_{11}+\alpha_{11}^2\right)}{\left(1-e^{(-\lambda\,h)}\right)\left(1-\alpha_{11}-e^{(-\lambda\,h)}\right)^2}$$

Hence, $\sum_{r=1}^{\infty} P(R=r)=1$ since $\left(1-\alpha_{11}-e^{-(h\,\lambda)}\right)^2=$

$$1-2\,\alpha_{11}+\alpha_{11}^2-2e^{-\lambda\,h}+2\alpha_{11}\,e^{-\lambda\,h}+e^{-2\,\lambda\,h}$$

C.2 The Moment Generating Function

We will now outline the steps in the derivation of $m(x')$, moment generating function for the probability distribution function for the variate R. x' is used as the dummy variable.

$$m(x')=$$

$$\sum_{r=1}^{\infty} e^{x' r}\sum_{i=0}^{r-1}\left(1-\alpha_{11}\right)^{-1-i+r}\alpha_{11}\left(-\frac{h\,\lambda}{e^{h\,(1+i)\,\lambda}}+\frac{\left(1-e^{-(h\,\lambda)}\right)\left(1+h\,i\,\lambda\right)}{e^{h\,i\,\lambda}}\right)$$

When $r=1$, $i=0$. Therefore,

$$m(x') = e^{x'}\alpha_{11}\left(1-e^{-(h\,\lambda)}-\lambda\,he^{-(h\,\lambda)}\right)$$

When $r=2$, $i=0$ or 1. Therefore,

$$\begin{aligned} m(x') &= e^{2x'}\left(\alpha_{11}(1-\alpha_{11})\left(1-e^{-(h\,\lambda)}-\lambda\,he^{-(h\,\lambda)}\right)\right)+ \\ & \quad e^{2x'}\left(\alpha_{11}\left((1-e^{-(h\,\lambda)})e^{-(h\,\lambda)}(1+h\,\lambda)-\lambda\,h\,e^{-(2\lambda\,h)}\right)\right) \end{aligned}$$

When $r=3$, $i=0$, 1 or 2. Therefore,

$$\begin{aligned} m(x') &= e^{3x'}\left(1-e^{-\lambda\,h}-\lambda\,he^{-\lambda\,h}\right)\alpha_{11}(1-\alpha_{11})^2+ \\ & \quad e^{3x'}\left(\left(1-e^{-\lambda\,h}\right)e^{-\lambda\,h}(1+\lambda\,h)-\lambda\,h\,e^{-(2\lambda\,h)}\right)\alpha_{11}(1-\alpha_{11})+ \\ & \quad e^{3x'}\left(\left(1-e^{-\lambda\,h}\right)e^{-2\,\lambda\,h}(1+2\,\lambda\,h)-\lambda\,h\,e^{-(3\lambda\,h)}\right)\alpha_{11} \end{aligned}$$

When $r = 4$, $i = 0$, 1, 2 or 3. Therefore,

$$
\begin{aligned}
m(x') &= e^{4x'}\left(1 - e^{-\lambda h} - \lambda h e^{-\lambda h}\right)\alpha_{11}(1-\alpha_{11})^3 + \\
&\quad e^{4x'}\left(\left(1 - e^{-\lambda h}\right)e^{-\lambda h}(1+\lambda h) - \lambda h e^{-(2\lambda h)}\right)\alpha_{11}(1-\alpha_{11})^2 + \\
&\quad e^{4x'}\left(\left(1 - e^{-\lambda h}\right)e^{-2\lambda h}(1+2\lambda h) - \lambda h e^{-(3\lambda h)}\right)\alpha_{11}(1-\alpha_{11}) + \\
&\quad e^{4x'}\left(\left(1 - e^{-\lambda h}\right)e^{-3\lambda h}(1+3\lambda h) - \lambda h e^{-(4\lambda h)}\right)\alpha_{11}
\end{aligned}
$$

Summation over all values of R yields

$$
\begin{aligned}
m(x') = \\
&\left(1 - e^{-\lambda h} - \lambda h e^{-\lambda h}\right) \times \\
&\left(\alpha_{11}e^{x'} + \alpha_{11}(1-\alpha_{11})\,e^{2x'} + \ldots + \alpha_{11}(1-\alpha_{11})^q\,e^{(q+1)x'}\right) \\
+ &\left(\left(1 - e^{-\lambda h}\right)e^{-\lambda h}(1+\lambda h) - \lambda h e^{-(2\lambda h)}\right) \times \\
&\left(\alpha_{11}e^{2x'} + \alpha_{11}(1-\alpha_{11})\,e^{3x'} + \ldots + \alpha_{11}(1-\alpha_{11})^q\,e^{(q+2)x'}\right) \\
+ &\left(\left(1 - e^{-\lambda h}\right)e^{-2\lambda h}(1+2\lambda h) - \lambda h e^{-(3\lambda h)}\right) \times \\
&\left(\alpha_{11}e^{3x'} + \alpha_{11}(1-\alpha_{11})\,e^{4x'} + \ldots + \alpha_{11}(1-\alpha_{11})^q\,e^{(q+3)x'}\right) \\
+ &\left(\left(1 - e^{-\lambda h}\right)e^{-3\lambda h}(1+3\lambda h) - \lambda h e^{-(4\lambda h)}\right) \times \\
&\left(\alpha_{11}e^{4x'} + \alpha_{11}(1-\alpha_{11})\,e^{5x'} + \ldots + \alpha_{11}(1-\alpha_{11})^q\,e^{(q+4)x'}\right) \\
+ &\left(\left(1 - e^{-\lambda h}\right)e^{-4\lambda h}(1+4\lambda h) - \lambda h e^{-(5\lambda h)}\right) \times \\
&\left(\alpha_{11}e^{5x'} + \alpha_{11}(1-\alpha_{11})\,e^{6x'} + \ldots + \alpha_{11}(1-\alpha_{11})^q\,e^{(q+5)x'}\right) \\
+ &\ \ldots \\
+ &\left(\left(1 - e^{-\lambda h}\right)e^{-(r-1)\lambda h}(1+(r-1)\lambda h) - \lambda h e^{-(r\lambda h)}\right) \times \\
&\left(\alpha_{11}e^{r x'} + \alpha_{11}(1-\alpha_{11})\,e^{(1+r)x'} + \ldots + \alpha_{11}(1-\alpha_{11})^q\,e^{(q+r)x'}\right) \\
+ &\ \ldots
\end{aligned}
$$

for $q = 0 \ldots \infty$. Further series summation gives

$$
\begin{aligned}
m(x') = \\
&\left(1 - e^{-\lambda h} - \lambda h e^{-\lambda h}\right)\frac{\alpha_{11}e^{x'}}{1-(1-\alpha_{11})e^{x'}} +
\end{aligned}
$$

$$\left(\left(1-e^{-\lambda h}\right)e^{-\lambda h}(1+\lambda h)-\lambda h\, e^{-(2\lambda h)}\right)\frac{\alpha_{11}e^{2x'}}{1-(1-\alpha_{11})e^{x'}}+$$
$$\left(\left(1-e^{-\lambda h}\right)e^{-2\lambda h}(1+2\lambda h)-\lambda h\, e^{-(3\lambda h)}\right)\frac{\alpha_{11}e^{3x'}}{1-(1-\alpha_{11})e^{x'}}+$$
$$\left(\left(1-e^{-\lambda h}\right)e^{-3\lambda h}(1+3\lambda h)-\lambda h\, e^{-(4\lambda h)}\right)\frac{\alpha_{11}e^{4x'}}{1-(1-\alpha_{11})e^{x'}}+\ldots$$
$$\left(\left(1-e^{-\lambda h}\right)e^{-(r-1)\lambda h}(1+(r-1)\lambda h)-\lambda h\, e^{-(r\lambda h)}\right)\frac{\alpha_{11}e^{r\,x'}}{1-(1-\alpha_{11})e^{x'}}+\ldots$$

Thus

$$m(x') =$$
$$\left(\frac{\alpha_{11}e^{x'}(1-e^{-\lambda h})}{1-(1-\alpha_{11})\,e^{x'}}\right)\left(1+e^{x'-\lambda h}+e^{2(x'-\lambda h)}+e^{3(x'-\lambda h)}+\ldots+e^{q(x'-\lambda h)}+\ldots\right)-$$
$$\left(\frac{\alpha_{11}\,\lambda\, h e^{(x'-\lambda h)}}{1-(1-\alpha_{11})\,e^{x'}}\right)\left(1+e^{x'-\lambda h}+e^{2(x'-\lambda h)}+e^{3(x'-\lambda h)}+\ldots+e^{q(x'-\lambda h)}+\ldots\right)+$$
$$\frac{\alpha_{11}e^{x'}\,\lambda\, h(1-e^{-(\lambda h)})}{1-(1-\alpha_{11})\,e^{x'}}\left(e^{x'-\lambda h}+2\,e^{2(x'-\lambda h)}+3\,e^{3(x'-\lambda h)}+\ldots+q\,e^{q(x'-\lambda h)}+\ldots\right)$$

which equals

$$\left(\frac{\alpha_{11}e^{x'}(1-e^{-\lambda h})}{1-(1-\alpha_{11})\,e^{x'}}\right)\left(\frac{1}{1-e^{x'-\lambda h}}\right)+$$
$$\left(\frac{\alpha_{11}ex'}{1-(1-\alpha_{11})\,e^{x'}}\right)\left(\frac{\lambda\, he^{x'-\lambda h}(1-e^{-\lambda h})}{(1-e^{x'-\lambda h})^2}-\frac{\lambda\, he^{-\lambda h}}{(1-e^{x'-\lambda h})}\right)$$

Thus, when the time to shift in the process mean is an Erlang(λ, 2)variate, the moment generating function for the variable R can be written as follows

$$m(x') = \frac{e^{x'}\left(\frac{1-e^{-(h\,\lambda)}}{1-e^{-(h\,\lambda)+x'}}+\frac{\left(-1+e^{x'}\right)h\,\lambda}{e^{h\,\lambda}\left(1-e^{-(h\,\lambda)+x'}\right)^2}\right)\alpha_{11}}{1-e^{x'}\,(1-\alpha_{11})}$$

C.3 Negative Moments about the Origin

We will now outline steps in the derivation of the negative moments for the variable R when the IC period is an Erlang(λ, 2)variate.

In order to obtain negative moments we use the following results of series summation presented by Prudnikov *et al* [61].

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = -\log(1-x)$$

$$\sum_{k=1}^{\infty} \frac{x^k}{k^n} = Li_n(x)$$

$$\begin{aligned} Li_n(x) &= \text{PolyLog}(n, x) \\ &= \frac{x\left(\int_0^\infty dt\, \frac{t^{-1+n}}{e^t - x}\right)}{\Gamma(n)} \\ \text{for} \quad & [n \geq 2, |x| \leq 1; n = 1, -1 \leq x < 1] \end{aligned}$$

For $|x| \leq 1$

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{x^k}{k^2} &= \int_0^x dt\, -\log\frac{1-t}{t} \\ &= \text{PolyLog}(2, x) \end{aligned}$$

Now,

$$E\left(R^{-q}\right) = \sum_{r=1}^{\infty} \left(\frac{1}{r}\right)^q P(R=r)$$

for $q = 1, 2, \ldots \infty$. We recall that

$$\begin{aligned} P(R=r) = & \frac{\left(1 - e^{-(h\lambda)} - \frac{h\lambda}{e^{h\lambda}}\right)\left(-e^{-(h\lambda r)} + (1-\alpha_{11})^r\right)\alpha_{11}}{1 - e^{-(h\lambda)} - \alpha_{11}} + \\ & \frac{\left(1 - e^{-(h\lambda)}\right) h\lambda\alpha_{11}\left(-e^{-(h\lambda) - h\lambda r} + \frac{(1-\alpha_{11})^r}{e^{h\lambda}}\right)}{\left(1 - e^{-(h\lambda)} - \alpha_{11}\right)^2} + \\ & \frac{\left(1 - e^{-(h\lambda)}\right) h\lambda\alpha_{11}\left(e^{-(h\lambda) - h\lambda r}\, r - \frac{(1-\alpha_{11})\, r}{e^{h\lambda r}}\right)}{\left(1 - e^{-(h\lambda)} - \alpha_{11}\right)^2} \end{aligned}$$

Derivation of the first negative moment, $E(R^{-1})$ uses the following results

$$\sum_{r=1}^{\infty} \frac{e^{-h\lambda r}}{r} = -\log(1 - e^{-h\lambda})$$

$$\sum_{r=1}^{\infty} \frac{(1-\alpha_{11})^r}{r} = -\log(\alpha_{11})$$

Therefore $E(R^{-1})$

$$= \frac{\left(1 - e^{-(h\,\lambda)} - \frac{h\,\lambda}{e^{h\,\lambda}}\right)\left(\log(1 - e^{-h\,\lambda}) - \log(\alpha_{11})\right)\,\alpha_{11}}{1 - e^{-(h\,\lambda)} - \alpha_{11}} +$$

$$\frac{\alpha\left(1 - e^{-(h\,\lambda)}\right)\,h\,\lambda}{\left(1 - \alpha_{11} - e^{-(h\,\lambda)}\right)^2} \times$$

$$\left(-e^{-h\,\lambda}\log(\alpha_{11}) + e^{-(h\,\lambda)}\log(1 - e^{-h\,\lambda}) - \frac{e^{-(h\,\lambda)}\left(1 - \alpha_{11} - e^{-(h\,\lambda)}\right)}{1 - e^{-(h\,\lambda)}}\right)$$

which gives

$$E(R^{-1}) = \alpha_{11}\left(-\frac{h\,\lambda}{e^{h\,\lambda}\left(1 - e^{-(h\,\lambda)} - \alpha_{11}\right)}\right) +$$

$$\alpha_{11}\left(\frac{\left(1 - e^{-(h\,\lambda)}\right)h\,\lambda\,\log\left(\frac{1-e^{-(h\,\lambda)}}{\alpha_{11}}\right)}{e^{h\,\lambda}\left(1 - e^{-(h\,\lambda)} - \alpha_{11}\right)^2} + \frac{\left(1 - e^{-(h\,\lambda)} - \frac{h\,\lambda}{e^{h\,\lambda}}\right)\log\left(\frac{1-e^{-(h\,\lambda)}}{\alpha_{11}}\right)}{1 - e^{-(h\,\lambda)} - \alpha_{11}}\right)$$

Derivation of the second negative moment, $E(R^{-2})$ uses the following results

$$\sum_{r=1}^{\infty} \frac{e^{-h\,\lambda\,r}}{r^2} = \mathrm{PolyLog}(2, e^{-h\,\lambda})$$

$$\sum_{r=1}^{\infty} \frac{(1-\alpha_{11})^r}{r^2} = \mathrm{PolyLog}(2, (1 - \alpha_{11}))$$

Therefore $E(R^{-2})$

$$= \frac{\left(1 - e^{-(h\,\lambda)} - \frac{h\,\lambda}{e^{h\,\lambda}}\right)\left(-\mathrm{PolyLog}(2, e^{-(h\,\lambda)}) + \mathrm{PolyLog}(2, (1 - \alpha_{11}))\right)\,\alpha_{11}}{1 - e^{-(h\,\lambda)} - \alpha_{11}} +$$

$$\frac{\alpha\,e^{-(h\,\lambda)}\left(1 - e^{-(h\,\lambda)}\right)\,h\,\lambda}{\left(1 - \alpha_{11} - e^{-(h\,\lambda)}\right)^2}\left(-\mathrm{PolyLog}(2, e^{-(h\,\lambda)}) + \mathrm{PolyLog}(2, 1 - \alpha_{11})\right) +$$

$$\frac{\alpha\left(1 - e^{-(h\,\lambda)}\right)\,h\,\lambda}{\left(1 - \alpha_{11} - e^{-(h\,\lambda)}\right)^2}\left(1 - \alpha_{11} - e^{-(h\,\lambda)}\right)\log(1 - e^{-(h\,\lambda)})$$

which gives

$$E(R^{-2}) = \alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\log(1-e^{-(h\,\lambda)})}{1-e^{-(h\,\lambda)}-\alpha_{11}}\right)+$$

$$\alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\left(-\mathrm{PolyLog}(2,e^{-(h\,\lambda)})+\mathrm{PolyLog}(2,1-\alpha_{11})\right)}{e^{h\,\lambda}\,(1-e^{-(h\,\lambda)}-\alpha_{11})^2}\right)+$$
$$\alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}-\frac{h\,\lambda}{e^{h\,\lambda}}\right)\left(-\mathrm{PolyLog}(2,e^{-(h\,\lambda)})+\mathrm{PolyLog}(2,1-\alpha_{11})\right)}{1-e^{-(h\,\lambda)}-\alpha_{11}}\right)$$

Derivation of the second negative moment, $E(R^{-3})$ uses the following results

$$\sum_{r=1}^{\infty}\frac{e^{-h\,\lambda\,r}}{r^3} = \mathrm{PolyLog}(3,e^{-h\,\lambda})$$

$$\sum_{r=1}^{\infty}\frac{(1-\alpha_{11})^r}{r^3} = \mathrm{PolyLog}(3,(1-\alpha_{11}))$$

Therefore $E(R^{-3})$

$$= \frac{\left(1-e^{-(h\,\lambda)}-\frac{h\,\lambda}{e^{h\,\lambda}}\right)\left(-\mathrm{PolyLog}(3,e^{-(h\,\lambda)})+\mathrm{PolyLog}(3,(1-\alpha_{11}))\right)\alpha_{11}}{1-e^{-(h\,\lambda)}-\alpha_{11}}+$$
$$\frac{\alpha\,e^{-(h\,\lambda)}\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda}{(1-\alpha_{11}-e^{-(h\,\lambda)})^2}\left(-\mathrm{PolyLog}(3,e^{-(h\,\lambda)})+\mathrm{PolyLog}(3,1-\alpha_{11})\right)-$$
$$\frac{\alpha\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda}{(1-\alpha_{11}-e^{-(h\,\lambda)})^2}\left(1-\alpha_{11}-e^{-(h\,\lambda)}\right)\mathrm{PolyLog}(2,e^{-(h\,\lambda)})$$

which gives

$$E(R^{-3}) = \alpha_{11}\left(-\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\mathrm{PolyLog}(2,e^{-(h\,\lambda)})}{1-e^{-(h\,\lambda)}-\alpha_{11}}\right)+$$

$$\alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\left(-\mathrm{PolyLog}(3,e^{-(h\,\lambda)})+\mathrm{PolyLog}(3,1-\alpha_{11})\right)}{e^{h\,\lambda}\,(1-e^{-(h\,\lambda)}-\alpha_{11})^2}\right)+$$
$$\alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}-\frac{h\,\lambda}{e^{h\,\lambda}}\right)\left(-\mathrm{PolyLog}(3,e^{-(h\,\lambda)})+\mathrm{PolyLog}(3,1-\alpha_{11})\right)}{1-e^{-(h\,\lambda)}-\alpha_{11}}\right)$$

Thus, we derive the following general result for $E(R^{-q})$, when the time to shift in the process mean is and Erlang(λ, 2) variate. For $q > 2$ the general result is

$$E(R^{-q}) \;=\; \alpha_{11}\left(-\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\mathrm{PolyLog}((q-1),e^{-(h\,\lambda)})}{1-e^{-(h\,\lambda)}-\alpha_{11}}\right)+$$

$$\alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}\right)\,h\,\lambda\,\left(-\mathrm{PolyLog}(q,e^{-(h\,\lambda)})+\mathrm{PolyLog}(q,1-\alpha_{11})\right)}{e^{h\,\lambda}\left(1-e^{-(h\,\lambda)}-\alpha_{11}\right)^2}\right)+$$
$$\alpha_{11}\left(\frac{\left(1-e^{-(h\,\lambda)}-\frac{h\,\lambda}{e^{h\,\lambda}}\right)\left(-\mathrm{PolyLog}(q,e^{-(h\,\lambda)})+\mathrm{PolyLog}(q,1-\alpha_{11})\right)}{1-e^{-(h\,\lambda)}-\alpha_{11}}\right)$$

Appendix D

Data set used to assess the ECPTU functions

In Table D.2,

r_{av} represents $\frac{a}{v}$ in $E(Z_1)$ and $E(Z_2)$.

r_{va} represents $\frac{v}{a}$ in $E(Z_3)$ and $E(Z_4)$.

Input Data [28]

No.	δ	λ	$c2$	d	$c3$	$c4$	b	e
1	2.0	0.01	100.00	2	50	25.0	0.5	0.1
2	2.0	0.02	100.00	2	50	25.0	0.5	0.1
3	2.0	0.03	100.00	2	50	25.0	0.5	0.1
4	2.0	0.02	50.00	2	50	25.0	0.5	0.1
5	2.0	0.01	1000.00	2	50	25.0	0.5	0.1
6	2.0	0.01	10000.00	2	50	25.0	0.5	0.1
7	2.0	0.01	100.00	2	50	25.0	0.5	0.1
8	2.0	0.01	100.00	20	50	25.0	0.5	0.1
9	2.0	0.01	100.00	2	5	2.5	0.5	0.1
10	2.0	0.01	100.00	2	500	250.0	0.5	0.1
11	2.0	0.01	100.00	2	5000	2500.0	0.5	0.1
12	2.0	0.01	100.00	2	50	25.0	5.0	0.1
13	2.0	0.01	100.00	2	50	25.0	0.5	1.0
14	2.0	0.01	100.00	2	50	25.0	0.5	10.0
15	2.0	0.01	1000.00	2	50	25.0	0.5	1.0
16	1.0	0.01	12.87	2	50	25.0	0.5	0.1
17	1.0	0.01	128.70	2	50	25.0	0.5	0.1
18	1.0	0.01	12.87	2	500	250.0	0.5	0.1
19	1.0	0.01	12.87	2	50	25.0	5.0	0.1
20	1.0	0.01	12.87	2	50	25.0	0.5	1.0
21	0.5	0.01	2.25	2	50	25.0	0.5	0.1
22	0.5	0.01	225.00	2	50	25.0	0.5	0.1
23	0.5	0.01	2.25	2	500	250.0	0.5	0.1
24	0.5	0.01	2.25	2	50	25.0	5.0	0.1
25	0.5	0.01	2.25	2	50	25.0	0.5	1.0

Table D.1: Data set [28] used to assess the ECPTU functions.

Input Data — randomly generated

No.	β	v	r_{av}	a	r_{va}
1	2	0.5298220	0.5265320	0.0872807	0.5265320
2	2	0.0518233	0.6027140	0.0204792	0.6027140
3	3	0.1641580	0.7762780	0.0165538	0.7762780
4	3	0.8702740	0.9144380	0.0113101	0.9144380
5	3	0.8625510	0.0536381	0.0854982	0.0536381
6	4	0.5298220	0.5265320	0.0872807	0.5265320
7	2	0.0518233	0.6027140	0.0204792	0.6027140
8	2	0.1641580	0.7762780	0.0165538	0.7762780
9	3	0.8702740	0.9144380	0.0113101	0.9144380
10	3	0.8625510	0.0536381	0.0854982	0.0536381
11	2	0.5298220	0.5265320	0.0872807	0.5265320
12	3	0.0518233	0.6027140	0.0204792	0.6027140
13	5	0.1641580	0.7762780	0.0165538	0.7762780
14	2	0.8702740	0.9144380	0.0113101	0.9144380
15	5	0.8625510	0.0536381	0.0854982	0.0536381
16	2	0.5298220	0.5265320	0.0872807	0.5265320
17	5	0.0518233	0.6027140	0.0204792	0.6027140
18	3	0.1641580	0.7762780	0.0165538	0.7762780
19	2	0.8702740	0.9144380	0.0113101	0.9144380
20	5	0.8625510	0.0536381	0.0854982	0.0536381
21	5	0.5298220	0.5265320	0.0872807	0.5265320
22	2	0.0518233	0.6027140	0.0204792	0.6027140
23	4	0.1641580	0.7762780	0.0165538	0.7762780
24	4	0.8702740	0.9144380	0.0113101	0.9144380
25	5	0.8625510	0.0536381	0.0854982	0.0536381

Table D.2: Additional data, randomly generated, used as input values for the $ECPTU_{Zi}$.

Appendix E

Results from $ECPTU_{a1}$, $ECPTU_b$ and $ECPTU_c$

Output Data — $ECPTU_{a1}$

No.	$k^*(4,0.5)$	$k^*(4,1)$	$k^*(4,8)$	$k^*(5,0.5)$	$k^*(5,1)$	$k^*(5,8)$
1	3.55670	3.34130	2.41549	3.86100	3.43286	2.738810
2	3.48214	NA	2.37955	NA	3.31032	NA
3	3.76976	3.50087	2.58855	3.90390	3.64157	2.777920
4	3.64874	3.37272	2.44963	3.78564	3.51812	2.650000
5	3.14953	2.84625	1.99593	3.30468	3.01838	2.114270
6	2.76561	2.44514	1.42897	2.94309	2.64642	1.726220
7	3.72592	3.45464	2.54569	3.86100	3.59723	2.738810
8	3.72213	3.45073	2.54127	3.85738	3.59345	2.734740
9	3.26694	2.96973	1.99593	3.41680	3.13437	2.236820
10	4.14475	3.89811	3.06568	4.27400	4.03005	3.224590
11	4.53809	4.31357	3.55884	4.66656	4.44193	3.697290
12	3.72581	3.45458	2.54568	3.86099	3.59717	2.738800
13	3.72583	3.45460	2.54568	3.86099	3.59717	2.738800
14	3.72492	3.45412	2.54561	3.85989	3.59659	2.738720
15	3.26695	2.96973	1.99593	3.41673	3.13438	2.236820
16	3.42455	3.17496	2.27664	3.51700	3.26732	2.372760
17	2.99451	2.70665	1.61607	3.08788	2.80092	1.727720
18	3.81722	3.59623	2.83145	3.90904	3.68727	2.920800
19	3.42374	3.17452	2.27657	3.51629	3.26688	2.372680
20	3.42390	3.17460	2.27658	3.51629	3.26688	2.372680
21	3.30294	3.07822	2.28653	3.35409	3.12918	2.337580
22	2.43742	2.13038	0.84191	2.49385	2.18797	0.919506
23	3.76123	3.56896	2.92278	3.80350	3.61027	2.960760
24	3.29835	3.07573	2.28611	3.34955	3.12672	2.337170
25	3.29926	3.07623	2.28619	3.34955	3.12672	2.337170

Table E.1: From $ECPTU_{a1}$, the optimum values of k when $n = 4$ or 5 and $h = 0.5$, 1 or 8 hours.

Output Data — $ECPTU_{a1}$

No.	$n^*(0.5)$	$n^*(1)$	$n^*(8)$	$h^*(5)$	$h^*(4)$
1	2	3	7	9.053650	6.951500
2	2	NA	8	5.279920	4.273780
3	1	4	7	3.884360	3.223980
4	2	3	7	6.275500	5.291540
5	4	5	9	6.484640	3.736860
6	6	8	11	5.940330	2.446570
7	2	3	7	9.053650	6.951500
8	2	3	7	9.018260	6.912820
9	2	3	7	8.844980	6.700450
10	2	3	7	10.669300	8.790460
11	1	2	7	17.701900	16.148500
12	2	3	7	13.283900	11.588300
13	1	1	5	13.283900	10.961700
14	NA	NA	2	25.136700	22.050800
15	2	3	7	7.925560	5.519580
16	2	3	13	4.342430	3.375950
17	4	7	23	1.358080	1.044550
18	1	3	13	6.474090	5.199780
19	2	3	13	9.837920	8.109720
20	NA	1	6	9.837920	7.377990
21	NA	2	16	4.305830	3.680370
22	12	21	79	0.332768	0.276582
23	NA	1	12	9.265060	8.381020
24	NA	2	16	15.338600	15.618100
25	NA	NA	3	15.338600	12.703600

Table E.2: From $ECPTU_{a1}$ for $k = 3$, the optimum values of n when $h = 0.5$, 1 or 8 hours and optimum h when $n = 4$ or 5.

Output Data — $ECPTU_{a1}$

No.	$ECPTU(n^*, h^*, k^*)$	k^*	h^*	n^*
1	33.23410	3.15169	10.92730	8
2	33.25870	3.15642	6.15540	8
3	33.26990	3.15485	4.40184	7
4	16.78310	3.14665	6.99594	7
5	330.54600	3.05555	8.89577	10
6	3303.27000	3.01966	8.52838	12
7	33.23410	3.15169	10.92730	8
8	31.95890	3.15119	10.89750	8
9	33.16580	2.35483	10.64400	6
10	33.74970	3.80004	11.09600	10
11	38.75260	4.33410	11.26920	12
12	33.57850	3.08425	14.69410	8
13	33.63320	2.41776	14.39370	5
14	34.99500	1.44046	20.02790	2
15	331.22500	2.35319	9.97254	6
16	4.49756	2.74866	19.51320	19
17	42.90070	2.72198	12.12630	25
18	5.02463	3.44047	20.41240	25
19	4.68831	2.71190	26.45660	21
20	4.91533	1.93514	28.02810	8
21	1.00625	2.34209	39.87440	38
22	75.26540	2.33602	13.14120	79
23	1.51851	3.10292	55.57960	59
24	1.10474	2.28524	52.31330	43
25	1.29407	1.45241	51.10310	7

Table E.3: The minimum $ECPTU_{a1}$ and optimum combination of n, h and k.

Output Data — $ECPTU_b$

No.	$k^*(4,0.5)$	$k^*(4,1)$	$k^*(4,8)$	$k^*(5,0.5)$	$k^*(5,1)$	$k^*(5,8)$
1	3.34992	3.05764	2.098430	3.49617	3.21721	2.32941
2	3.20633	2.90649	1.929950	3.35849	3.07449	2.17664
3	3.12074	2.81659	1.830000	3.27687	2.99011	2.08629
4	3.35164	3.05967	2.100180	3.49761	3.21885	2.33062
5	2.85513	2.53866	1.532560	3.02664	2.73243	1.81912
6	2.32274	1.98889	0.959325	2.53417	2.23048	1.30552
7	3.34992	3.05764	2.098430	3.49617	3.21721	2.32941
8	3.34915	3.05701	2.096750	3.49540	3.21662	2.32795
9	2.85506	2.53862	1.532560	3.02657	2.73239	1.81912
10	3.80587	3.53990	2.649810	3.93915	3.67936	2.83423
11	4.26031	4.02115	3.218130	4.38846	4.15070	3.36744
12	3.34956	3.05745	2.098400	3.49582	3.21703	2.32938
13	3.34963	3.05749	2.098410	3.49582	3.21703	2.32938
14	3.34674	3.05596	2.098190	3.49236	3.21522	2.32914
15	2.85510	2.53864	1.532560	3.02661	2.73241	1.81912
16	3.08243	2.80528	1.772720	3.17387	2.89694	1.87562
17	2.59396	2.26324	0.944730	2.68843	2.36054	1.09188
18	3.54228	3.30552	2.473170	3.63086	3.39290	2.55823
19	3.07972	2.80378	1.772450	3.17117	2.89545	1.87535
20	3.08026	2.80408	1.772500	3.17117	2.89545	1.87535
21	3.02525	2.78461	1.909320	3.07231	2.83131	1.95641
22	2.01073	1.64367	NA	2.06844	1.70419	NA
23	NA	NA	NA	NA	NA	NA
24	3.00898	2.77547	1.907640	3.05636	2.82235	1.95476
25	3.01213	2.77727	1.907980	3.05636	2.82235	1.95476

Table E.4: From $ECPTU_b$, the optimum values of k when $n = 4$ or 5 and $h = 0.5$, 1 or 8 hours.

Output Data — $ECPTU_b$

No.	$n^*(0.5)$	$n^*(1)$	$n^*(8)$	$h^*(5)$	$h^*(4)$
1	3	5	9	1.414000	1.235910
2	4	5	9	1.003290	0.877253
3	4	6	10	0.821433	0.718440
4	3	5	9	1.427200	1.248570
5	6	7	11	0.444846	0.388580
6	8	9	13	0.140478	0.122686
7	3	5	9	1.414000	1.235910
8	3	5	9	1.412050	1.234140
9	3	5	9	1.334270	1.159350
10	3	5	9	2.062920	1.848800
11	2	4	9	5.913780	5.415840
12	3	5	9	3.173870	2.884080
13	1	2	7	3.173870	2.635620
14	NA	1	4	9.777120	8.035080
15	3	5	9	0.993478	0.824293
16	3	6	21	1.592820	1.274160
17	8	13	31	0.479035	0.380660
18	3	5	20	2.610360	2.156790
19	3	6	21	3.842530	3.256290
20	NA	2	11	3.842530	2.933650
21	2	5	31	2.238010	1.989390
22	25	41	111	0.129036	0.107723
23	NA	NA	NA	NA	NA
24	2	5	31	49.173300	NA
25	NA	NA	NA	49.173300	100.974000

Table E.5: From $ECPTU_b$ for $k = 3$, optimum values of n when $h =$ 0.5, 1 or 8 hours and optimum h when $n = 4$ or 5.

Output Data — $ECPTU_b$

No.	$ECPTU(n^*, h^*, k^*)$	k^*	h^*	n^*
1	1.80501	3.19083	1.452090	6
2	2.63940	3.18945	1.030660	6
3	3.29659	3.18367	0.841726	6
4	2.00068	3.18630	1.464490	6
5	5.19441	3.19949	0.457970	6
6	15.91310	3.21589	0.146001	6
7	1.80501	3.19083	1.452090	6
8	1.53607	3.19082	1.450100	6
9	1.47063	2.43183	1.345140	4
10	4.09524	3.81155	1.568360	7
11	26.03740	4.34232	1.885210	9
12	3.60297	3.13795	3.433770	8
13	3.52940	2.47201	2.687170	3
14	7.79173	1.51415	5.489710	1
15	10.70420	2.48024	0.843256	3
16	1.07889	2.76623	5.811420	17
17	2.93436	2.77812	1.791700	17
18	3.29387	3.45463	7.367320	23
19	1.60084	2.70216	11.005300	21
20	2.13553	1.89387	12.328500	8
21	0.76267	2.22275	24.181400	44
22	6.13384	2.29891	2.052650	46
23	NA	NA	NA	NA
24	0.90762	2.13025	37.862900	53
25	NA	NA	NA	NA

Table E.6: The minimum $ECPTU_b$ and optimum combination of n, h and k.

Output Data — $ECPTU_c$

No.	$k^*(4,0.5)$	$k^*(4,1)$	$k^*(4,8)$	$k^*(5,0.5)$	$k^*(5,1)$	$k^*(5,8)$
1	3.48969	3.20511	2.264440	3.63111	3.35750	2.48065
2	3.34951	3.05718	2.097780	3.49578	3.21677	2.32882
3	3.26574	2.96895	1.999150	3.41534	3.13333	2.23930
4	3.49000	3.20553	2.264870	3.63128	3.35772	2.48075
5	3.00814	2.69840	1.703910	3.17061	2.88040	1.97322
6	2.48630	2.15677	1.132460	2.68427	2.38282	1.46047
7	3.48969	3.20511	2.264440	3.63111	3.35750	2.48065
8	3.48969	3.20511	2.264440	3.63111	3.35750	2.48065
9	3.00814	2.69840	1.703910	3.17061	2.88040	1.97322
10	3.92865	3.66976	2.801200	4.06013	3.80597	2.97554
11	4.32847	4.09284	3.301770	4.45674	4.22215	3.44859
12	3.48969	3.20511	2.264440	3.63111	3.35750	2.48065
13	3.48969	3.20511	2.264440	3.63111	3.35750	2.48065
14	3.48969	3.20511	2.264440	3.63111	3.35750	2.48065
15	3.00814	2.69840	1.703910	3.17061	2.88040	1.97322
16	3.20747	2.94059	1.961630	3.29960	3.03280	2.06176
17	2.74495	2.43146	1.209430	2.83893	2.52740	1.33875
18	3.61590	3.38246	2.565530	3.70654	3.47222	2.65412
19	3.20747	2.94059	1.961630	3.29960	3.03280	2.06176
20	3.20747	2.94059	1.961630	3.29960	3.03280	2.06176
21	3.10801	2.86997	2.017070	3.15809	2.91977	2.06744
22	2.17246	1.83064	0.150303	2.22969	1.88988	0.26696
23	3.52750	3.32266	2.626520	3.56891	3.36299	2.66358
24	3.10801	2.86997	2.017070	3.15809	2.91977	2.06744
25	3.10801	2.86997	2.017070	3.15809	2.91977	2.06744

Table E.7: From $ECPTU_c$, the optimum values of k for $n = 4$ or 5 and $h = 0.5$, 1 or 8 hours.

Output Data — $ECPTU_c$

No.	$n^*(0.5)$	$n^*(1)$	$n^*(8)$	$h^*(5)$	$h^*(4)$
1	2	4	8	1.996870	1.744240
2	3	5	9	1.415360	1.236430
3	4	5	9	1.157760	1.011480
4	2	4	8	2.008390	1.754760
5	5	6	10	0.628994	0.549324
6	7	9	12	0.198660	0.173487
7	2	4	8	1.996870	1.744240
8	2	4	8	1.996870	1.744240
9	2	4	8	1.886380	1.638080
10	2	4	8	2.880450	2.578950
11	2	4	8	7.241630	6.611580
12	2	4	8	4.481020	4.065680
13	1	2	6	4.481020	3.716070
14	NA	1	3	13.787500	11.286000
15	2	4	8	1.404640	1.164940
16	3	5	18	2.192880	1.749020
17	7	11	28	0.673355	0.534622
18	2	4	18	3.213550	2.638140
19	3	5	18	5.163970	4.333330
20	1	2	9	5.163970	3.923520
21	1	4	26	2.442530	2.108250
22	19	32	100	0.180297	0.150344
23	NA	2	25	3.971720	3.589880
24	1	4	26	10.744900	12.200800
25	NA	NA	6	10.744900	9.115960

Table E.8: From $ECPTU_c$ for $k = 3$, the optimum values of n when $h = 0.5$, 1 or 8 hours and optimum h when $n = 4$ or 5.

Output Data — $ECPTU_c$

No.	$ECPTU(n^*, h^*, k^*)$	k^*	h^*	n^*
1	1.126550	3.193200	2.053750	6
2	1.591090	3.190770	1.454630	6
3	1.946720	3.189260	1.190110	6
4	1.122990	3.190080	2.065610	6
5	3.570150	3.193980	0.646718	6
6	11.298100	3.209310	0.204775	6
7	1.126550	3.193200	2.053750	6
8	1.126550	3.193200	2.053750	6
9	1.043230	2.433220	1.903840	4
10	1.200830	3.814910	2.192020	7
11	1.268720	4.354780	2.322270	9
12	2.423630	3.144640	4.854240	8
13	2.374270	2.474930	3.812280	3
14	5.476940	1.519910	7.819950	1
15	7.547020	2.480580	1.192240	3
16	0.608176	2.766950	8.134780	17
17	1.942350	2.779670	2.533220	17
18	0.677744	3.456890	9.310400	23
19	0.987073	2.725440	15.491300	22
20	1.388350	1.910580	17.234400	8
21	0.402516	2.295330	32.268700	49
22	4.253980	2.317600	2.971070	48
23	0.468180	2.986430	39.134000	68
24	0.513075	2.154820	45.610700	51
25	0.814754	0.977454	78.645800	14

Table E.9: The minimum $ECPTU_c$ and optimum combination of n, h and k.

Appendix F

Results from the $ECPTU_{Zi}$ $(i = 1 \ldots 4)$

Output Data — $ECPTU_{Z1}$

No.	$ECPTU(n^*, h^*, k^*)$	k^*	h^*	n^*
1	1.130770	3.19134	2.052820	6
2	1.753920	3.19055	1.462250	6
3	2.112200	3.19023	1.196540	6
4	1.137460	3.18805	2.065760	6
5	3.573800	3.19447	0.646698	6
6	NA	NA	NA	NA
7	1.169510	3.19135	2.055500	6
8	1.140070	3.19135	2.053450	6
9	1.043460	2.43425	1.903890	4
10	1.241810	3.81511	2.194950	7
11	1.708070	4.35593	2.357720	9
12	2.482160	3.14485	4.858000	8
13	2.405140	2.47483	3.807490	3
14	5.478780	1.52147	7.768630	1
15	7.551890	2.48069	1.192510	3
16	0.612102	2.77846	8.224230	17
17	2.044610	2.77942	2.538780	17
18	0.867037	3.46010	9.663060	24
19	0.989240	2.70275	15.449700	21
20	1.393540	1.91043	16.923400	8
21	0.409732	2.25704	34.395000	48
22	4.290120	2.31373	2.958450	47
23	0.645437	3.00252	49.296900	74
24	0.515693	2.18653	51.236500	57
25	NA	NA	NA	NA

Table F.1: The minimum $ECPTU_{Z1}$ and optimum combination of n, h and k.

Output Data — $ECPTU_{Z2}$

No.	$ECPTU(n^*, h^*, k^*)$	k^*	h^*	n^*
1	1.128420	3.19134	2.052650	6
2	1.663530	3.19054	1.458180	6
3	1.996230	3.19029	1.192310	6
4	1.127300	3.18806	2.064570	6
5	3.571240	3.19447	0.646691	6
6	NA	NA	NA	NA
7	1.145610	3.19135	2.053840	6
8	1.132540	3.19135	2.052930	6
9	1.043300	2.43425	1.903880	4
10	1.213030	3.81568	2.193410	7
11	1.463460	4.35577	2.338060	9
12	2.441080	3.14486	4.852250	8
13	2.379940	2.47485	3.804670	3
14	5.477740	1.52147	7.768410	1
15	7.547920	2.48069	1.192490	3
16	0.609928	2.77846	8.220610	17
17	1.961200	2.77975	2.533710	17
18	0.734211	3.46029	9.396940	24
19	0.988034	2.70276	15.446300	21
20	1.389360	1.91048	16.911500	8
21	0.403323	2.24462	33.650600	47
22	4.269800	2.30332	2.906010	46
23	0.509334	2.99278	40.354400	69
24	0.513093	2.19522	50.294800	57
25	NA	NA	NA	NA

Table F.2: The minimum $ECPTU_{Z2}$ and optimum combination of n, h and k.

Output Data — $ECPTU_{Z3}$

No.	$ECPTU(n^*, h^*, k^*)$	k^*	h^*	n^*
1	3.55725	3.19169	2.077980	6
2	13.04420	3.19117	1.543640	6
3	18.65960	3.18737	1.296760	6
4	18.32570	3.18459	2.519380	6
5	6.76240	3.19478	0.647824	6
6	NA	NA	NA	NA
7	8.89914	3.18862	2.135040	6
8	10.04410	3.19084	2.137880	6
9	1.86884	2.42996	1.910560	4
10	38.10730	3.80638	2.766830	7
11	NA	NA	NA	NA
12	11.86920	3.14208	5.099450	8
13	15.88060	2.48098	4.099410	3
14	14.45940	1.52019	8.168720	1
15	11.48260	2.48043	1.195240	3
16	3.04860	2.77900	9.152850	17
17	14.54090	2.79859	2.715490	17
18	NA	NA	NA	NA
19	11.85440	2.65494	50.427200	21
20	6.46191	1.89389	22.239300	8
21	NA	NA	NA NA	
22	10.99140	2.31861	3.011940	48
23	NA	NA	NA	NA
24	NA	NA	NA	NA
25	NA	NA	NA	NA

Table F.3: The minimum $ECPTU_{Z3}$ and optimum combination of n, h and k.

Output Data — $ECPTU_{Z4}$

No.	$ECPTU(n^*, h^*, k^*)$	k^*	h^*	n^*
1	2.26838	3.19144	2.06433	6
2	8.56715	3.19107	1.50717	6
3	12.08320	3.18725	1.25053	6
4	11.49160	3.17936	2.30205	6
5	NA	NA	NA	NA
6	NA	NA	NA	NA
7	5.33361	3.18927	2.09504	6
8	6.11611	3.18909	2.10320	6
9	1.45148	2.43331	1.90824	4
10	13.46980	3.81376	2.34589	7
11	NA	NA	NA	NA
12	6.40573	3.14309	4.95336	8
13	7.21309	2.47464	3.89910	3
14	10.82940	1.51902	8.06742	1
15	8.43278	2.48071	1.19302	3
16	1.75416	2.77990	8.61608	17
17	6.11907	2.77827	2.57391	17
18	NA	NA	NA	NA
19	7.48429	2.70571	22.70060	22
20	2.52538	1.90821	17.80160	8
21	1.49509	2.19247	57.59140	48
22	7.89814	2.30534	2.94728	46
23	NA	NA	NA	NA
24	NA	NA	NA	NA
25	NA	NA	NA	NA

Table F.4: The minimum $ECPTU_{Z4}$ and optimum combination of n, h and k.

Appendix G

Use of Partial Derivatives to Find n^*, h^* and k^*

Let x be the vector of parameters.

$$x = \begin{pmatrix} n \\ h \\ k \end{pmatrix}$$

$ECPTU(x)$ is the scalar-valued function of x.

$ECPTU_x(x)$ is the vector of first partial derivatives of $ECPTU(x)$.

$$ECPTU_x(x) = \begin{pmatrix} \frac{\partial ECPTU(x)}{\partial n} \\ \frac{\partial ECPTU(x)}{\partial h} \\ \frac{\partial ECPTU(x)}{\partial k} \end{pmatrix}$$

$ECPTU_{xx}(x)$ is the matrix of second partial derivatives of $ECPTU(x)$.

$$ECPTU_{xx}(x) = \begin{pmatrix} \frac{\partial^2 ECPTU(x)}{\partial n^2} & \frac{\partial^2 ECPTU(x)}{\partial n \partial h} & \frac{\partial^2 ECPTU(x)}{\partial n \partial k} \\ \frac{\partial^2 ECPTU(x)}{\partial h \partial n} & \frac{\partial^2 ECPTU(x)}{\partial h^2} & \frac{\partial^2 ECPTU(x)}{\partial h \partial k} \\ \frac{\partial^2 ECPTU(x)}{\partial k \partial n} & \frac{\partial^2 ECPTU(x)}{\partial k \partial h} & \frac{\partial^2 ECPTU(x)}{\partial k^2} \end{pmatrix}$$

$$\Delta = \begin{pmatrix} \Delta_n \\ \Delta_h \\ \Delta_k \end{pmatrix}$$

Δ_n, Δ_h and Δ_k are close to zero.

For scalar variables

$$ECPTU(x+\Delta) = ECPTU(x) + \Delta ECPTU'(x) + \frac{1}{2}\Delta^2 ECPTU'(x)$$

For vector variables

$$ECPTU(x+\Delta) = ECPTU(x) + ECPTU_x(x)\Delta + \frac{1}{2}\Delta' ECPTU_{xx}(x)\Delta$$

To find the point at which $ECPTU(x+\Delta)$ is stationary we equate its gradient to zero. That is,

$$\frac{\partial^2 ECPTU(x+\Delta)}{\partial x} = ECPTU(x) + ECPTU_x(x)\Delta = 0 \qquad (G.1)$$

In Equation G.1 $ECPTU(x)$ is the vector of first partial derivatives and $ECPTU_x(x)$ is the matrix of second partial derivatives. Generally, any function $F'(x)$ denotes the gradient of the function $f(x)$. Thus,

$$ECPTU'(x) = \frac{\partial^2 ECPTU(x+\Delta)}{\partial x}$$

The optimum parameter values make $ECPTU'(x) = 0$ so that

$$\Delta = -[ECPTU_x(x)]^{-1} ECPTU(x)$$

In a numeric search for the optimum parameter value Δ is the vector of increments in the starting vector of parameter values. The search is stopped when Δ equals zero and $ECPTU_x(x) > 0$. When the vector of increments is zero the optimum parameter values have been found.

That is, for any vector x_i of starting parameter values, $x_{i+1} = x_i + \Delta$.