A UNIFIED METHODOLOGY FOR PROJECT PLANNING RISK

BY

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ABSTRACT

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It is essential for project managers to be in a position to assess the costs and time involved in almost any project that has to be considered by their companies. The existence of their companies and institutions might depend on how accurate these assessments are and how they are used to quantify the risk involved in the venture.

The Thesis reviews the existing literature of risk assessment methodologies and evaluates the advantages and the disadvantages of the approaches which have been used. Many papers have been produced which have illustrated and explained most of the current methods and techniques used in the assessment of risk and also suggestions have been made about how and when to use these methods. Chapman, C. for example, has employed the "Controlled Interval" approach in his work over the last ten years, with only limited accuracy involved in his final estimates of the total time/cost of a given project. His co-operation with Cooper, D. has resulted in producing the "PERK" software package.

We have fully examined and studied in this Thesis other methodologies, like Functional and Numerical Integration, and some improvements have been suggested. Substantial work has been done on the Moments Method and further developments on the applicability of the method to parallel activities has been carried out. The accuracy of the estimates obtained with the use of Pearson's curves, has been investigated.

The particular contribution of this work to Project Planning Risk Assessment involves the development and assessment of mathematical techniques which have been used in Project Management Planning. We develop an approach which is mathematically justifiable and practically feasible to project managers.

A new Unified Methodology is tested and found to be appropriate in a variety of project management contexts.

Our new approach -The New Unified Methodology- makes use of The Progressive Reduction Procedure to collapse activity networks and uses the first four moments of Erlang activity time distributions to fit Pearson curves to the distribution of project completion time.

OBJECTIVES:

The main objectives of the research described in this Thesis are:

(i) To review the existing literature on risk analysis and evaluate the advantages and disadvantages of the available methodologies.

(ii) To develop an approach to risk assessment which is mathematically justifiable and practically feasible.

(iii) To explore new developments on the Moments Method, with further enhancement on the applicability of the method to networks, with parallel activities.

(iv) To develop a new methodology based on the moments method, to take account of multi-modal activity times.

(v) To validate a new unified methodology and it's appropriateness in a variety of project management contexts.

ADVANCED STUDIES

The following advanced studies were undertaken in connection with the programme of the research for this Thesis:

(i) Attendance and participation in research seminars are arranged by the Department local group at The Nottingham Trent University.

(ii) Attendance at risk analysis lectures series given by The Nottingham Trent University.

(iii) Attendance at the following symposiums and conferences:

Reliability and Process safety, Reliability and The Reduction of Risk, Heathrow March 1988.

PMS/Bridge A Project Model Simulator, Birmingham July 1988.

Risk Analysis Event 708, Birmingham January 1989.

Managing R & D Projects Problems And Approaches, London April 1989.

Quality and Statistical Process Control, Nottingham September 1989.

World Quality Day, Nottingham November 1989.

(iv) Participation in the following conference:

NOAS 89 OR and Decision Support Systems, Linkoping, Sweden, June 1989.

(v) Participation and attendance in the following conference:

OR31 Conference, Southampton, September 1989.

(vi) Presentation and discussion of research findings with Industrial collaborators.

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DECLARATION

During the period of registration for PhD, the author has not been registered as a candidate for any other award of CNAA, nor any award of a University.

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CHAPTER ONE

INTRODUCTION

1.1 DEFINITIONS OF RISK

Risk has been defined in many different ways, depending on the area of involvement or the type of analysis used. Most definitions of risk link it with "Uncertainty". Chapman, C. and Cooper, D. (1983) gave risk many definitions, "An undesirable implication of uncertainty", also risk was defined as "Lack of certainty". On the other hand Johnson, D. (1986) defined risk as "A state of knowledge and awareness that hardly ever exists". Others preferred not to define risk at all. My definition of risk in the project management field would be, the probability of completing part or the whole of a project within a given period of time, or within given cost constraints.

1.2 TYPE OF RISK

The type of risk of interest to the current study is stochastic, and concerned with time and cost variation in project planning. This is distinct from risk studies associated with safety in high risk industries, such as the chemical industry.

1.3 USES

The methodologies used in the assessment of risk are becoming increasingly important in the field of project planning and management. This is particularly evident in construction industries, where escalation of costs and time are factors of prime importance. The methodologies of risk assessment are also important in military planning. Project planners would want to assess risk in terms of probability of achieving a targeted completion date. A similar assessment of financial risk would need to be evaluated.

As a good example of miscalculation of risk assessment in a big project is the Channel Tunnel, see John, D. (1991), which is one of the most badly assessed. Risk could have been reduced by further analysis of similar projects elsewhere and knowledge could have been used to further reduce risk, see Laurance, B. (1992). Proper continuous risk assessment would have provided the project manager with good idea of the amount of risk at specific time, see Davidson, A. (1990).

1.4 FEATURES

Most decision makers would try to reduce the uncertainty by means of identification, evaluation, control and management of risk. In most of risk analysis the above four stages are summarized as planning and scheduling, which is very well known as Critical Path Analysis, CPA. A project could be structured between start and finish activities. The project then could be described and viewed in a logical diagram which would show the arrangement of the activities in a network model. Such a technique is sometimes known as PERT (Program Evaluation and Review Technique).

1.5 CURRENT METHODOLOGIES

The work on risk assessment that has already been done is somewhat fragmented in that each of the different methodologies is applicable to a particular situation. No one methodology has been developed to deal with a comprehensive variety of projects. Chapman, C. and Cooper, D. (1983) claim that the C.I.(Controlled Interval) Approach, is superior to others in it's simplicity in reducing specification errors. However, this claim is not substantiated by the limited illustrations presented.

The Moments Method that has been widely employed in other applications, e.g. Kottas, J.F. And Lau, H.S. (1978) has good potential here in risk assessment. It is clear from a review of existing literature that some conditions that might well be found in a practical situation can not be adequately catered for with existing methodologies, or at least the degree of accuracy attained is unacceptable. Sometimes, as with the Functional or Numerical Integration Approach, Cox, D.R. And Smith, W.L. (1961), the mathematics may become unmanageable, but such well established tools, can provide a basis for the formulation of a new approach that will be designed to cater for any situation presented by the practical experience of the construction consultants of our cooperating establishment. Computer packages, e.g. Macsyma or Mathematica, reduce the amount of effort and time required to build most of the mathematical models for a given work. Sullivan, R.S. And Hayya, J.C. (1980) and others, have used Monte Carlo Simulation and come to the conclusion that this method is the most flexible and easy; but not the most efficient, even when effort is minimised by the use of computer packages, e.g. SLAMII/TESS, NAG and STATGRAPHICS.

1.6 THE PROBLEM

The review of present methodologies revealed that there is no one method for all types of projects and also revealed that all methodologies are limited.

The risk associated with project planning is often revealed in terms of escalation of cost and time. The decision maker may or may not be in a position to control the uncertainties; but he should be able to estimate the risk involved in the venture. This estimate is usually based upon the probability distributions of the total project cost and total project time. The total project direct cost can be regarded simply as the sum of the cost of the individual activities and so relatively easy to deal with. The project time evaluated from a network diagram of activities which in general will be made up of parallel and series activities, is more difficult to assess, bearing in mind that each activity time is regarded as a random variable.

Since 'Time' is a continuous random variable, it must be treated as such in any accurate assessment of risk.

At the heart of the problem of risk assessment, is the need to evaluate the distribution of the sum (for a series only network) of two random variables and the distribution of the maximum (for parallel activities) of two random variables. The activity times are taken to be independent.

1.7 OBJECTIVES

The main effort in the present research is to identify one methodology for assessing Project Planning Risk, which takes account of as many practical situations as possible, some of which are supplied by the collaborating establishment. This unified methodology is designed to cater for most of the conditions encountered in Project Planning Techniques and to allow for a variety of pdf shapes for activity times: unimodal, symmetric, J-shaped and multimodal.

CHAPTER TWO

REVIEW OF EXISTING METHODOLOGIES

2.1 INTRODUCTION

A significant part in the analysis of Project Planning Risk is the determination of the distributions of total completion time and total cost. These distributions enable the practitioner to assess the probability of meeting budgets and achieving deadlines under various strategies. The methodologies of the analysis concentrate on stochastic PERT network models. It is well known that the standard PERT analysis based on three time estimates, a Beta distribution of activity times and a Normal completion time, produces biased results and ignores the possibility of a change in critical path.

A great deal of research has been carried out on alternative methodologies for estimating project cost and time distributions, and the methods can be broadly classified into four main approaches: Analytical, Numerical Approximation, Use of Moments, and Simulation. It would be fair to say that no one methodology is appropriate for every project planning network. A review of procedures is presented and suggestions are made as to when each method might be suitable.

2.2 THE ANALYTICAL APPROACH

2.2.1 INTRODUCTION

It is reasonable to start with the analytical approach, because if it is possible to find the exact cost and completion time distributions, then they can only be derived in this way. Ringer, L.J. (1969) developed the algorithm of Hartly, H.O. and Wortham, A.W. (1966), to show that the cumulative distribution function, (cdf) of completion time could be expressed as a multiple integral involving the product of the cdf's of individual activity times. This expression for the probability of $u_i < t$ for all i, u_i being the length of path i can be evaluated in functional form in a limited number of cases. Ringer, L.J. (1969) considers small networks which involve Wheatstone Bridge, or Criss-Cross subnetworks. The functional approach assumes that the input distributions (of individual activity times) are known and can be expressed in functional form. For an exact result the continuous nature of time as a random variable demands a functional form for the density in any case. Any data presented in discrete form that has to be expressed in functional form, say by use of polynomials, Martin, J.J. (1965), would probably involve approximations.

Another limitation of a purely analytical approach is the fact that the cdf's may not exist in closed form and if they do, their products may not be integrable. The size of the network would also be limited to four or five activities.

The analytical approach can be simplified to some extent, when approximating assumptions are made. Difficulties arise because paths through a network are not independent. Waving this (exceptional) condition can still produce useful results and at the same time simplify the mathematics involved. The components of the network are reduced to either activities in series or activities in parallel.

2.2.2 SERIES ACTIVITIES

These give rise to a density function involving a convolution of the type:

$$g(T) = \int_0^T f_1(t)\ f_2(T-t)\ dt \qquad (2.1)$$

where g(T) is the pdf of $T=t_1+t_2$, the pdf of t_1 being $f_1(t)$ and t_2 being $f_2(t)$.

By the Central Limit Theorem, the distribution of the sum, approaches the Normal Distribution when the number of random variables is large, and they are independent and identically distributed. The evaluation of the convolution is required therefore only when the number of random variables is small i.e for small series networks with less than ten activities.

The distribution of the project time will be an n-fold convolution. Table 2.1 shows some combinations of pdf's, whose convolution has or has not been evaluated analytically.

2.2.3 PARALLEL ACTIVITIES

When two activities are arranged in parallel, and have times t_1 and t_2, the time to completion is the maximum of t_1 and t_2. The pdf of the maximum, evaluated by a consideration of order statistics is given by:

$$f_1(T)F_2(T) + f_2(T)F_1(T) \tag{2.2}$$

where F_1 and F_2 are the distribution functions of t_1 and t_2. For three parallel activities, the pdf of the maximum of t_1, t_2 and t_3 is:

$$f_1(T)F_2(T)F_3(T) + f_2(T)F_1(T)F_3(T) + f_3(T)F_1(T)F_2(T) \tag{2.3}$$

The difficulty here, is that the distribution functions may not always be defined in functional form. The integral involved can not always be evaluated in closed form. A case in point is the Normal Distribution.

2.2.4 PRACTICAL CONSIDERATIONS

In reality, data on times and costs will be either subjective or based on past experience. In both cases, the data may be presented as a discrete probability distribution: the activity times having the same distribution type. A goodness-of-fit test might be used on these distributions from which a continuous pdf might be found. Again integrability can be a problem that is compounded since the total completion time distribution is found by a sequential reduction of the network model involving repeated use of expressions (2.1) and (2.2). However, the increased use of advanced computer packages, such as MACSYMA and MATHEMATICA, Wolfran, S. (1988), make the analytical approach more feasible for small and medium sized networks.

Example 1:

To highlight the method and complications involved, see Tables 2.2, 2.3, 2.4 and Figure 2.1 .

From Table 2.4, we can see that the total float of non-critical activities has been chosen to be large enough, to ensure critical path does not change; where as if the critical path changes, then equation (2.2), for parallel activities would have to be applied as well.

$S_2 = t_1 + t_2$ has distribution function:

$$F(S_2) = \int\int_{t_1+t_2<s_2} \left(\frac{1}{2}\right) \left(\frac{1}{4}\right) dt_2 \; dt_1 \tag{2.4}$$

where $0 < t_1 < 2$ and $3 < t_2 < 7$, see Fig 2.2 .

(a) For $3 < S_2 < 5$ i.e. area P,

$$F(S_2) = \int_0^{S_2-3} \int_3^{S_2-t_1} \left(\frac{1}{8}\right) dt_2 \; dt_1 \tag{2.5}$$

and by differentiation:

$$f(S_2) = \frac{(S_2-3)}{8}$$

(b) For $5 < S_2 < 7$ i.e. area Q,

$$F(S_2) = \int_0^2 \int_{5-t_1}^{S_2-t_1} \left(\frac{1}{8}\right) dt_2 \; dt_1 \qquad (2.6)$$

again by differentiation,

$$f(S_2) = \frac{1}{4}$$

(c) For $7 < S_2 < 9$ i.e. area R,

$$F(S_2) = \int_{S_2-7}^{2} \int_{7-t_1}^{S_2-t_1} \left(\frac{1}{8}\right) dt_2 \; dt_1 + \int_0^{S_2-7} \int_{7-t_1}^{7} \left(\frac{1}{8}\right) dt_2 \; dt_1 \qquad (2.7)$$

by differentiation:

$$f(S_2) = \frac{(16-2S_2)}{8} + \frac{(S_2-7)}{8}$$

$$f(S_2) = \frac{(9-S_2)}{8} \quad .$$

The pdf of $S_2 = t_1 + t_2$ is :

$$f(S_2) = \frac{(S_2-3)}{8} \qquad for \ 3<S_2<5 \tag{2.8}$$

$$f(S_2) = \frac{1}{4} \qquad 5<S_2<7 \tag{2.9}$$

$$f(S_2) = \frac{(9-S_2)}{8} \qquad 7<S_2<9 \tag{2.10}$$

see Figure 2.3 for plotting of the density function $f(S_2)$.

The pdf of $S_3=S_2+t_3$ would give rise to $3x3=9$ $f(S_3)$ functions with 9 corresponding ranges for S_3. In general the total project time S_5 would involve $3^4=81$ ranges for S_5 each with an associated $f(S_5)$ as pdf of project completion time. The difficulties over multiple-ranges are eliminated if we assume an exponential distribution for each activity time.

Example 2:

This second example, assumes the same network as in example 1 with:

$$f_A(t_1) = e^{-t_1} \quad , \qquad t_1 > 0 \qquad (2.11)$$

$$f_C(t_2) = \frac{1}{5}e^{-\frac{t_2}{5}} \quad , \qquad t_2 > 0 \qquad (2.12)$$

$$f_H(t_3) = \frac{1}{6}e^{-\frac{t_3}{6}} \quad , \qquad t_3 > 0 \qquad (2.13)$$

$$f_I(t_4) = \frac{1}{8}e^{-\frac{t_4}{8}} \quad , \qquad t_4 > 0 \qquad (2.14)$$

$$f_J(t_5) = \frac{1}{2}e^{-\frac{t_5}{2}} \quad , \qquad t_5 > 0 \qquad (2.15)$$

The pdf of $S_2 = t_1 + t_2$ is :

$$f(S_2) = \int_0^{S_2} e^{-t_1} \, \frac{1}{5}e^{-\frac{1}{5}(S_2 - t_1)} \, dt_1 \qquad (2.16)$$

$$f(S_2) = \frac{1}{5}e^{-\frac{1}{5}S_2} \int_0^{S_2} e^{-\frac{4}{5}t_1} \, dt_1$$

$$f(S_2) = \frac{1}{5}e^{-\frac{1}{5}S_2}\left[-\frac{5}{4}e^{-\frac{4}{5}S_2}+\frac{5}{4}\right]$$

$$f(S_2) = \frac{1}{4}\left[e^{-\frac{S_2}{5}}-e^{-S_2}\right] \qquad for \quad S_2>0.$$

The pdf of $S_3 = S_2 + t_3$

$$f(S_3) = \int_0^{S_3}\frac{1}{4}\left[e^{-\frac{1}{5}S_2}-e^{-S_2}\right]\frac{1}{6}e^{-\frac{1}{6}(S_3-S_2)}dS_2 \tag{2.17}$$

$$f(S_3) = \frac{1}{4}\left[\frac{24}{5}e^{-\frac{1}{6}S_3}+\frac{1}{5}e^{-S_3}-5e^{-\frac{1}{5}S_3}\right], \qquad for \quad S_3>0.$$

And so on till we get the pdf of Project time (S_5) which is:

$$f(S_5)=\frac{1}{12}[-\frac{324}{5}e^{-\frac{1}{6}S_5}+\frac{3}{35}e^{-S_5}+\frac{250}{9}e^{-\frac{1}{20}S_5}+$$
$$\frac{512}{21}e^{-\frac{1}{8}S_5}+\frac{3955}{315}e^{-\frac{1}{2}S_5}], \qquad for \quad S_5>0 \quad (2.18)$$

see Figure 2.4 for plotting of the density function $f(S_5)$.

2.2.5 COMMENTS ON THE ANALYTICAL APPROACH

The uniform distribution is more difficult to handle than the exponential distribution. For a series only and small network, the normal assumption for each activity time would enable the project time and total project cost distributions to be readily found since they would be normal with mean and variance equal to the sum of the means and variances which make up the critical path. However, when parallel activities are present, the normal assumption for activity times would create problems when evaluating the distribution of the maximum. In our example these problems have been avoided since non-critical total floats are large, and so are not likely to become critical.

Clearly, the number of occasions in which functional integration can be carried out successfully in a real situation is very limited; but numerical integration can sometimes be used when an analytical approach fails.

2.2.6 ADVANTAGES

The main advantage of the analytical approach in risk assessment is, that exact results can be obtained. The nature of the cost and completion time and the effect of changes in input parameters can be more easily assessed.

2.2.7 MINIMUM APPLICABILITY CONDITIONS

The following are the conditions which have to be satisfied so the method is applicable:

(I)	Small networks.
(II)	Analytic form for pdf's and cdf's of input distributions.
(III)	Product of cdf's integrable.

2.2.8 DISADVANTAGES

To derive the pdf of total time of a network with different input distributions, would be a very difficult task to deal with and in most cases, is not worth doing; because of the complexity of the mathematical term. In cases like these, the analytical work would be partially or totally substituted by numerical solutions which would result in the reduction in the accuracy of the computations.

The following is an example of a small network, consisting of three activities, two are in series, see Figure 2.5 where the first is an activity with a Normal pdf:

$$f_1(\mu,\sigma^2) = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} e^{\left(-\frac{(t_1-\mu)^2}{2\sigma^2}\right)} \tag{2.19}$$

the second activity with a Gamma pdf:

$$f_2(c,b) = \left(\frac{t_2}{b}\right)^{c-1} \frac{e^{-\frac{t_2}{b}}}{b\Gamma(c)} \tag{2.20}$$

and the third activity is in parallel with the other two and has a Beta pdf:

$$f_3(\gamma,\omega) = \frac{t_3^{\gamma-1}(1-t_3)^{\omega-1}}{\beta(\gamma,\omega)} \tag{2.21}$$

The pdf of $s_2 = t_1 + t_2$ is :

$$f(S_2) = \int_0^{s_2} f_1(t_1) f_2(s_2-t_1)\ dt_1 \tag{2.22}$$

$$f(S_2) = \int_{-\infty}^{s_2}\left[\frac{1}{\sigma(2\pi)^{\frac{1}{2}}}e^{\left(-\frac{(t_1-\mu)^2}{2\sigma^2}\right)}\right]$$

$$\left[\left(\frac{s_2-t_1}{b}\right)^{c-1}\frac{e^{\left(-\frac{(s_2-t_1)}{b}\right)}}{b\Gamma(c)}\right] dt_1 \qquad (2.23)$$

the pdf of the distribution of completion time is of the form:

$$g(T) = F(s_2)\ f(t_3) + f(s_2)\ F(t_3) \qquad (2.24)$$

$$g(T) = \{\int_{-\infty}^{T}\int_{-\infty}^{s_2}\left[\frac{1}{\sigma(2\pi)^{\frac{1}{2}}}e^{\left(-\frac{(t_1-\mu)^2}{2\sigma^2}\right)}\right]$$

$$\left[\left(\frac{s_2-t_1}{b}\right)^{c-1}\frac{e^{\left(-\frac{(s_2-t_1)}{b}\right)}}{b\Gamma(c)}\right]dt_1 ds_2\}\ \left[\frac{t_3^{\gamma-1}(1-t_3)^{\omega-1}}{\beta(\gamma,\omega)}\right] +$$

$$\left[\int_{-\infty}^{s_2}\left[\frac{1}{\sigma(2\pi)^{\frac{1}{2}}}e^{\left(-\frac{(t_1-\mu)^2}{2\sigma^2}\right)}\right]\left[\left(\frac{s_2-t_1}{b}\right)^{c-1}\frac{e^{\left(-\frac{(s_2-t_1)}{b}\right)}}{b\Gamma(c)}\right]dt_1\right]$$

$$\left[\int_{-\infty}^{T}\frac{t_3^{\gamma-1}(1-t_3)^{\omega-1}}{\beta(\gamma,\omega)}\ dt_3\right] . \qquad (2.25)$$

The above, is clearly very complicated and difficult to handle even with the aid of mathematical packages e.g MATHEMATICA, to produce the required answers in closed form. The software in such cases may not be able to compute the required mathematical answers. See Table 2.1 for various combinations of pdf's are evaluated analytically. In any case this example involves the integral of e^{-t^2} between finite limits which cannot be found in closed form.

2.3 NUMERICAL INTEGRATION APPROXIMATION

2.3.1 INTRODUCTION

One of the main drawbacks of the analytical approach is the need for integrable pdf's and cdf's of the input distributions. Numerical Integration can provide approximations for the values of the integrals and the use of computer packages can give a high degree of accuracy. Ringer, L.J. (1969) produced software for his numerical algorithm for some subnetworks.

2.3.2 DISADVANTAGES

If numerical integration is used to evaluate the convolution integral, the result is in tabular form. This in turn will again have to be used as input for further numerical integration when the sequential reduction method is being applied. In this case the degree of interpolation required for an acceptable degree of accuracy makes numerical integration unacceptable for use in the progressive reduction method and when input distributions are from empirical data.

In real life problems, where big and complicated networks would be involved, the construction of the distribution of total project time/cost by using this approach would be a slow, and impractical process. Updating and manipulating the network would cause further difficulties.

2.3.3 MINIMUM APPLICABILITY CONDITIONS

The method is applicable if the following conditions are met:

(I)	Small and medium networks.
(II)	Analytic form for pdf's and cdf's of input distributions.
(III)	Available software.

2.4 THE MOMENTS METHOD

2.4.1 INTRODUCTION

Some of the problems associated with the analytical method can be overcome by using moments of the input distributions rather than having them completely defined in functional form. Under certain assumptions about the input distributions, the moments of the maximum and sum of two random variables can be expressed in terms of the moments of the input distributions. The activity network is again reduced by assuming path independence and by replacing parallel and series activities by a single arc. This will enable the moments of project cost and completion time to be evaluated from which probability points can then be read off from tables of Pearson curves. The first four central moments of the input distributions will be required in this case.

Under certain simplifying assumptions Sculli, D. (1983) proposed an approximation for completion time mean and variance. His method for normally and independently distributed activity times made use of Clark's, C.E. (1961) formula for the mean and variance of:

$$T = \max(t_1, t_2) + t_3, \tag{2.26}$$

t_i , i=1,2,3 being activity times. Only the first two moments for each activity arc are required and tables giving parameters of the maximum of two normal variates, enable completion time mean and variance to be computed manually, even for large networks. A comparison of the results with Monte Carlo Simulation showed a 0.6% or less error in the estimate of the mean completion time, and an error of 8.8% or less in the standard deviation of completion time. These errors were the worst of the three examples.

An underlying assumption inherent in moment approaches is that the distribution of the maximum and sum of two activity times is of the same type as that of the individual activity times. Golenko-Ginzburg, D. (1989) defines such distributions as stable with respect to maximisation and convolution. For normal activity times, Greer, W.R. and La Cava (1979) found the stability assumption acceptable for a considerable range of parameters. Sculli, D. and Wong, K.L. (1985) also found the assumption acceptable for a good range of parameters of the Beta distribution.

2.4.2 SERIES ACTIVITY NETWORKS

The approach for series activities has been discussed and illustrated by Kottas, J.F. and Lau, H. (1978). One advantage of using this approach is that the method does not impose restrictions on the form of the input distributions in the network, other than unimodal or J-shaped, which may be in an empirical form; but not necessarily so. The method in general is based on the computation of the first four moments of the distributions. By the use of simple mathematical equations, the first four moments of the output "combined", distribution would be generated as following:

if $S_2 = t_1 + t_2$ then

$$\mu_i(S_2) = \mu_i(t_1) + \mu_i(t_2) \qquad for \quad i=1,2,3 \tag{2.27}$$

$$\mu_4(S_2) = \mu_4(t_1) + 6\ \mu_2(t_1)\mu_2(t_2) + \mu_4(t_2) \tag{2.28}$$

where μ_i is the i^{th} moment about the mean, and t_1 and t_2 might represent the time to carry out two independent activities in series.

2.4.2.1 ILLUSTRATION OF METHOD FOR SERIES CASE

To illustrate the approach further, take the first four central moments of the Normal distribution N(5.0,1.0), which are:

$$\mu_1(t_1) = 5.0,$$

$$\mu_2(t_1) = 1.0,$$

$$\mu_3(t_1) = 0.0,$$

$$\mu_4(t_1) = 3.0 \quad ,$$

and the first four central moments of the Exponential distribution Exp(1.0), which are:

$$\mu_1(t_2) = 1.0,$$

$$\mu_2(t_2) = 1.0,$$

$$\mu_3(t_2) = 2.0,$$

$\mu_4(t_2) = 9.0$.

The first four central moments of the 'combined' distribution of $S_2 = t_1 + t_2$ could be evaluated as the following:

$$\mu_1(S_2) = 5.0 + 1.0 = 6.0 \tag{2.29}$$

$$\mu_2(S_2) = 1.0 + 1.0 = 2.0 \tag{2.30}$$

$$\mu_3(S_2) = 0.0 + 2.0 = 2.0 \tag{2.31}$$

$$\mu_4(S_2) = 3.0 + (6.0)\ (1.0)\ (1.0) + 9.0 = 18.0 \tag{2.32}$$

Pearson curve parameters are:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(2.0)^2}{(2.0)^3} = 0.5 \tag{2.33}$$

$$\sqrt{\beta_1} = \sqrt{0.5} = 0.70711 \tag{2.34}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{18.0}{(2.0)^2} = 4.5 \tag{2.35}$$

and we know that $\sigma = \sqrt{\mu_2} = \sqrt{2.0} = 1.41421$.

To estimate $P(S_2 < 7)$, we need to standardise our units:

$$z = \frac{(x - \mu_1)}{\sigma} = \frac{(7.0 - 6.0)}{1.41421} = 0.70711. \qquad (2.36)$$

To find the probability of $P(S_2 < 7)$ from tables, locate the tabulations of $\sqrt{\beta_1} = 0.70711$ and $\beta_2 = 4.5$, where z falls between a probability of 0.75 and 0.90. In general interpolation is not recommended; but ignoring this condition would compute the interpolated probability to be 0.78. To get better approximation of the probability, computer software is used, see Davis, C.S. and Stephens, M.A. (1983), to evaluate the corresponding probability to be $P(S_2 < 7) = 0.83$. The corresponding probability of $(S_2 < 7)$, evaluated analytically is approximately (0.81), which indicates that the Moments is a reasonable approach to be used when Numerical Integration or other approaches are not applicable.

2.4.3 PARALLEL ACTIVITY NETWORKS

When the moments method is applied to networks which contain activities in parallel, there are two main problems to be overcome. The first is that the distribution of the activity times should be stable so that the distribution of the several parallel activities can be replaced by one of the same type as an original input distribution. The second problem is that of evaluating the moments integral:

$$\mu_r' = \int_0^\infty (t^r f_1 F_2 + t^r f_2 F_1) \; dt \tag{2.37}$$

where f is the pdf of the input distribution. These two problems are discussed more fully in chapter three.

2.4.4 ADVANTAGES

The moments method has many advantages over the other methods where it:

(I)	Can be used when the input distribution is in the form of an empirical frequency distribution, since the first four moments can be readily evaluated.
(II)	The method can be applied when the activity time distributions are skewed.
(III)	Is fairly quick and easy to be applied.

2.4.5 MINIMUM APPLICABILITY CONDITIONS

Applications of the moments method is restricted to the following conditions:

(I)	First two moments of input distributions must be known or estimated.
(II)	Either, a look-up table for distribution of maximum and sum, or appropriate software required.
(III)	For use of Pearson's curves, the output distribution has to be Unimodal, J-shaped or U-shaped.

2.5 METHOD OF SIMULATION

2.5.1 INTRODUCTION

The use of simulation provides the most flexible approach for the estimation of project cost and completion time. The method is flexible; because any kind of input distribution can be used and even large networks can be simulated. The main disadvantages are the amount and cost of computer time and the need for computer software. Many professional software packages have been developed for scheduling, (e.g PERTMASTER and PMW), and network analysis. SLAM II software, Pritsker, A. (1986), is designed as a computer simulation system and is suitable for simulating network models.

Most of the research in this area concentrates on ways of reducing computer time while still achieving accuracy. Several researchers have suggested numerous ways of decreasing computer simulation times and there is still potential for further work in this area.

Burt, J.M. and Garman, M.B. (1971) suggested ways of reducing computational effort. In their conditional Monte

Carlo Simulation, activities common to two or more paths were fixed at a sample value, thus reducing the number of computations required.

Cook, T.M. and Jennings, R.H (1979) in their paper on "Intelligent Simulation Methods", compare three methods of reducing computer time. Two of these methods, the min-max heuristic and path deletion, were first suggested by Van Slyke, R.M.(1963). His min-max heuristic approach first assumes all activity times to be their optimistic times. The critical path for this network is regarded as a minimum completion time. All activity times are then assumed to be their pessimistic times and any path whose length is less than the minimum completion time based on the optimistic estimates, is rejected as non-critical and disregarded from the simulation.

The path deletion approach similarly, disregards certain paths: namely those that after one hundred iterations have never been critical.

The third method considered by Cook, T.M. and Jennings, R.H. (1979), was the dynamic shutoff method. This is a crude simulation, which is terminated when there is no significant change between the current cdf and one hundred iterations earlier.

The rest of the research showed that no one method was consistently better than another; but at least simulation methods were shown to be not always cost prohibitive, even for very large networks. Computer time could be greatly reduced by "Intelligent Simulation" and at the same time a high degree of accuracy can be achieved.

2.5.2 COMMENTS ON THE SIMULATION APPROACH

The approach would be used if other methods are impracticable, i.e. the minimum applicability conditions for the other methods are not met. Simulating a complete network, would necessitate a specialised professional package, SLA-MII/TESS for example.

2.5.3 ADVANTAGES

The main advantages of the simulation approach are:

(I)	Parallel and conditional activities can be easily simulated.
(II)	An animated simulation package is readily accepted and understood.
(III)	It's flexibility ensures wide application.

2.5.4 DISADVANTAGES

Four of the main disadvantages of the simulation approach are:

(I)	The approach is machine dependent.
(II)	Simulating big and complex networks might take a great amount of the CPU time and computer memory.
(III)	The prices of some of these software packages are sometimes off-putting.
(IV)	The need for experience in using the software packages.

2.6 <u>OTHER METHODOLOGIES</u>

2.6.1 <u>MULTIVARIATE APPROACH</u>

Recently, work by Anklesaria, K.P. (1986) et.al. has included a multivariate approach to estimating the completion time of PERT networks. Like Ringer L.J. (1969), the cdf of completion time is expressed as a multi-dimensional integral; but the set of all complete paths is treated as a multivariate distribution. The model takes into account the possible dependence of activity times. The multivariate distribution, by the Central Limit Theory, is approximated to a multivariate Normal Distribution. Drezner, Z. (1986) divised a simple calculation for evaluating this which, for a network with fewer than seven complete paths is done in "reasonable computer time". An approximation for the calculation is considered by discounting some paths that are not likely to effect completion time too much. Comparison of the results with a simulation (requiring more computer time), showed a good degree of accuracy in the two problems that were solved. The method seems to have great potential for medium sized networks with dependent activity times.

2.6.2 THE CONTROLLED INTERVAL (C.I. APPROACH)

Chapman, C.B. and Cooper, D.F. (1983) considered a technique for deriving the distribution of the sum of two or more dependent random variables. The data was assumed to be discrete and in empirical form. It was also assumed that each of the input distributions had intervals of equal width. The convolution integral is replaced by a summation; but the evaluations have inherent bias due to computational error. Consideration is given to ways of controlling this error and although the techniques were incorporated into computer software, the paper is written more for the purpose of gaining insight into the problems of modelling risk rather than to obtain a project cost and time distribution.

2.6.3 PMS/BRIDGE (A PROJECT MODEL SIMULATOR)

This expert system, Hoskyns (1988) allows information to be input either interactively from the user, that being based on personal judgements, or from previous data. The package uses this information to evaluate the size of the project, risk and complexity, scheduling and control of the system. The software has portfolio management functions allowing for a balance of the priorities of a portfolio of projects. The system uses subjective information rather than functional input information.

2.7 CONCLUSIONS

Table 2.5 gives a summary of the relative merits of the five methodologies associated with project planning risk. It is intended to give general guidelines for one aspect of risk management, namely the estimation and control of project cost and time.

Table 2.1 Configurations on evaluating pdf's analytically of the Uniform, Exponential, Normal and Gamma distributions.

The symbols are used to represent:

X = Has been evaluated analytically.

– = Has not been evaluated analytically.

Table 2.1

	Uniform	Exponential	Normal	Gamma
Uniform	X	X	–	X
Exponential	X	X	–	X
Normal	–	–	X	–
Gamma	X	X	–	X

Table 2.2 The mean and range of ten Uniform activity times.

Table 2.2

Activity	Predecessor	Distribution	Mean	Range
A	–	Uniform	1	0 – 2
B	A	Uniform	2	1 – 3
C	A	Uniform	5	3 – 7
D	B	Uniform	1	0 – 2
E	B	Uniform	3	1 – 5
F	D	Uniform	2	1 – 3
G	E	Uniform	2	1 – 3
H	C	Uniform	6	4 – 8
I	H	Uniform	8	6 – 10
J	F,G,I	Uniform	2	1 – 3

Table 2.3 The range and pdf of the five critical activity times.

Table 2.3

Critical Activities	Range	pdf
A	$0 < t_1 < 2$	$f_A(t) = 1/2$
C	$3 < t_2 < 7$	$f_C(t) = 1/4$
H	$4 < t_3 < 8$	$f_H(t) = 1/4$
I	$6 < t_4 < 10$	$f_I(t) = 1/4$
J	$1 < t_5 < 3$	$f_J(t) = 1/2$

Table 2.4 The total floats of the ten activity times.

Table 2.4

Activity	Total float
A	0
B	12
C	0
D	14
E	12
F	14
G	12
H	0
I	0
J	0

Table 2.5 Comparison of methodologies for risk assessments in project management.

Table 2.5

Method	Manual / computer based	Maximum size of network	Accuracy	Minimum applicability conditions	Software Packages: Name	Software Packages: Execution time	Possible alternative
Analytical approach	Manual	Very small	Highest	1- Small networks 2- Pdf`s in functional form 3- Product of cdf`s is integrable	Macsyma	Short if possible	Numerical integration
	Computer based	Medium			Mathematica		
Numerical integra-tion	Computer based	Large	High	1- Small networks 2- Pdf`s in functional form 3- Availability of software	NAG, etc. Self written programs	Medium	Moments methods
Moments method	Manual	Medium	Medium	1- First two moments of input distributions are known 2- Availability of software or look-up tables of moments of distribution of maximum 3- For Pearson`s curves the completion time distribution must be unimodal, U-shaped, J-shaped	Self written programs	Short	Simulation
	Computer based	Large					
Simulation	Computer based	Large	Medium / High	1- Availability of software/hardware 2- Experties of user	SLAM II/TESS, Hocus, SIMAN, NAG and self written programs	Long	-
PERT analysis	Manual and computer based	Large	Poor	1- Three time estimates 2- Non-critical activities remain so 3- Total project time is assumed to be normal	PMW, Pertmaster	Medium	-

Figure 2.1 Network diagram representing the ten activity times of the first example.

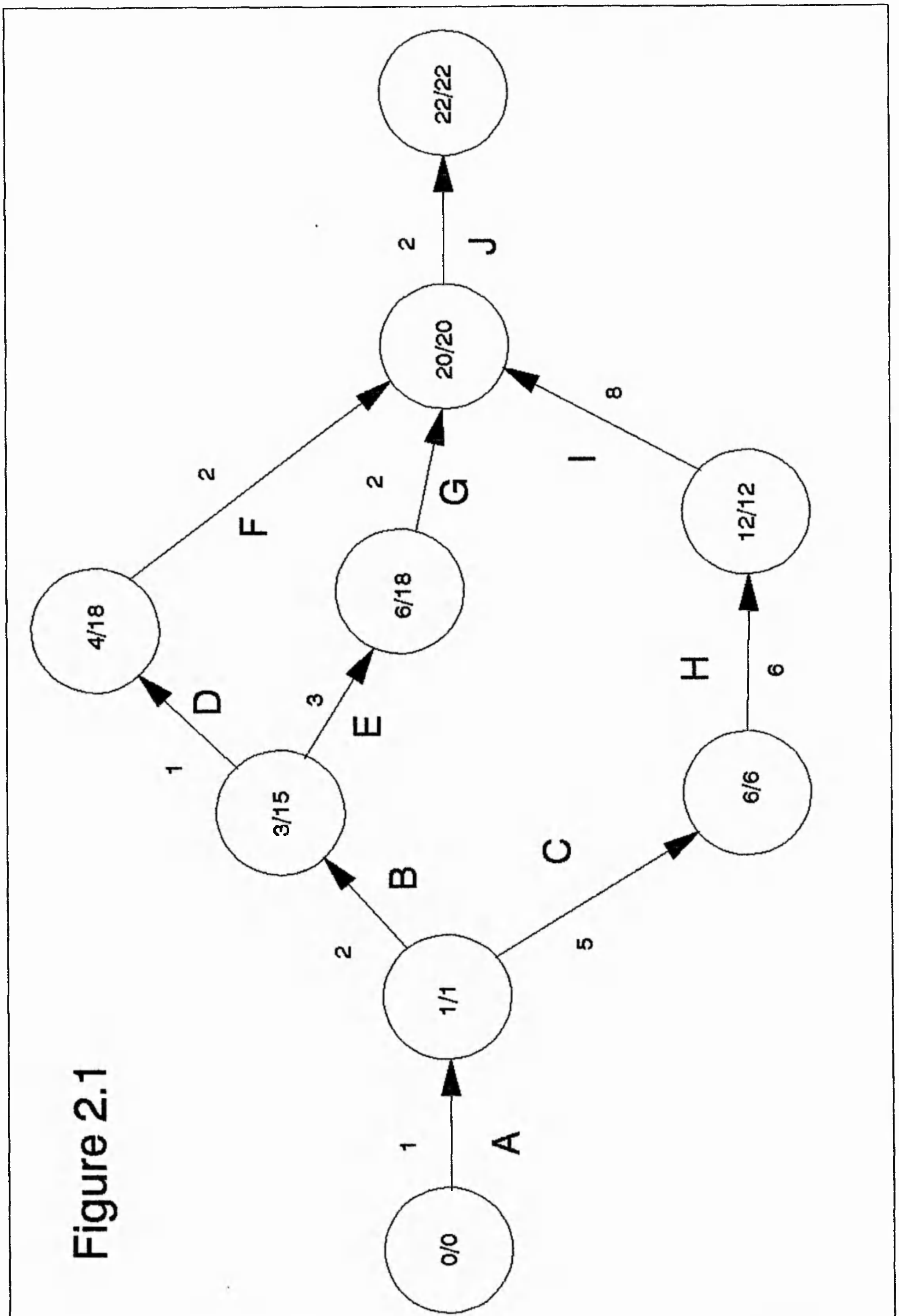


Figure 2.1

Figure 2.2 A diagram showing the three areas (P,Q,R) to be integrated.

Figure 2.2

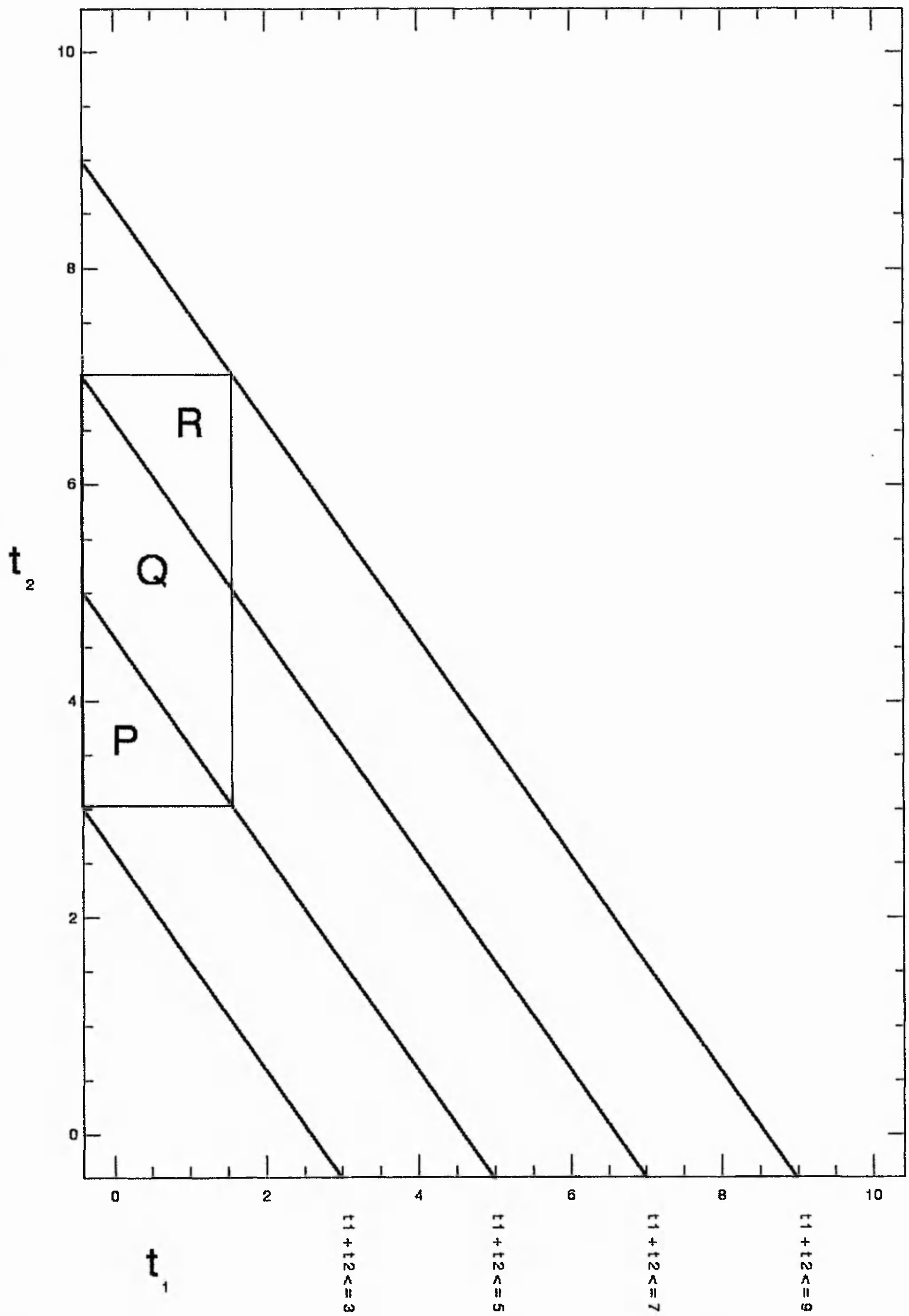


Figure 2.3 A diagram showing the density function $f(S_2)$ of project completion time.

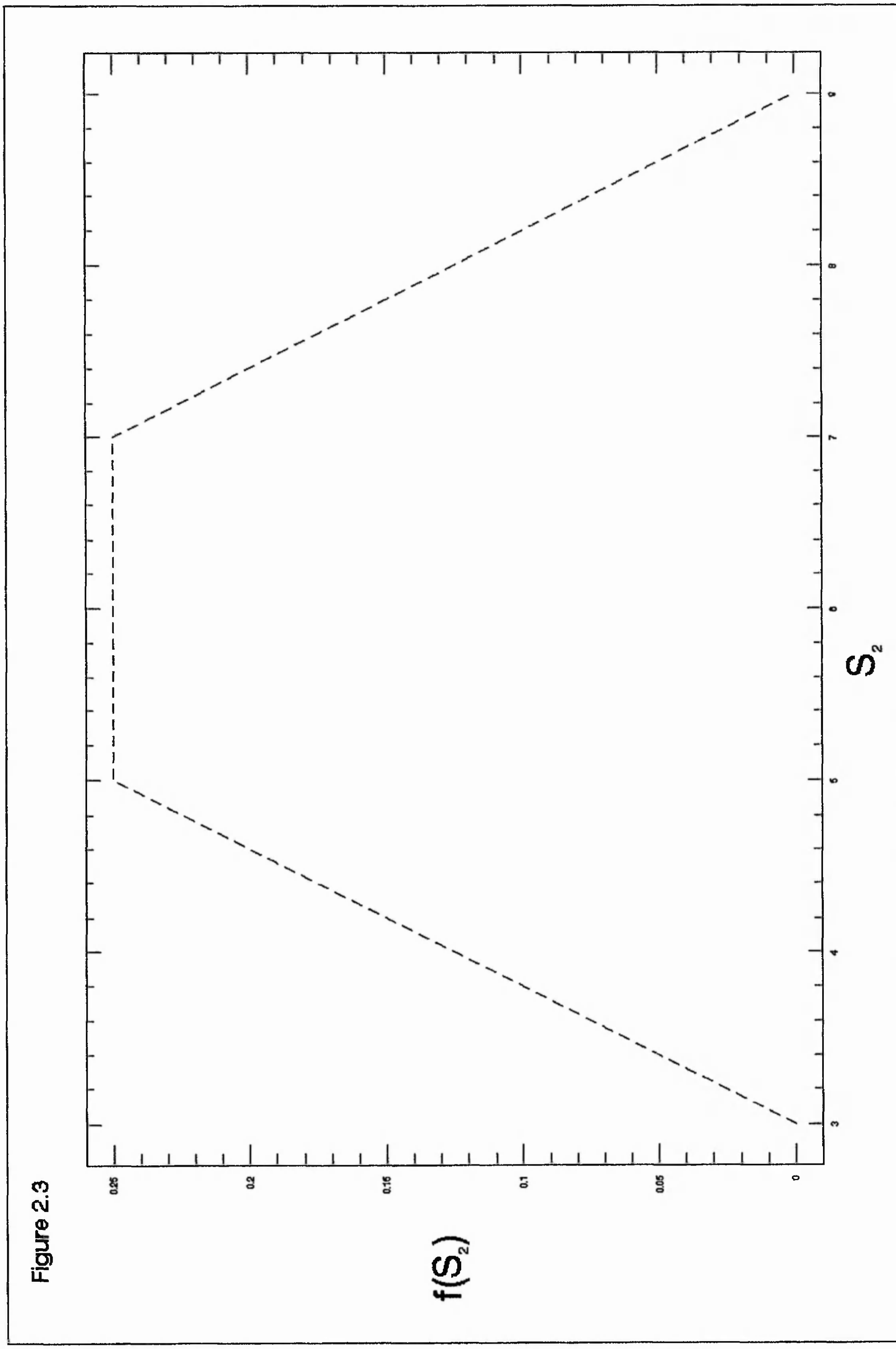


Figure 2.3

Figure 2.4 A graph of the density function $f(S_5)$.

Figure 2.4

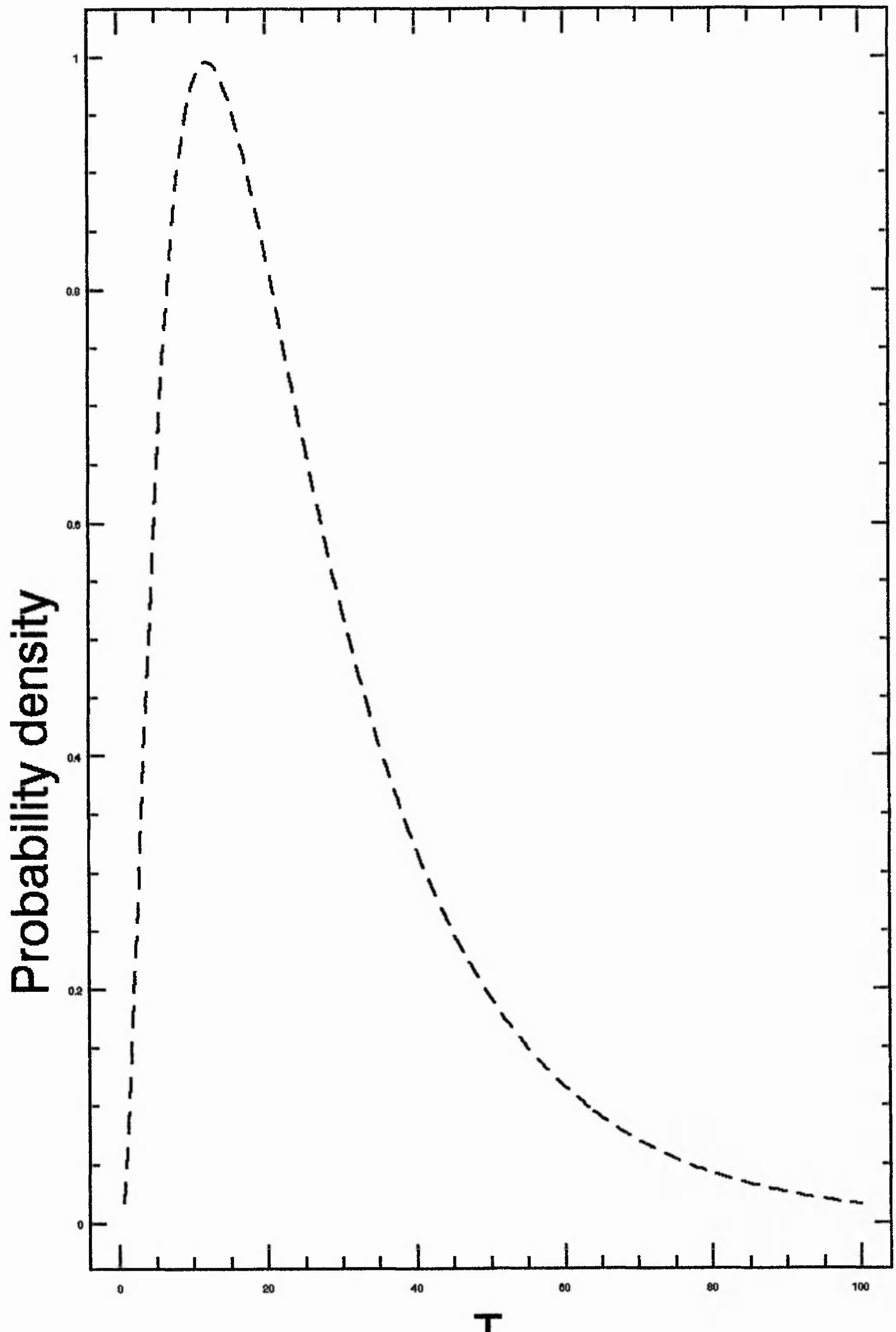


Figure 2.5 A network diagram of two activity times (Normal and Gamma) in series and one (Beta) in parallel, with them.

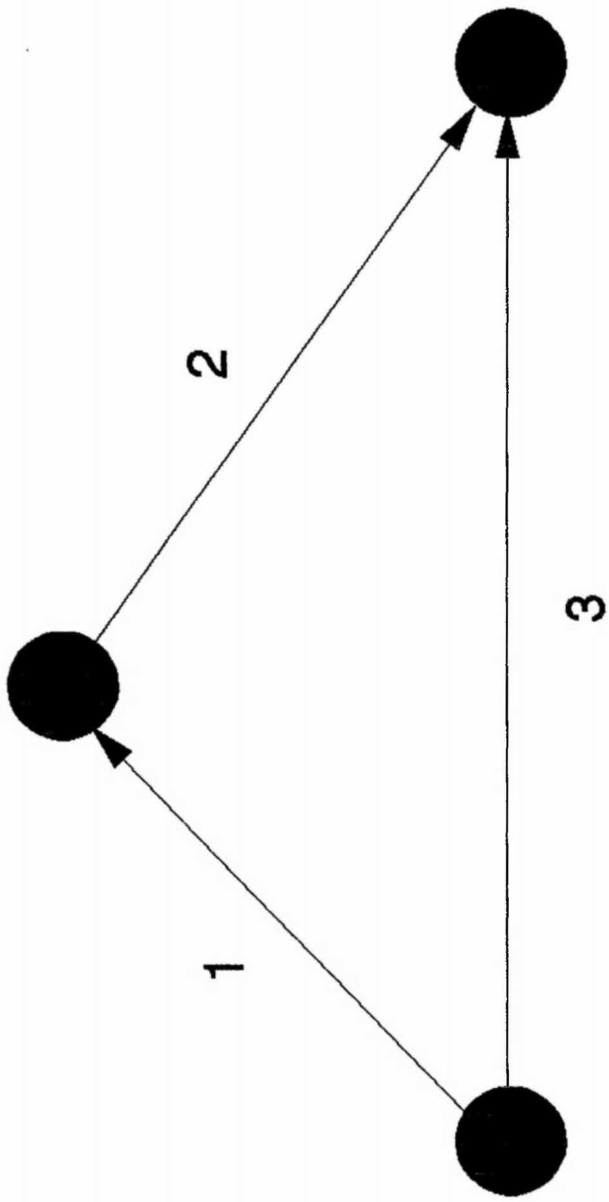


Figure 2.5

CHAPTER THREE

MOMENTS METHOD (UNIMODAL INPUT DISTRIBUTION)

3.1 A UNIFIED METHODOLOGY

Having reviewed the relative merits of current methodologies we now turn our attention to a consideration of a method of risk assessment that gives acceptably accurate results and that can be applied to most if not all practical situations i.e. we investigate the possibility of a unified methodology.

One of the main difficulties in risk assessment is the gathering of information on the input distributions i.e. the distributions of time and cost for each activity in the project. Whether that initial data is in the form of an empirical distribution, -perhaps obtained from historical data- or whether it comprises of purely subjective estimates of certain parameters, it must be regarded as sample data with a random element present. This data has to be used to estimate the characteristics of the population distribution from which such sample data is taken. The initial assumption that we make about input distributions is that they are from the Erlang family:- an assumption which is justified for the reasons set out below in paragraph 3.1.1 . The distribution associated with the maximum of two Erlang variables has a pdf which is not easy to handle algebraically and the difficulty is compounded when the maximum of three or more random variables is required. The problem is overcome by concentrating on the moments of these distributions rather than handling the probability density function itself. There is a price to be paid for this in loss of accuracy but in most cases the resulting errors were small and insignificant in comparison to the error of the input distributions. Where the errors were not small, considerations were given to how they could be reduced to an acceptable level. The proposed unified methodology then is essentially one based on the moments of the input distributions.

In the assessment of project planning risk, the use of the method of moments to find the distribution of project completion time, is particularly useful when activity time distributions are in empirical form or when the analytical method can not be used because the mathematics becomes unmanageable. The moments method is conditional upon being able to evaluate the moments of the distribution of the sum and maximum of activity times. Formulae for the first four central moments of the distribution of the maximum of two normal variables have been derived by Clark, C.E. (1961) and numerical approximation methods have been used by Sculli, D. and Wong, K.L. (1985), to find the mean and variance of the maximum of two Beta variables. This Chapter presents the derivation of formulae, for the moments of the distribution of the maximum of two Erlang variables.

3.1.1 CHOICE OF ERLANG INPUT DISTRIBUTION

The Erlang family of distributions each of which has only two parameters is appropriate for describing activity times; because frequently, in practice activity times will be skewed and not normal. An appropriate choice of parameters can be found to give a good fit to any unimodal activity time distribution.

The Erlang distribution has been taken as a model to fit the distribution of input activity times in most of the projects discussed in this thesis. The next three sub-sections explain the reasons behind this choice.

3.1.1.1 SKEWED INPUT DISTRIBUTIONS

The size of projects we are dealing with are small to medium. The shape of the activity time distributions is usually skewed. The Erlang distribution has been found to be one of the most suitable models for such studies, since the degree of skewness can be controlled by a suitable choice of parameters.

3.1.1.2 ONLY TWO PARAMETERS REQUIRED

Using the Erlang distribution to fit an activity time requires only the first two moments, which are easy to compute and to be understood, unlike other models which require more moments to be computed and probably would be very difficult for the project manager, to comprehend.

3.1.1.3 AMENABLE TO MATHEMATICAL MANIPULATIONS

Another advantage of using the Erlang distribution is that analytical expressions for the first two moments can be derived without the need to resort to special functions. This is also true for even the third and fourth moments.

3.2 MOMENTS FORMULAE FOR PARALLEL ACTIVITIES

Let t_1 and t_2 be Erlang, and independently distributed activity times with means μ_1 and μ_2 respectively, and integer shape parameters c_1 and c_2 respectively. The density function, (pdf) of $\max(t_1, t_2)$ is of the form:

$$f_1 \quad F_2 \quad + \quad f_2 \quad F_1 \tag{3.1}$$

see David, H.A. (1981), where f and F are Erlang pdf and cdf functions respectively. It can be shown algebraically that if

$$p = \frac{c_1\mu_2}{(c_1\mu_2 + c_2\mu_1)} \quad \text{and} \tag{3.2}$$

$$q = \frac{c_2\mu_1}{(c_1\mu_2 + c_2\mu_1)} = 1 - p \tag{3.3}$$

then the first four central moments of the distribution of the maximum are:

$$\mu_{jM} = \mu'_{j,1}\left[1 - p^{c_1+j}\sum_{i=0}^{c_2-1}\binom{c_1+i+j-1}{i}q^i\right] +$$

$$\mu'_{j,2}\left[1 - q^{c_2+j}\sum_{i=0}^{c_1-1}\binom{c_2+i+j-1}{i}p^i\right] - A_j \quad \text{for} \quad j = 1,2,3,4 \qquad (3.4)$$

where $\mu'_{j,1} = \mu_1^j \prod_{i=0}^{j-1} \frac{c_1 + i}{c_1}$

is the j^{th} moment about the origin of the activity time distribution with parameters μ_1, c_1 etc... and

$$A_1 = 0 \qquad (3.5)$$

$$A_2 = \mu_{1M}^2 \qquad (3.6)$$

$$A_3 = 3\mu_{1M}\mu_{2M} + \mu_{1M}^3 \qquad (3.7)$$

$$A_4 = 4\mu_{1M}\mu_{3M} + 6\mu_{1M}^2\mu_{2M} + \mu_{1M}^4 \qquad (3.8)$$

3.3 ALGEBRAIC DERIVATION OF THE SECOND MOMENT OF DISTRIBUTION OF THE MAXIMUM OF TWO RANDOM VARIABLES

The derivation of the second central moment of the distribution of the maximum of two Erlang variables is shown. Moments of other orders are derived similarly.

In our case

$$f_1 = \frac{\lambda_1^{c_1} x^{c_1-1} e^{(-\lambda_1 x)}}{(c_1-1)!} , \qquad 0<x<\infty \quad (3.9)$$

and

$$F_1 = 1-e^{(-\lambda_1 x)}\left[\sum_{i=0}^{c_1-1} \frac{(\lambda_1 x)^i}{i!}\right] \quad (3.10)$$

$$where \ \lambda_1 = \frac{c_1}{\mu_1} = \frac{ShapeParameter}{Mean} .$$

Similarly, we may define f_2 and F_2.

Therefore the second moment about the mean of the distribution of the maximum of two Erlang variables, is

$$\mu_{2M} = \int_0^\infty \frac{\lambda_1^{c_1} x^{c_1+1} e^{(-\lambda_1 x)}}{(c_1-1)!} \; [1 - e^{(-\lambda_2 x)} - \lambda_2 x e^{(-\lambda_2 x)} -$$

$$\frac{\lambda_2^2 x^2 e^{(-\lambda_2 x)}}{2!} - \ldots - \frac{\lambda_2^{c_2-1} x^{c_2-1} e^{(-\lambda_2 x)}}{(c_2-1)!}] dx \;\; +$$

$$\int_0^\infty \frac{\lambda_2^{c_2} x^{c_2+1} e^{(-\lambda_2 x)}}{(c_2-1)!} \; [1 - e^{(-\lambda_1 x)} - \lambda_1 x e^{(-\lambda_1 x)} -$$

$$\frac{\lambda_1^2 x^2 e^{(-\lambda_1 x)}}{2!} - \ldots - \frac{\lambda_1^{c_1-1} x^{c_1-1} e^{(-\lambda_1 x)}}{(c_1-1)!}] dx \;\; - \;\; \mu_{1M}^2 \qquad (3.11)$$

$$\mu_{2M} = G(\lambda_1, c_1, \lambda_2, c_2) \;\; + \;\; G(\lambda_2, c_2, \lambda_1, c_1) \;\; - \;\; \mu_{1M}^2 \qquad (3.12)$$

where μ_{1M} is the mean of the distribution of the maximum, of two Erlang variables.

Noting that:

$$Mean \;=\; \frac{c}{\lambda}, \qquad Variance \;=\; \frac{c}{\lambda^2}$$

and the j^{th} moment about origin,

$$\mu'_{j,1} \;=\; \int_0^\infty x^j f_1 dx \;=\; \left(\frac{1}{\lambda_1}\right)^j \prod_{i=0}^{j-1} (c_1 + i) \qquad (3.13)$$

we have:

$$G(\lambda_1, c_1, \lambda_2, c_2) = \left(\frac{c_1+1}{c_1}\right)\mu_1^2 -$$

$$\sum_{i=0}^{c_2-1} \frac{\lambda_1^{c_1}\lambda_2^{i}}{i!(\lambda_1+\lambda_2)^{c_1}} \int_0^\infty \frac{(\lambda_1+\lambda_2)^{c_1} x^{c_1+1+i} e^{(-x(\lambda_1+\lambda_2))}}{(c_1-1)!} \, dx \qquad (3.14)$$

$$G(\lambda_1, c_1, \lambda_2, c_2) = \left(\frac{c_1+1}{c_1}\right)\mu_1^2 -$$

$$\sum_{i=0}^{c_2-1} \frac{\lambda_1^{c_1}\lambda_2^{i}}{i!(\lambda_1+\lambda_2)^{c_1}} \cdot \frac{c_1(c_1+1)\ldots(c_1+1+i)}{(\lambda_1+\lambda_2)^{i+2}}$$

$$G(\lambda_1, c_1, \lambda_2, c_2) = \left(\frac{c_1+1}{c_1}\right)\mu_1^2 -$$

$$\frac{\lambda_1^{c_1} c_1(c_1+1)}{(\lambda_1+\lambda_2)^{c_1+2}} \sum_{i=0}^{c_2-1} \left\{\begin{matrix} c_1+1+i \\ i \end{matrix}\right\} \left[\frac{\lambda_2}{\lambda_1+\lambda_2}\right]^i$$

$$G(\lambda_1, c_1, \lambda_2, c_2) = \left(\frac{c_1+1}{c_1}\right)\mu_1^2 -$$

$$\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{c_1+2} \mu_1^2 \left(\frac{c_1+1}{c_1}\right) \sum_{i=0}^{c_2-1} \left\{\begin{matrix} c_1+1+i \\ i \end{matrix}\right\} \left[\frac{\lambda_2}{\lambda_1+\lambda_2}\right]^i$$

$$G(\lambda_1, c_1, \lambda_2, c_2) = \mu'_{2,1}\left[1 - p^{c_1+2} \sum_{i=0}^{c_2-1} \left\{\begin{matrix} c_1+1+i \\ i \end{matrix}\right\} q^i \right] .$$

Hence from (3.12) the result is proved.

3.4 MOMENTS FORMULAE FOR SERIES ACTIVITIES

The first four moments of the distribution of the sum of two independent random variables $t_1 + t_2$ are easily evaluated in terms of the activity time distribution parameters, see section 2.4.2.

3.5 APPLICATION OF THE FORMULAE

The moments formulae presented can be readily used to find the moments of the distribution of project completion time. By assuming path independence, the network being made up of series and parallel activity arcs can be reduced sequentially to a single activity arc whose activity time will be the project completion time. The process will involve a repeated use of the moments formulae which implies that the Erlang distribution must be assumed to be stable under maximisation, see Golenko-Ginzburg, D. (1989). The distribution of the maximum of two or more Erlang variables will not be exactly Erlang; but the formulae gave good approximations to the moments obtained by simulation as is shown in Table 3.1 . The results were typical for a good range of input parameters.

If a further assumption is made, namely that the project completion time is Erlang distributed, then we need only find the first two central moments, since a good fit to an Erlang curve can be made when these two moments are known. If on the other hand we choose to use the first four moments, the percentage points of the completion time distribution can be read off from tables of Pearson's curves or by the use of appropriate software Amos, D.E. and Daniel, S.L. (1971). In this case one would expect to obtain a more accurate estimate of the completion time distribution.

3.6 FITTING AN ERLANG CURVE

If historical data are available in empirical form, the shape parameter of the Erlang distribution, C is found by rounding:

$$\frac{(Mean)^2}{Variance} \tag{3.15}$$

to the nearest integer. The error incurred by rounding was found to have little effect on the accuracy of the results.

3.7 ILLUSTRATIVE EXAMPLES

In the first example with network in Figure 3.1 and input values in Table 3.2, the activity times were Erlang distributed. The data was chosen so that each of the complete paths was likely to become critical. To determine the accuracy of the results, the network was simulated 10000 times using simulated Erlang input distributions. The accuracy of the simulation is discussed in section 3.13.

The data of the second example was in empirical form as might well be the case in practice. The distribution of the activity times varied; some unimodal and J-shaped, again to reflect a practical situation based on historical data. The data was used to estimate Erlang parameters before the moments formulae were applied. The simulated activity times used for checking the accuracy of the results, were drawn from the empirical data. Again, data was chosen so that there was a real possibility of each path becoming critical. The network diagram and input values for the network for this second example are shown in Figure 3.2 and Table 3.3 .

A third example was taken from Sculli, D. (1983). This network was for Normally distributed activity times, see Figure 3.3 and Table 3.4 .

The results of the three examples are summarized in Tables 3.5, 3.6 and 3.7 and graphically in Figures 3.4, 3.5 and 3.6.

3.8 COMMENTS ON EXAMPLES

The graphs showed that in each of the three examples, the moments methods -both for the 2-moments Erlang fit and the 4-moments Pearson fit- gave significantly more accurate estimates of risk than the PERT approach. In the tails of the completion time distribution, there were only slight discrepancies between the 2-moments and the 4-moments approaches; but for the rest of the completion time distribution, the 4-moments approach was better. However, an advantage of the 2-moments approach that assumes Erlang completion time, is that Pearson tables are not required. Also the production of look-up tables of the first two moments of the distribution of the maximum is feasible, thus doing away with the need for computer software.

The examples showed that the errors incurred by the rounding of the shape parameter, by assuming the stability of the distribution under maximisation and by assuming path independence, do not significantly effect the estimates of project completion time. A variety of input distributions in a single network system still gave acceptably accurate results as the second example showed. All the percentage errors in the mean and standard deviation, derived from the results of the moments method were acceptably small, especially compared to those of the PERT approach. The 4-moments Pearson tables method, when compared with the 2-moments approach, gave marginally more accurate estimates of completion time mean as they were expected to, and the same accuracy is expected with the standard deviation; but in some cases this expectation is marginally violated probably because of the number of numerical approximations involved. The distributions of completion times of the three examples are close to normal and this is because the shape parameters are large. The shape and scale parameters of completion times for the 2-moments method are computed in Table 3.8 .

The normal distribution gives a good approximation to Erlang distribution when the shape parameter C is big, see the plotted Erlang's pdf with C=28.72404 and b=4.07519 in Figure 3.7 and the plotted normal's pdf with mean=117.05590 and standard deviation=21.84090 in Figure 3.8 . In general, we assumed that the activity times are independent, identically distributed and also the number of activity times is big, but for the central limit theorem to apply, activities must be in series and have same mean and variance, even so the total project time (Erlang) seems to be asymptotically normal.

In conclusion, the use of the moments method for risk assessment with assumed Erlang input distributions is to be recommended particularly for activity times that have skew distributions and where historical data is in empirical form.

3.9 ILLUSTRATIVE EXAMPLE ON THE LIMITATIONS OF THE MOMENTS METHOD

The aim of this section is to show, with the use of an example, the effect on the errors as the order of the moments and the number of parallel activities increases.

This example consists of up to twenty activities in parallel and all activities have different Erlang parameters. The errors in the mean, standard deviation, and coefficient of skewness and kurtosis, when comparing moments method with simulation are listed in Table 3.9 . It is clear from the table that the errors are generally increasing with the increase of the moment's order, also they are likely to be higher when the number of parallel activities are small and this maybe because the errors cancel each other when the number of activities are increased.

3.10 REVIEW OF UNDERLYING ASSUMPTIONS

The following three sub-sections are to review the assumptions made for the type of networks that are studied in this thesis.

3.10.1 INDEPENDENT ACTIVITIES

Activities in a network can be dependent or independent of each other. In the work that has been addressed in this thesis we have assumed that activities are independent no matter what the arrangement of the network. This assumption has been adopted to simplify the mathematical and analytical work involved in the estimation of the completion time of a project.

3.10.2 PATH INDEPENDENCE

Two methods have been used, method (A) and method (B) to collapse a network, see Chapter five. In both methods, network paths are assumed to be independent, i.e. they have no common activities.

3.10.3 STABILITY OF DISTRIBUTION OF MAXIMUM

The Erlang distribution has been assumed to be stable under maximization, see Golenko-Ginzburg, D.(1989). Even though the distribution of the maximum of two or more Erlang variates is not exactly Erlang, the derived moments formulae gave a good approximation to the moments obtained by simulation.

3.11 ROBUSTNESS OF THE MOMENTS METHOD

When applying the moments method, we assumed that the activities are independent of each other and also all possible paths in the network are independent. Also the distribution of the maximum is assumed to be stable. From many examples and real life data, it has been found that violation of the above assumptions still has resulted in a good estimation of the completion time of a project, see Chapter five and Chapter six.

3.12 ERRORS IN PERCENTILES OF COMPLETION TIMES

The percentage errors compared to simulation, in the 5,10,50,90 and 95 percentiles of the moments method, using the first four moments, is tabulated in Table 3.10, for the three examples 1, 2 and 3 . The table shows no obvious pattern of errors. As a conclusion on these three examples, it is not feasible to attempt to add a correction term to reduce the percentage errors, which are acceptably small anyway.

3.13 RELIABILITY OF THE SIMULATION

The reliability of the computer simulation has been examined by comparison with the results from analytical work. Several computer simulations have been carried out to examine their reliability, and we illustrate by using the small network in Figure 3.9, which contains four activities. The distributions of the activities are exponential with means 1.0, 5.0, 6.0 and 10.0 . The distribution function of project completion time has been found analytically to be:

$$F(x) = -\frac{36}{5}e^{-\frac{1}{6}x} - \frac{1}{20}e^{-x} + \frac{25}{4}e^{-\frac{1}{5}x} + \frac{36}{5}e^{-\frac{4}{15}x} + \frac{1}{20}e^{-\frac{11}{10}x} - \frac{25}{4}e^{-\frac{3}{10}x} - e^{-\frac{1}{10}x} + 1.0 \qquad (3.16)$$

The errors in the simulation (of 100000 values) calculated by comparison with the exact analytical approach are listed in Table 3.11 . It is clear that the errors are well within sampling error range and show that computer simulation provides an acceptable standard by which other methods can be assessed. The simulated values are generally correct to three decimal places and so provide a good standard by which the unified methodology can be judged.

Table 3.1 Comparison of moments of the distribution of the maximum by formulae and simulation.

Table 3.1

Number of parallel activities	Parameters of addition		μ_{1m}		μ_{2m}	
	C	μ	Formula	Simulation	Formula	Simulation
1	4.0	4.0				
2	1.0	3.0	4.94922	4.94341	6.89586	6.80735
3	2.0	5.0	6.59201	6.56174	9.85234	10.31701
4	5.0	4.5	7.21037	7.11237	8.87295	8.92557
5	4.0	6.0	8.31834	8.23206	8.35435	9.19678

Table 3.2 The shape parameter C and the mean of nine Erlang activity times, for the first example.

Table 3.2

Activity	C	μ
A	9.0	9.0
B	2.0	6.0
C	5.0	25.0
D	6.0	24.0
E	8.0	32.0
F	8.0	40.0
G	10.0	80.0
H	4.0	24.0
I	9.0	27.0

Table 3.3 Frequency tabulation of ten activity times, for the second example.

Table 3.3 (1)

Activity	t	Probability
A	2	0.10
	3	0.50
	4	0.40
B	1	0.60
	2	0.30
	3	0.10
C	6	0.50
	7	0.20
	8	0.20
	9	0.10
D	4	0.10
	5	0.20
	6	0.70
E	6	0.20
	7	0.30
	8	0.50

[continued]

Table 3.3 (2)

Activity	t	Probability
F	3	0.30
	4	0.20
	5	0.10
	6	0.10
	7	0.30
G	13	0.10
	14	0.10
	15	0.10
	16	0.50
	17	0.20
H	9	0.20
	10	0.20
	11	0.60
I	12	0.20
	13	0.60
	14	0.10
	15	0.10
J	3	0.10
	4	0.20
	5	0.70

Table 3.4 The mean and variance of the thirteen Normal activity times of the third example.

Table 3.4

Activity	Mean	Variance
A	9.00	1.80
B	6.00	1.20
C	8.00	1.60
D	6.00	1.20
E	9.00	1.80
F	12.00	2.40
G	5.00	1.00
H	3.00	0.60
I	7.00	1.40
J	3.00	0.60
K	7.00	1.40
L	4.00	0.80
M	11.00	2.20

Table 3.5 The probability of completion time < T, for the first example on the moments method, simulation and PERT.

Table 3.5

T	Moments Method		Simulation	PERT
	2 Moments	4 Moments		
72.1061	0.0102	0.0025	0.0058	0.1172
74.5802	0.0152	0.0050	0.0098	0.1357
77.4109	0.0233	0.0100	0.0171	0.1593
81.8933	0.0422	0.0250	0.0338	0.2017
86.0847	0.0686	0.0500	0.0644	0.2470
91.3349	0.1156	0.1000	0.1159	0.3106
101.2271	0.2474	0.2500	0.2706	0.4457
114.1185	0.4758	0.5000	0.5234	0.6292
129.3987	0.7319	0.7500	0.7640	0.8111
145.6064	0.9003	0.9000	0.9049	0.9290
156.6077	0.9561	0.9500	0.9529	0.9690
167.0446	0.9818	0.9750	0.9752	0.9876
180.3429	0.9948	0.9900	0.9897	0.9968
190.1848	0.9981	0.9950	0.9939	0.9990
199.9283	0.9993	0.9975	0.9971	0.9997
Mean	117.0559	117.0282	115.8882	106.6634
% Error in the mean	1.0100	0.9800		7.9600
Standard deviation	21.8409	21.8162	22.1499	25.2002
% Error in the standard deviation	1.4000	1.5100		13.7700

Table 3.6 The probability of completion time < T, for the second example on the moments method, simulation and PERT.

Table 3.6

T	Moments Method		Simulation	PERT
	2 Moments	4 Moments		
23.08346	0.0055	0.0025	0.0151	0.1544
23.30122	0.0091	0.0050	0.0193	0.1917
23.54288	0.0153	0.0100	0.0234	0.2390
23.91055	0.0316	0.0250	0.0356	0.3218
24.23839	0.0564	0.0500	0.0556	0.4041
24.62936	0.1035	0.1000	0.0897	0.5079
25.31220	0.2432	0.2500	0.2108	0.6837
26.11128	0.4864	0.5000	0.4949	0.8448
26.94802	0.7451	0.7500	0.7821	0.9425
27.72899	0.9042	0.9000	0.9314	0.9821
28.20723	0.9557	0.9500	0.9635	0.9923
28.62766	0.9799	0.9750	0.9790	0.9966
29.12207	0.9931	0.9900	0.9892	0.9988
29.46162	0.9969	0.9950	0.9924	0.9994
29.77824	0.9987	0.9975	0.9956	0.9997
Mean	26.1568	26.1560	26.1271	24.7154
% Error in the mean	0.1137	0.1106		5.4000
Standard deviation	1.2034	1.2051	1.1607	1.3247
% Error in the standard deviation	3.6788	3.8253		14.1294

Table 3.7 The probability of completion time < T, for the third example on the moments method, simulation and PERT.

Table 3.7

T	Moments Method		Simulation	PERT
	2 Moments	4 Moments		
22.5885	0.0423	0.0025	0.0771	0.6105
22.6262	0.0448	0.0050	0.0805	0.6174
22.6847	0.0488	0.0100	0.0854	0.6279
22.8146	0.0589	0.0250	0.1009	0.6511
22.9800	0.0739	0.0500	0.1232	0.6798
23.2457	0.1040	0.1000	0.1669	0.7237
23.8898	0.2110	0.2500	0.3025	0.8162
24.8768	0.4586	0.5000	0.5584	0.9149
26.0241	0.7607	0.7500	0.8223	0.9725
27.0338	0.9227	0.9000	0.9390	0.9918
27.5700	0.9643	0.9500	0.9694	0.9960
27.9752	0.9817	0.9750	0.9815	0.9978
28.3676	0.9911	0.9900	0.9909	0.9988
28.5862	0.9942	0.9950	0.9940	0.9992
28.7550	0.9959	0.9975	1.0000	0.9994
Mean	25.0552	25.0366	24.7402	23.1641
% Error in the mean	1.2700	1.2000		6.3700
Standard deviation	1.3531	1.4010	1.3913	1.0392
% Error in the standard deviation	2.7500	0.7000		25.3100

Table 3.8 The shape parameter (C) and the scale parameter (b), using the 2-moments method, for the three examples.

Table 3.8

Example number	Shape parameter (C)	Scale parameter (b)
1	28.72404	4.07519
2	472.44274	0.05537
3	342.87510	0.07307

Table 3.9 Percentage errors in the mean, standard deviation, coefficient of skewness and kurtosis for the twenty Erlang activity times in parallel.

Table 3.9

No. of Parallel activities	% Error in			
	Mean	Standard deviation	Coefficient of	
			Skewness	Kurtosis
2	0.66	2.88	1.33	2.35
3	1.02	6.64	6.49	10.90
4	1.76	5.95	7.31	5.17
5	2.28	5.33	7.48	13.19
6	0.89	0.57	3.02	2.08
7	0.66	0.50	2.22	1.19
8	0.57	1.27	0.67	2.70
9	0.64	0.34	0.21	0.13
10	0.60	0.62	1.47	0.13
11	0.06	1.01	1.52	0.07
12	0.35	1.20	0.77	0.94
13	0.28	0.45	1.01	1.02
14	0.19	0.16	0.79	2.29
15	0.26	0.03	4.24	1.90
16	0.47	1.11	1.19	1.00
17	0.13	0.62	1.99	0.38
18	0.25	1.21	1.29	0.72
19	0.30	1.13	3.57	0.40
20	0.37	0.22	1.78	1.09
μ	0.61790	1.64421	2.54474	2.50789

Table 3.10 Percentage errors in percentiles for examples 1, 2 and 3.

Table 3.10

Percentile	% Error in percentiles of example		
	1	2	3
5	-2.34498	-0.38018	-
10	-1.80679	0.23534	-1.92353
50	-1.05669	0.05703	-0.91394
90	-0.38861	-0.59569	-1.26422
95	-0.42622	-0.71806	-1.25680

Table 3.11 The distribution of the completion time using the analytical approach against simulation, and also the percentages of error between the two.

Table 3.11

	Pr(Completion) < T		
T	Simulation	Analytical	% Error
2.00000	0.0042900000	0.0042975472	0.1759243434
4.00000	0.0369200000	0.0365242487	1.0719157834
6.00000	0.1063300000	0.1054011709	0.8735343812
8.00000	0.2008700000	0.2004096033	0.2292013347
10.00000	0.3074400000	0.3071719752	0.0871795406
12.00000	0.4153300000	0.4140933455	0.2977522666
14.00000	0.5148400000	0.5137189210	0.2177528936
16.00000	0.6046700000	0.6021554459	0.4158556052
18.00000	0.6800000000	0.6780330321	0.2892599924
20.00000	0.7432500000	0.7415537465	0.2282211261
22.00000	0.7948500000	0.7937774267	0.1349403405
24.00000	0.8380800000	0.8361424168	0.2311931044
26.00000	0.8716100000	0.8701714322	0.1650471887
28.00000	0.8978000000	0.8973076993	0.0548341173
30.00000	0.9189100000	0.9188359340	0.0080602030
32.00000	0.9361800000	0.9358549292	0.0347231116
34.00000	0.9492000000	0.9492792387	0.0083479438
36.00000	0.9598100000	0.9598555732	0.0047481470
38.00000	0.9681300000	0.9681852085	0.0057025921
40.00000	0.9747900000	0.9747474634	0.0043636715
42.00000	0.9799300000	0.9799216729	0.0008497627
44.00000	0.9840900000	0.9840065111	0.0084838689
46.00000	0.9872000000	0.9872363304	0.0036801424
48.00000	0.9898000000	0.9897946205	0.0005434930
50.00000	0.9920600000	0.9918248915	0.0236990163
52.00000	0.9933800000	0.9934393580	0.0059753595
54.00000	0.9946100000	0.9947258027	0.0116430284
56.00000	0.9955700000	0.9957529640	0.0183778129
58.00000	0.9963800000	0.9965747439	0.0195451473
60.00000	0.9969400000	0.9972334847	0.0294385542
62.00000	0.9975200000	0.9977625142	0.0243117173
64.00000	0.9979900000	0.9981881222	0.0198521237
66.00000	0.9982500000	0.9985310938	0.0281586604
68.00000	0.9987000000	0.9988079008	0.0108041233
70.00000	0.9989800000	0.9990316271	0.0051679801
72.00000	0.9991700000	0.9992126900	0.0042725441
74.00000	0.9993300000	0.9993594022	0.0029421939
76.00000	0.9994600000	0.9994784118	0.0018421705
78.00000	0.9996000000	0.9995750462	0.0024963831
80.00000	0.9996700000	0.9996535833	0.0016422123

Figure 3.1 A network diagram of nine unimodal activity times.

Figure 3.1

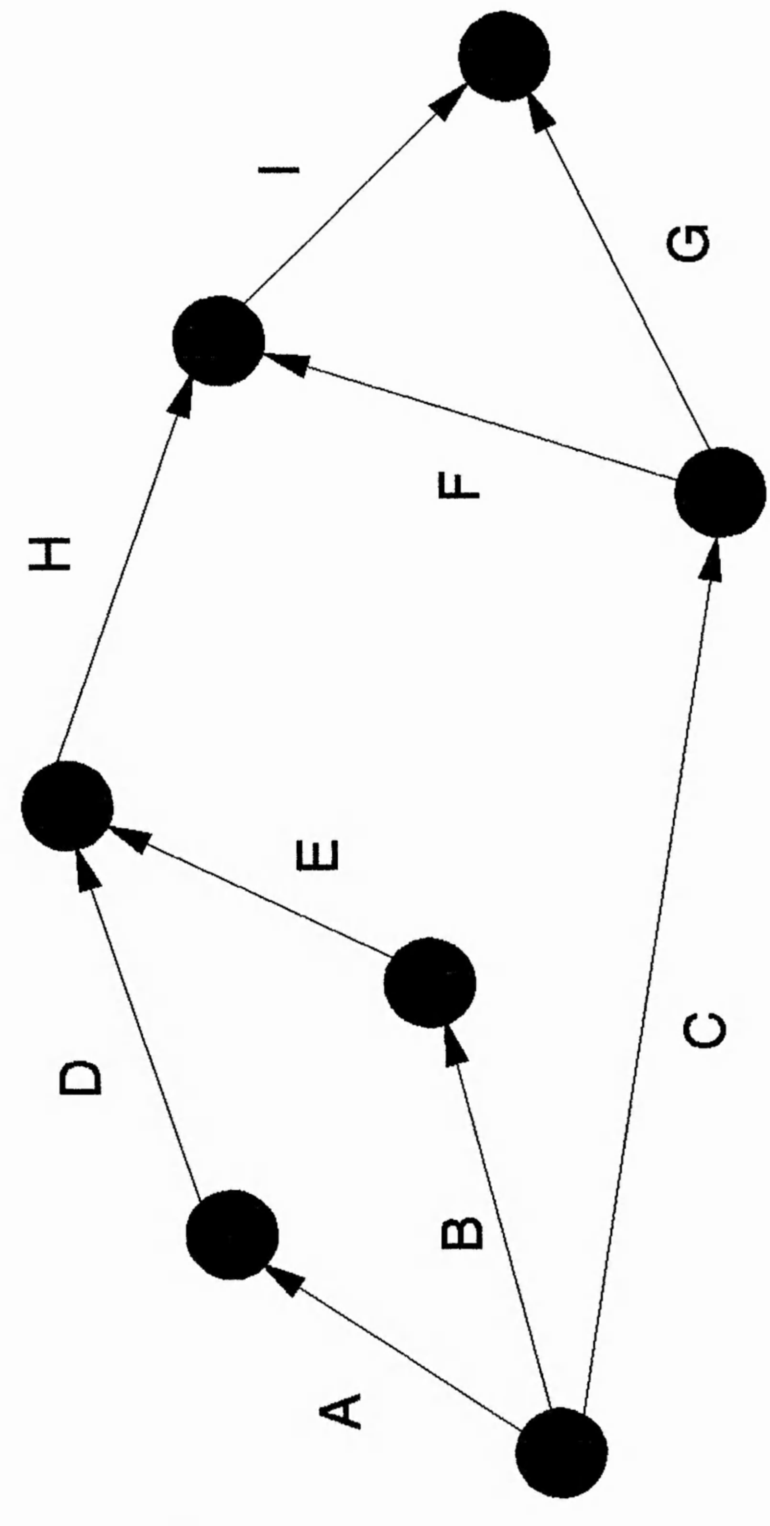


Figure 3.2 A network diagram represents ten empirical activity times of the second example.

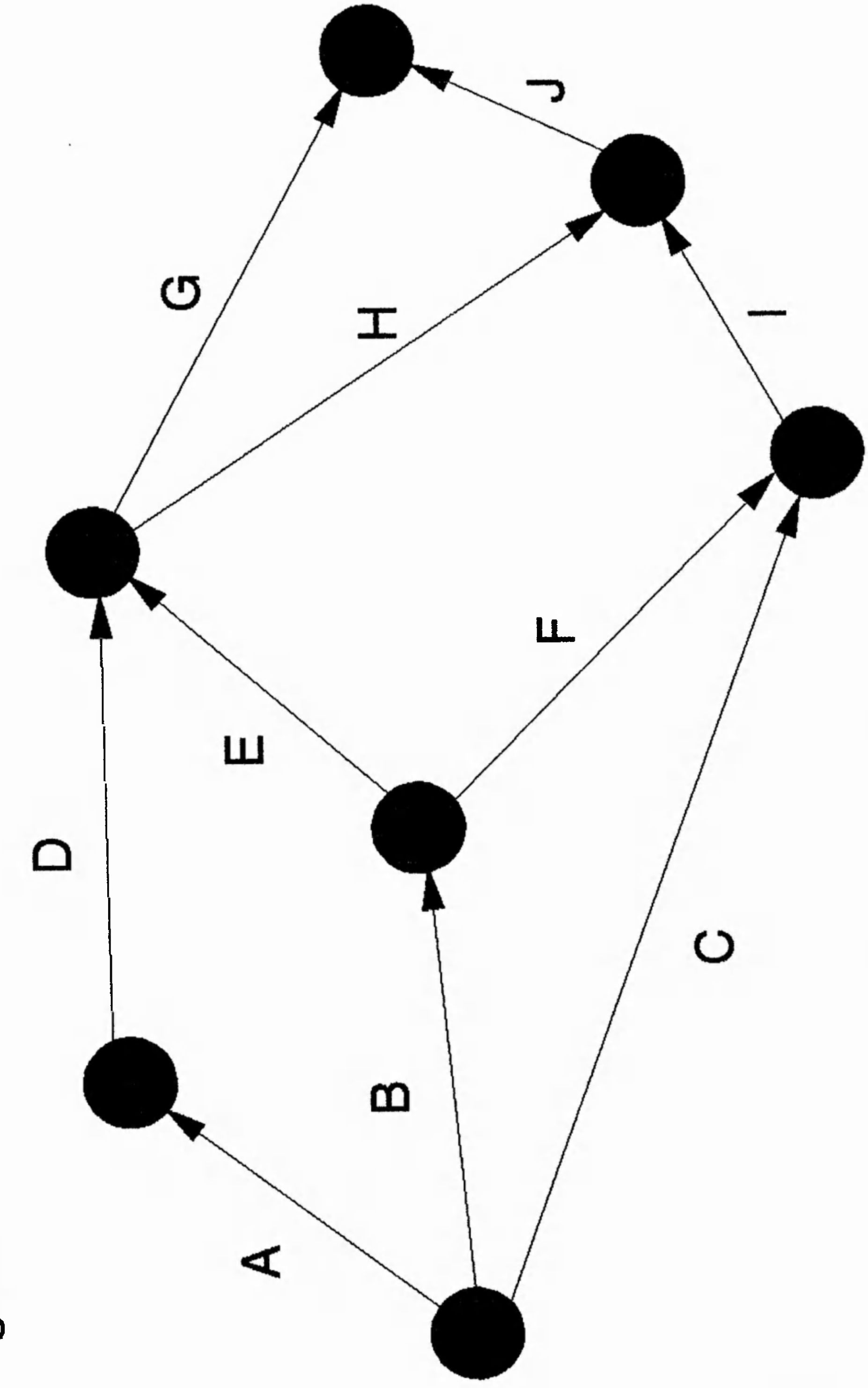


Figure 3.2

Figure 3.3 A network diagram of thirteen Normal activity times of the third example.

Figure 3.3

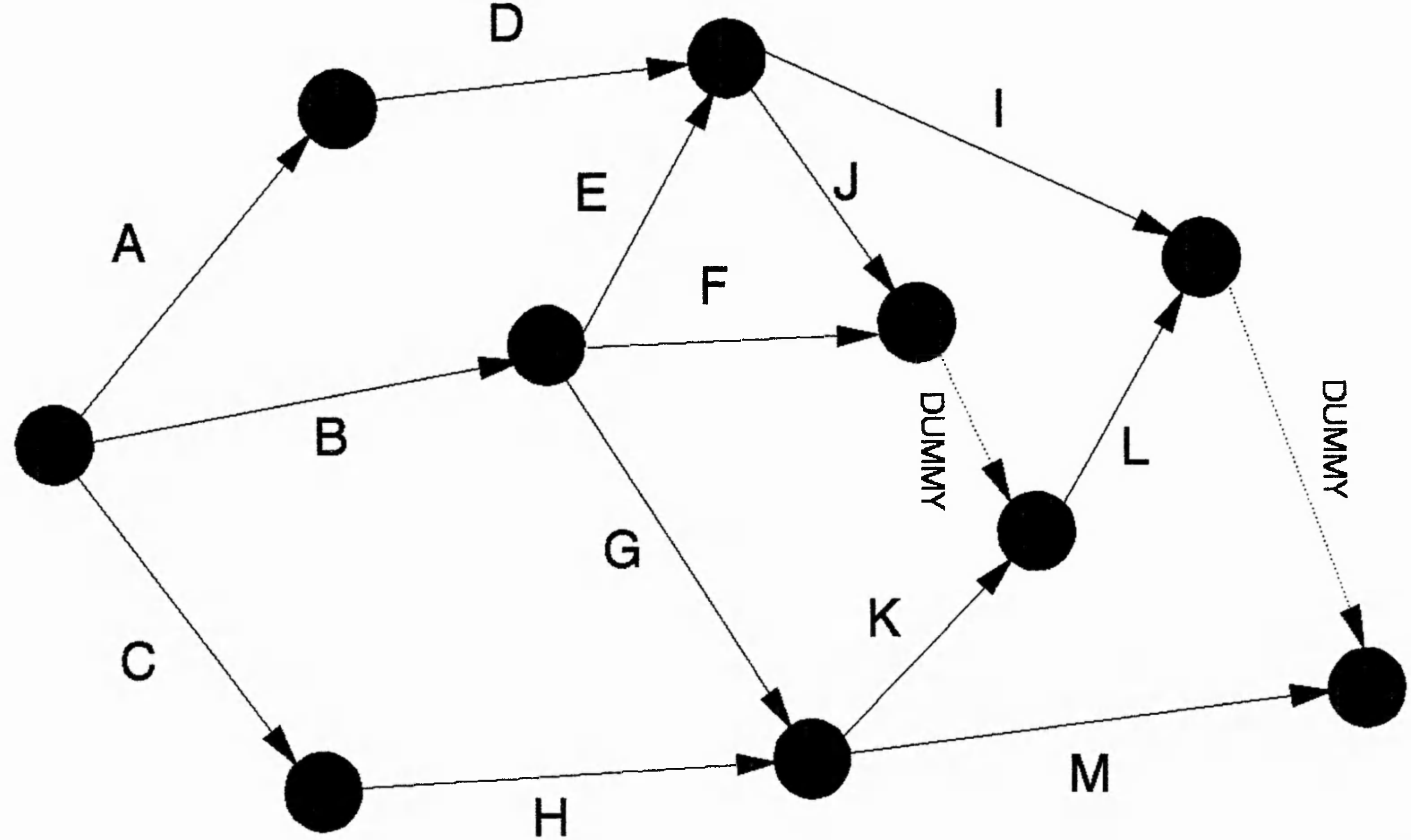


Figure 3.4 A graph showing the probability of completion time <T, for the first example on the moments method, simulation and PERT.

Key:

------- = 2 Moments

.... = 4 Moments

_______ = Simulation

_.._.._ = PERT

Figure 3.4

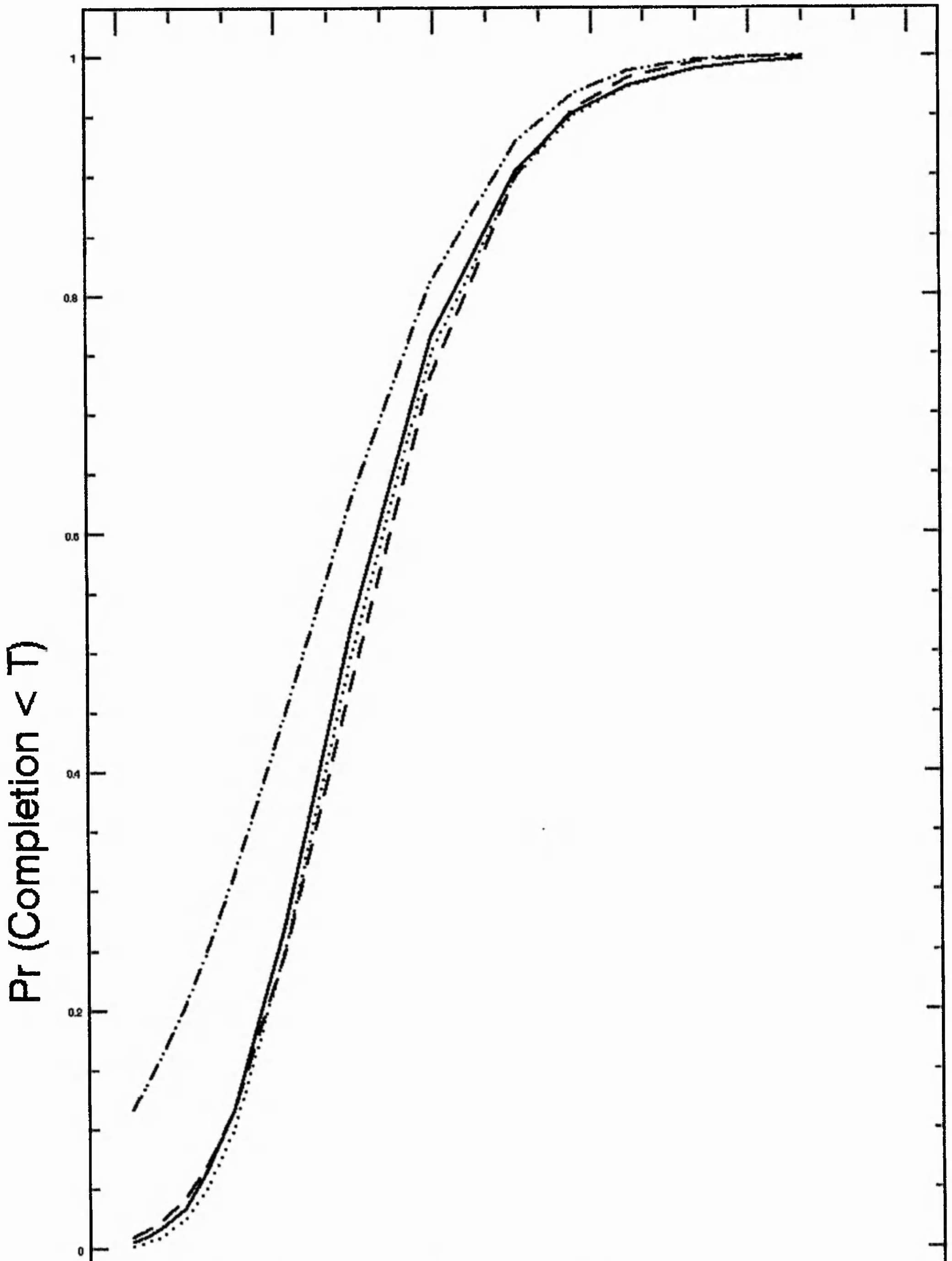


Figure 3.5 A graph showing the probability of completion time <T, for the second example on the moments method, simulation and PERT.

Key:

-------- = 2 Moments

.... = 4 Moments

________ = Simulation

_.._.._ = PERT

Figure 3.5

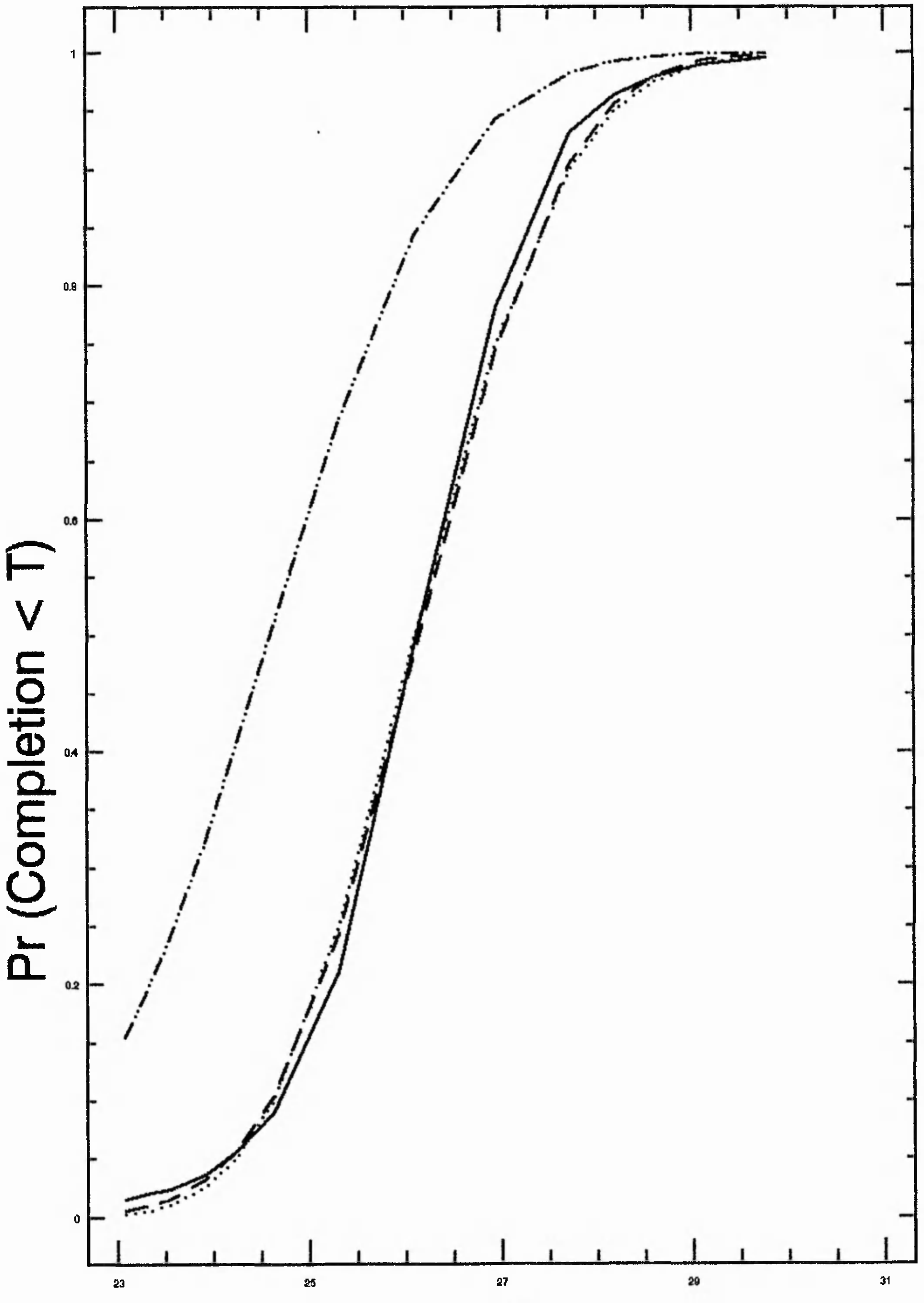


Figure 3.6 A graph showing the probability of completion time <T, for the third example on the moments method, simulation and PERT.

Key:

------- = 2 Moments

.... = 4 Moments

_______ = Simulation

_.._.._ = PERT

Figure 3.6

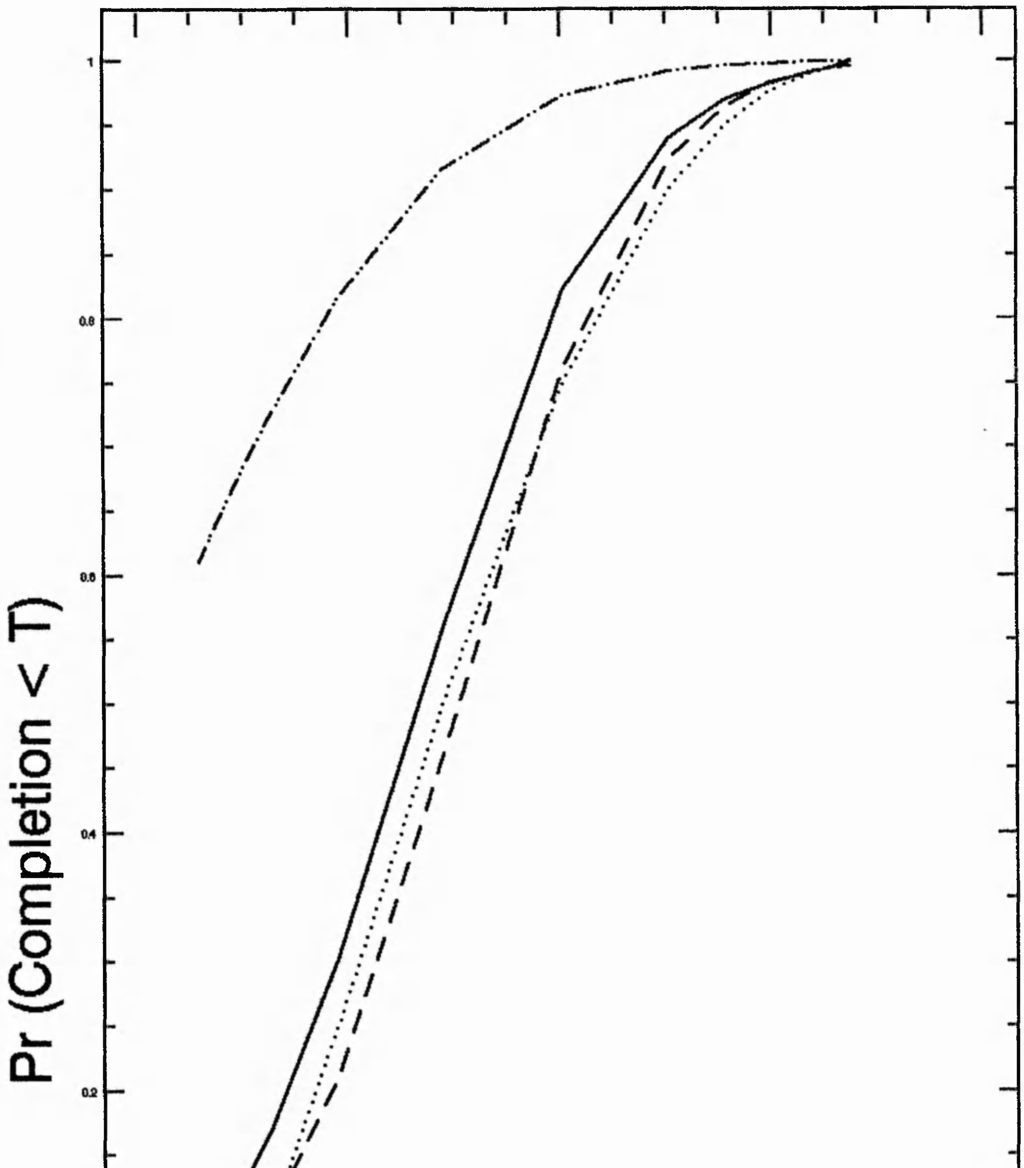
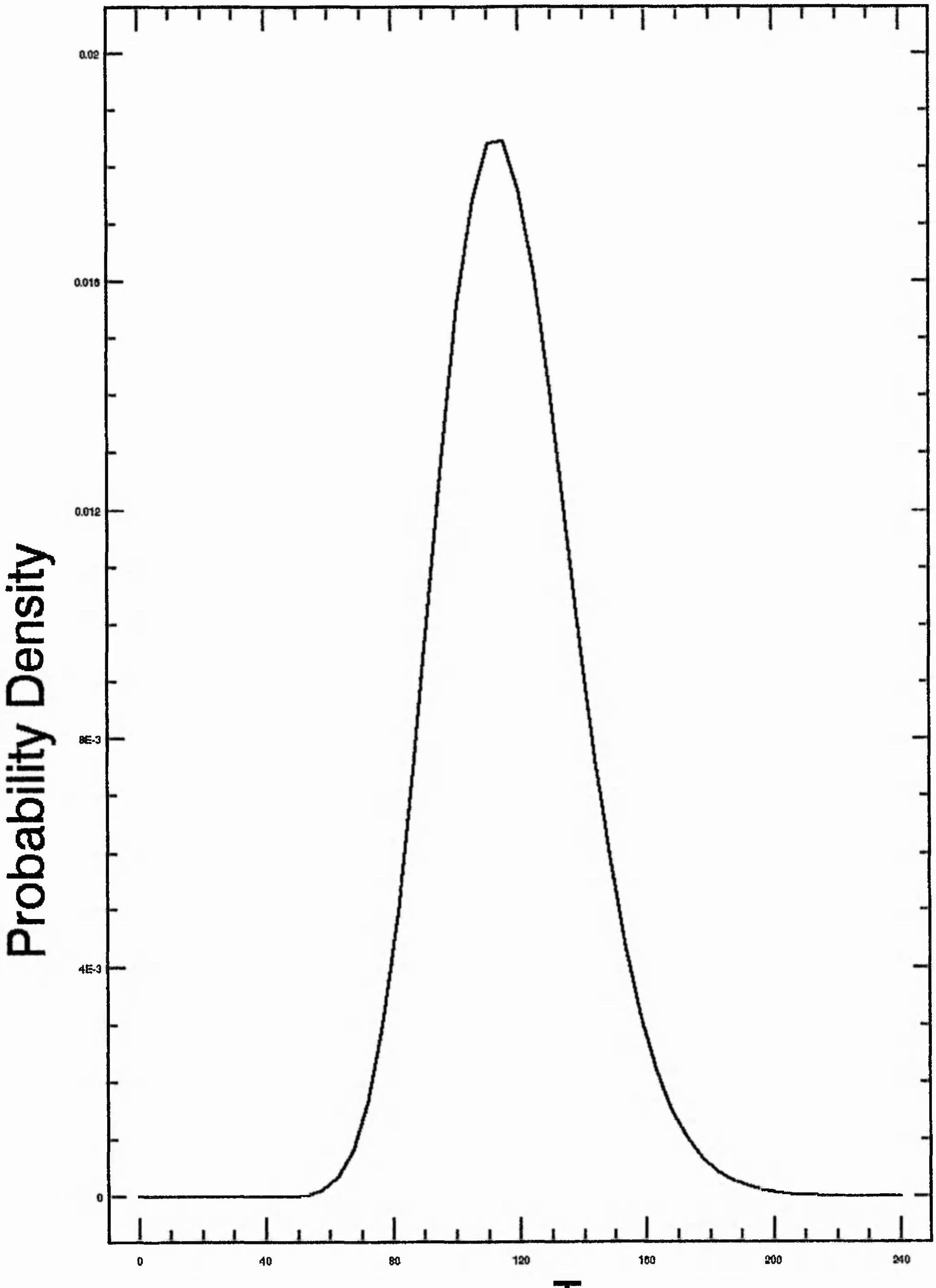


Figure 3.7 The graph of an Erlang pdf with shape parameter C=28.72404 and a scale parameter b=4.07519.

Figure 3.7



Probability Density
0.02
0.016
0.012
8E-3
4E-3
0
0
40
80
120
160
200
240
T

Figure 3.8 The graph of a normal pdf with mean=117.05590 and standard deviation=21.84090.

Figure 3.8

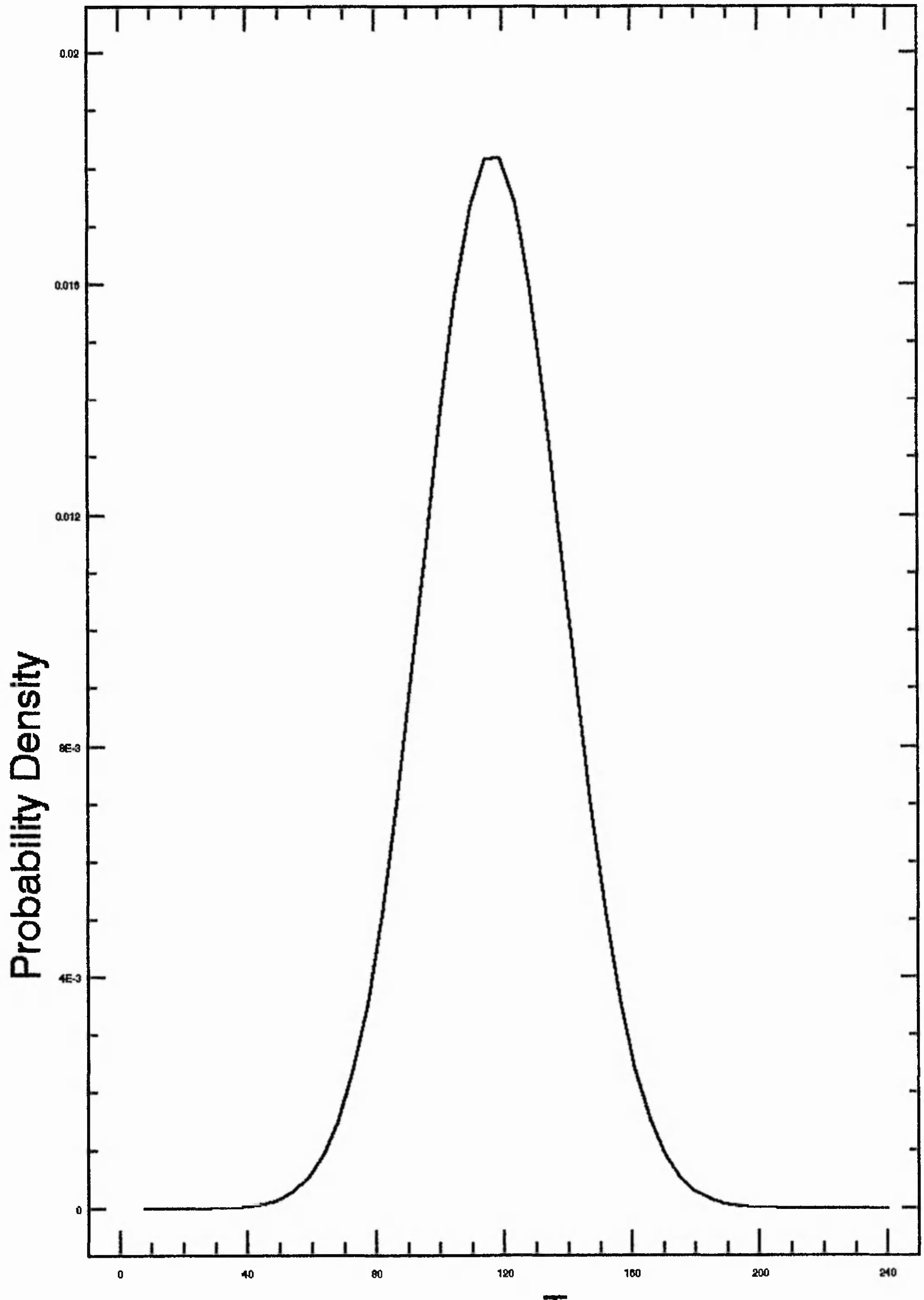
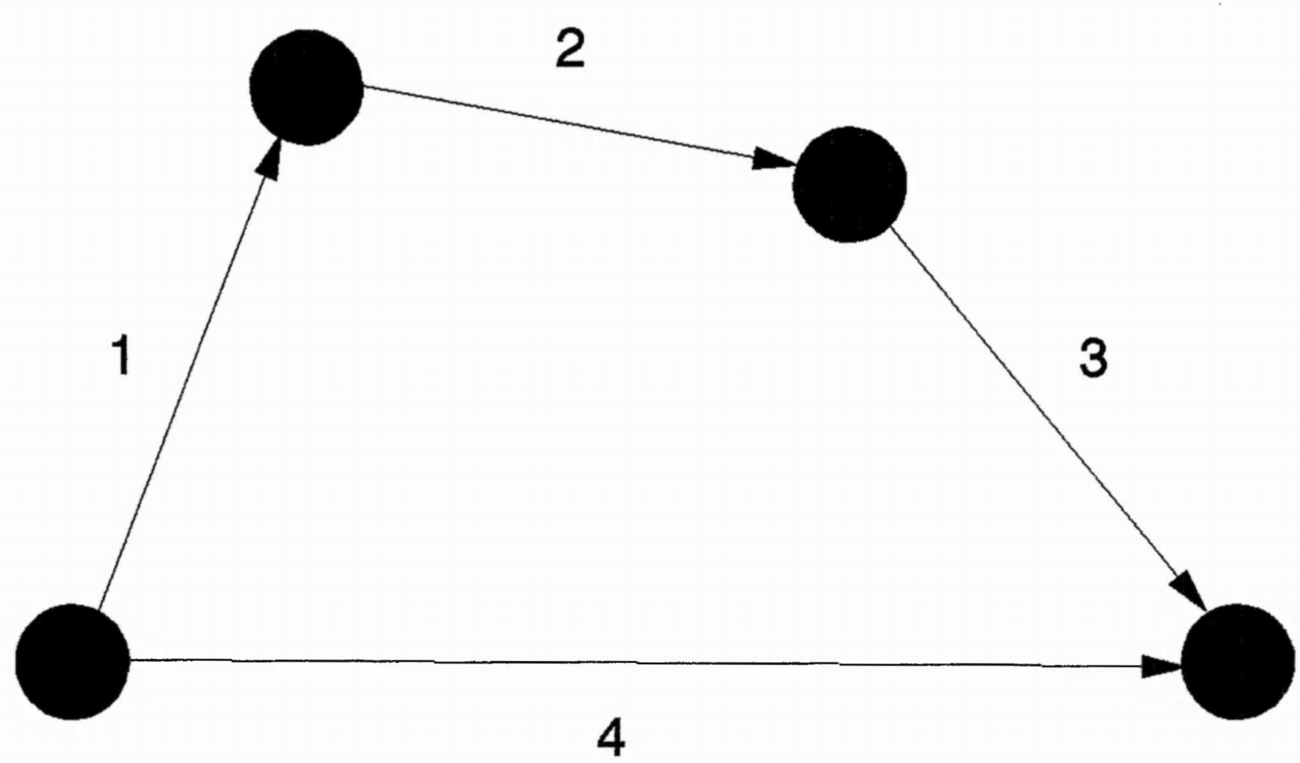

0.02
0.016
0.012
8E-3
4E-3
0
Probability Density
0
40
80
120
160
200
240
T

Figure 3.9 The network diagram of four exponential activity times with means, $u(t_1)=1.0$, $u(t_2)=5.0$, $u(t_3)=6.0$ and $u(t_4)=10.0$.

Figure 3.9



CHAPTER FOUR

MOMENTS METHOD (BIMODAL INPUT DISTRIBUTION)

4.1 BIMODAL INPUT DISTRIBUTION

While the most common activity time distribution will be unimodal and skew, there are occasions when the distribution suggested by the limited activity time information is bimodal. This situation will arise when the data comes from two separate sources, or the same source but with two possible scenarios e.g fine weather and foul weather. Estimates of the time to carry out a task may come from two distributions each suggesting a unimodal task time, but the distribution formed by pooling the estimates could give rise to the weighted average of two unimodal distributions, which may be bimodal. The combining of the two Erlang distributions, see Figure 4.1, would give rise to a pdf in the form of a weighted average of two Erlang pdf's :-

$$\pi \; f(t;\lambda_1,C_1) + (1-\pi) \; f(t;\lambda_2,C_2) \tag{4.1}$$

π being the weighting factor.

For $\pi = 0.5$, $\lambda_1 = 3.0$, $C_1 = 3$, $\lambda_2 = 4.5$, $C_2 = 10$, the probability distribution curve is shown in Figure 4.1. The figure shows the bimodal activity time (weighted sum) is made of two unimodal activity times, the first Erlang activity time with $\lambda_1 = 3.0$, $C_1 = 3.0$ and the second activity time with $\lambda_2 = 4.5$, $C_2 = 10.0$, combined they give rise to the bimodal activity time. Bimodal activity times in this situation may thus be regarded as being made up of two unimodal Erlang activity times.

When a bimodal distribution is suggested by the input data, the parameters $\pi, \lambda_1, C_1, \lambda_2, C_2$ are estimated using a maximum likelihood procedure, and the f in (4.1) is taken to be the more general Gamma pdf.

In this Chapter, it is shown that the pdf of the sum (or maximum) of two bimodal activity times can be expressed as the weighted average of the pdf's of the sum (or maximum) of unimodal activity times. This powerful result implies that bimodal and, in fact multimodal distributions can be dealt with by reducing them to cases in which only unimodal distributions are present.

4.2 THE PDF AND THE FIRST FOUR CENTRAL MOMENTS OF TWO BIMODAL ACTIVITIES IN SERIES

Suppose two independent bimodal activities in series have pdfs of their times of the form:

$$\pi_1 \ f(t;\lambda_1,C_1) \ + \ (1-\pi_1) \ f(t;\lambda_2,C_2) \tag{4.2}$$

and

$$\pi_2 \ f(t;\lambda_3,C_3) \ + \ (1-\pi_2) \ f(t;\lambda_4,C_4) \tag{4.3}$$

respectively, which will be denoted by $\pi_1 f_1+(1-\pi_1)f_2$ and $\pi_2 f_3+(1-\pi_2)f_4$. The pdf of the sum of the two activity times ϕ_S will be given by the convolution:

$$\phi_S=\int_0^S [\pi_1 f_1+(1-\pi_1)f_2]$$
$$[\pi_2 f_3(S-t)+(1-\pi_2)f_4(S-t)] \ dt \tag{4.4}$$

$$\phi_S = \int_0^S [\pi_1 \pi_2 f_1 f_3(S-t) +$$

$$\pi_1(1-\pi_2) f_1 f_4(S-t) +$$

$$\pi_2(1-\pi_1) f_2 f_3(S-t) +$$

$$(1-\pi_1)(1-\pi_2) f_2 f_4(S-t)] \; dt$$

i.e

$$\phi_s = \pi_1 \pi_2 f_{1+3} + \pi_1(1-\pi_2) f_{1+4} +$$

$$\pi_2(1-\pi_1) f_{2+3} + (1-\pi_1)(1-\pi_2) f_{2+4} \tag{4.5}$$

where f_{1+3} is the pdf of the sum of two Erlang variables whose separate pdfs are f_1 and f_3 respectively, etc.

The four terms of the integral are themselves convolutions showing that the pdf of two bimodal activity times in series is the weighted average of the pdf of four pairs of unimodal activity times, each pair being made up of two activity times in series. This is illustrated in Figure 4.2 and Figure 4.3. The pdf of two bimodal activities in series, see Figure 4.2, is equivalent to the weighted average of the pdfs of the four series pairs shown in Figure 4.3 .

THE FIRST FOUR CENTRAL MOMENTS (SERIES ACTIVITIES)

4.2.1 THE FIRST MOMENT

The mean of the sum of two bimodal independent activity times μ_s will be the sum of weighted means of the above four pairs, i.e

$$\pi_1\pi_2(\mu_1+\mu_3) + \pi_1(1-\pi_2)(\mu_1+\mu_4) + \pi_2(1-\pi_1)(\mu_2+\mu_3) + (1-\pi_1)(1-\pi_2)(\mu_2+\mu_4) \tag{4.6}$$

$$\mu_s = \pi_1\mu_1 + (1-\pi_1)\mu_2 + \pi_2\mu_3 + (1-\pi_2)\mu_4 \tag{4.7}$$

where μ_i = mean of distribution whose pdf = f_i and i=1,2,3,4. Notice that (4.7) may also be derived, alternatively as:

$$\mu_s = Mean\ of\ t_1 + Mean\ of\ t_2$$

$$\mu_s = \mu(t_1) + \mu(t_2) \tag{4.8}$$

$$\mu_s = \pi_1\mu_1 + (1-\pi_1)\mu_2 + \pi_2\mu_3 + (1-\pi_2)\mu_4 \tag{4.9}$$

4.2.2 THE SECOND CENTRAL MOMENT

The variance of the sum of two bimodal activity times is:

$$\sigma_s^2 \;=\; Variance\ of\ t_1 \;+\; Variance\ of\ t_2 \qquad (4.10)$$

$$\sigma_s^2 \;=\; \sigma^2(t_1) \;+\; \sigma^2(t_2)$$

$$\text{Variance of } t_1 = \mu_2'(t_1) - \mu^2(t_1)$$

$$\therefore\ \sigma^2(t_1) = \pi_1(\sigma_1^2 + \mu_1^2) \;+\; (1-\pi_1)(\sigma_2^2 + \mu_2^2) \;-$$

$$(\pi_1\mu_1 + (1-\pi_1)\mu_2)^2 \qquad (4.11)$$

$$\sigma^2(t_1) = \pi_1\sigma_1^2 \;+\; (1-\pi_1)\sigma_2^2 \;+\; \pi_1\mu_1^2 \;+\; (1-\pi_1)\mu_2^2 \;-$$

$$\pi_1^2\mu_1^2 \;-\; 2\pi_1(1-\pi_1)\mu_1\mu_2 \;-\; (1-\pi_1)^2\mu_2^2$$

$$\sigma^2(t_1) = \pi_1\sigma_1^2 \;+\; (1-\pi_1)\sigma_2^2 \;+\; \pi_1(1-\pi_1)(\mu_1-\mu_2)^2$$

$$\therefore \quad \sigma_s^2 = \pi_1\sigma_1^2 + (1-\pi_1)\sigma_2^2 +$$
$$\pi_2\sigma_3^2 + (1-\pi_2)\sigma_4^2 +$$
$$\pi_1(1-\pi_1)(\mu_1-\mu_2)^2 +$$
$$\pi_2(1-\pi_2)(\mu_3-\mu_4)^2 \qquad (4.12)$$

where σ_i^2 is the Variance of the distribution, whose pdf is f_i, $i=1,2,3,4$.

4.2.3 THE THIRD CENTRAL MOMENT

The skewness of the distribution of the sum of two bimodals is:

$$\mu_{S3} = \textit{Third central moment of } t_1 + \textit{Third central moment of } t_2 \tag{4.13}$$

$$\mu_3 = \mu_3(t_1) + \mu_3(t_2).$$

The third central moment of t_1 :

$$\mu_3(t_1) = \mu_3'(t_1) - 3\mu(t_1)\sigma^2(t_1) - \mu^3(t_1) \tag{4.14}$$

$$\mu_3(t_1) = \pi_1\mu_{31}' + (1-\pi_1)\mu_{32}' - 3\mu(t_1)\sigma^2(t_1) - \mu^3(t_1)$$

$$\therefore \mu_{S3} = \pi_1\mu_{31}' + (1-\pi_1)\mu_{32}' + \pi_2\mu_{33}' + (1-\pi_2)\mu_{34}' - 3\mu(t_1)\sigma^2(t_1) - \mu^3(t_1) - 3\mu(t_2)\sigma^2(t_2) - \mu^3(t_2) \tag{4.15}$$

where μ_{3i}' is the third moment about the origin of the distribution, whose pdf is f_i, $i = 1,2,3,4$.

4.2.4 THE FOURTH CENTRAL MOMENT

The kurtosis of the distribution of the sum of two bimodals is:

$$\mu_{s4} = \textit{Fourth central moment of } t_1 + \textit{Fourth central moment of } t_2 + 6\ (\textit{Variance of } t_1)\ (\textit{Variance of } t_2) \tag{4.16}$$

The fourth central moment of t_1 :

$$\mu_4(t_1) = \mu_4'(t_1) - 4\mu(t_1)\mu_3(t_1) - 6\mu^2(t_1)\sigma^2(t_1) - \mu^4(t_1) \tag{4.17}$$

$$\mu_4(t_1) = \pi_1\mu_{41}' + (1-\pi_1)\mu_{42}' - 4\mu(t_1)\mu_3(t_1) - 6\mu^2(t_1)\sigma^2(t_1) - \mu^4(t_1)$$

$$\begin{aligned}\therefore \mu_{s4} = {} & \pi_1\mu_{41}' + (1-\pi_1)\mu_{42}' - 4\mu(t_1)\mu_3(t_1) - \\ & 6\mu^2(t_1)\sigma^2(t_1) - \mu^4(t_1) + \\ & \pi_2\mu_{43}' + (1-\pi_2)\mu_{44}' - 4\mu(t_2)\mu_3(t_2) - \\ & 6\mu^2(t_2)\sigma^2(t_2) - \mu^4(t_2) + \\ & 6\sigma^2(t_1)\sigma^2(t_2)\end{aligned} \tag{4.18}$$

where μ_{4i}' is the fourth moment about the origin of the distribution whose pdf is f_i .

4.2.5 SUMMARY OF RESULTS

$$\mu_s = \mu(t_1) + \mu(t_2) \tag{4.19}$$

$$\mu_s = \pi_1\mu_1 + (1-\pi_1)\mu_2 + \pi_2\mu_3 + (1-\pi_2)\mu_4$$

$$\sigma_s^2 = \sigma^2(t_1) + \sigma^2(t_2) \tag{4.20}$$

$$\sigma_s^2 = \pi_1\mu'_{21} + (1-\pi_1)\mu'_{22} + \pi_2\mu'_{23} + (1-\pi_2)\mu'_{24} - \mu^2(t_1) - \mu^2(t_2)$$

or use (4.12) .

$$\mu_{s3} = \mu_3(t_1) + \mu_3(t_2) \tag{4.21}$$

$$\mu_{s3} = \pi_1\mu'_{31} + (1-\pi_1)\mu'_{32} + \pi_2\mu'_{33} + (1-\pi_2)\mu'_{34} - 3\mu(t_1)\sigma^2(t_1) - \mu^3(t_1) - 3\mu(t_2)\sigma^2(t_2) - \mu^3(t_2) \tag{4.22}$$

$$\mu_{s4} = \mu_4(t_1) + \mu_4(t_2) + 6\sigma^2(t_1)\sigma^2(t_2) \tag{4.23}$$

$$\mu_{s4} = \pi_1\mu'_{41} + (1-\pi_1)\mu'_{42} + \pi_2\mu'_{43} + (1-\pi_2)\mu'_{44} -$$
$$4\mu(t_1)\mu_3(t_1) - 6\mu^2(t_1)\sigma^2(t_1) - \mu^4(t_1) -$$
$$4\mu(t_2)\mu_3(t_2) - 6\mu^2(t_2)\sigma^2(t_2) - \mu^4(t_2) +$$
$$6\sigma^2(t_1)\sigma^2(t_2)$$

Note:

$$\mu(t_1) = \pi_1\mu_1 + (1-\pi_1)\mu_2 \qquad (4.24)$$

$$\sigma^2(t_1) = \pi_1\mu'_{21} + (1-\pi_1)\mu'_{22} - \mu^2(t_1) \qquad (4.25)$$

$$\mu_3(t_1) = \pi_1\mu'_{31} + (1-\pi_1)\mu'_{32} - 3\mu(t_1)\sigma^2(t_1) - \mu^3(t_1) \qquad (4.26)$$

similarly for $\mu(t_2)$, $\sigma^2(t_2)$ and $\mu_3(t_2)$.

4.3 THE PDF AND THE FIRST FOUR CENTRAL MOMENTS OF TWO BIMODAL ACTIVITIES IN PARALLEL

We now consider the independent bimodal activity times that are associated with parallel activities. We require the pdf of the maximum of the two times, denoted by ϕ_m. Using the same notation as before with F_i, $i = 1,2,3,4$, denoting cdfs we have:

$$\phi_m = [\pi_1 f_1 + (1-\pi_1) f_2][\pi_2 F_3 + (1-\pi_2) F_4] + [\pi_2 f_3 + (1-\pi_2) f_4][\pi_1 F_1 + (1-\pi_1) F_2] \tag{4.27}$$

$$\begin{aligned}\phi_m = {} & \pi_1 \pi_2 (f_1 F_3 + f_3 F_1) + \\ & \pi_1 (1-\pi_2)(f_1 F_4 + f_4 F_1) + \\ & \pi_2 (1-\pi_1)(f_2 F_3 + f_3 F_2) + \\ & (1-\pi_1)(1-\pi_2)(f_2 F_4 + f_4 F_2)\end{aligned}$$

$$\therefore \quad \phi_m = \pi_1 \pi_2 f_{m13} + \pi_1 (1-\pi_2) f_{m14} + \pi_2 (1-\pi_1) f_{m23} + (1-\pi_1)(1-\pi_2) f_{m24} \tag{4.28}$$

where f_{mij} is the pdf of the maximum of two Erlang random variables whose pdfs are f_i and f_j. So the pdf of two bimodal activities times in parallel is the weighted average of the pdf of four pairs of unimodal activity times, each being made up of two activity times in parallel, see Figures 4.4 and 4.5 . The pdf of the two bimodal activity times in parallel, Figure 4.4 is equivalent to the weighted average of the pdfs of four parallel pairs, see Figure 4.5.

THE FIRST FOUR CENTRAL MOMENTS (PARALLEL ACTIVITIES)

4.3.1 THE FIRST MOMENT

The mean of the maximum of two bimodal activity times, μ_M will be weighted sum of the means of the above four pairs, i.e

$$\mu_M = \pi_1\pi_2\mu_{(13)} + \pi_1(1-\pi_2)\mu_{(14)} + \pi_2(1-\pi_1)\mu_{(23)} + (1-\pi_1)(1-\pi_2)\mu_{(24)} \tag{4.29}$$

where $\mu_{(ij)}$ is the mean of the distribution of the maximum of two times whose unimodal pdfs are f_i and f_j, respectively.

4.3.2 THE SECOND CENTRAL MOMENT

The variance of the maximum of two bimodal activity times is:

$$\sigma_M^2 = \int_0^\infty t^2\phi_M dt - \mu_M^2 \tag{4.30}$$

$$\sigma_M^2 = \pi_1\pi_2\mu'_{2(13)} + \pi_1(1-\pi_2)\mu'_{2(14)} + \pi_2(1-\pi_1)\mu'_{2(23)} + (1-\pi_1)(1-\pi_2)\mu'_{2(24)} - \mu_M^2 \tag{4.31}$$

where for example $\mu'_{2(13)}$ is the second moment about the origin of the distribution of the maximum of two random variables whose unimodal pdfs are f_1 and f_3 respectively.

4.3.3 THE THIRD CENTRAL MOMENT

The skewness of the maximum of two bimodal activity times is:

$$\mu_{M3} = \int_0^\infty t^3 \phi_M dt - 3\mu_M \sigma_M^2 - \mu_M^3 \tag{4.32}$$

$$\mu_{M3} = \pi_1 \pi_2 \mu'_{3(13)} + \pi_1(1-\pi_2)\mu'_{3(14)} + \pi_2(1-\pi_1)\mu'_{3(23)} + (1-\pi_1)(1-\pi_2)\mu'_{3(24)} - 3\mu_M \sigma_M^2 - \mu_M^3 \tag{4.33}$$

4.3.4 THE FOURTH CENTRAL MOMENT

The last central moment, the kurtosis, of the maximum of two bimodal activity times could be derived as:

$$\mu_{M4} = \int_0^\infty t^4 \phi_M dt \quad - \quad 4\mu_M \mu_{M3} \quad - \quad 6\mu_M^2 \sigma_M^2 \quad - \quad \mu_M^4 \tag{4.34}$$

$$\begin{aligned}\mu_{M4} = {} & \pi_1 \pi_2 \mu'_{4(13)} \; + \; \pi_1 (1-\pi_2) \mu'_{4(14)} \; + \\ & \pi_2 (1-\pi_1) \mu'_{4(23)} \; + \; (1-\pi_1)(1-\pi_2) \mu'_{4(24)} \; - \\ & 4\mu_M \mu_{M3} \; - \; 6\mu_M^2 \sigma_M^2 \; - \; \mu_M^4 \end{aligned} \tag{4.35}$$

Figure 4.1 A diagram showing two Unimodal Erlang pdfs A and B and their weighted average pdf C with

$\pi = 0.5$,

$for\, A,\ \ \lambda_1 = 3.0,\ \ C_1 = 3.0,$

$for\, B,\ \ \lambda_2 = 4.5,\ \ C_2 = 10.0.$

Figure 4.1

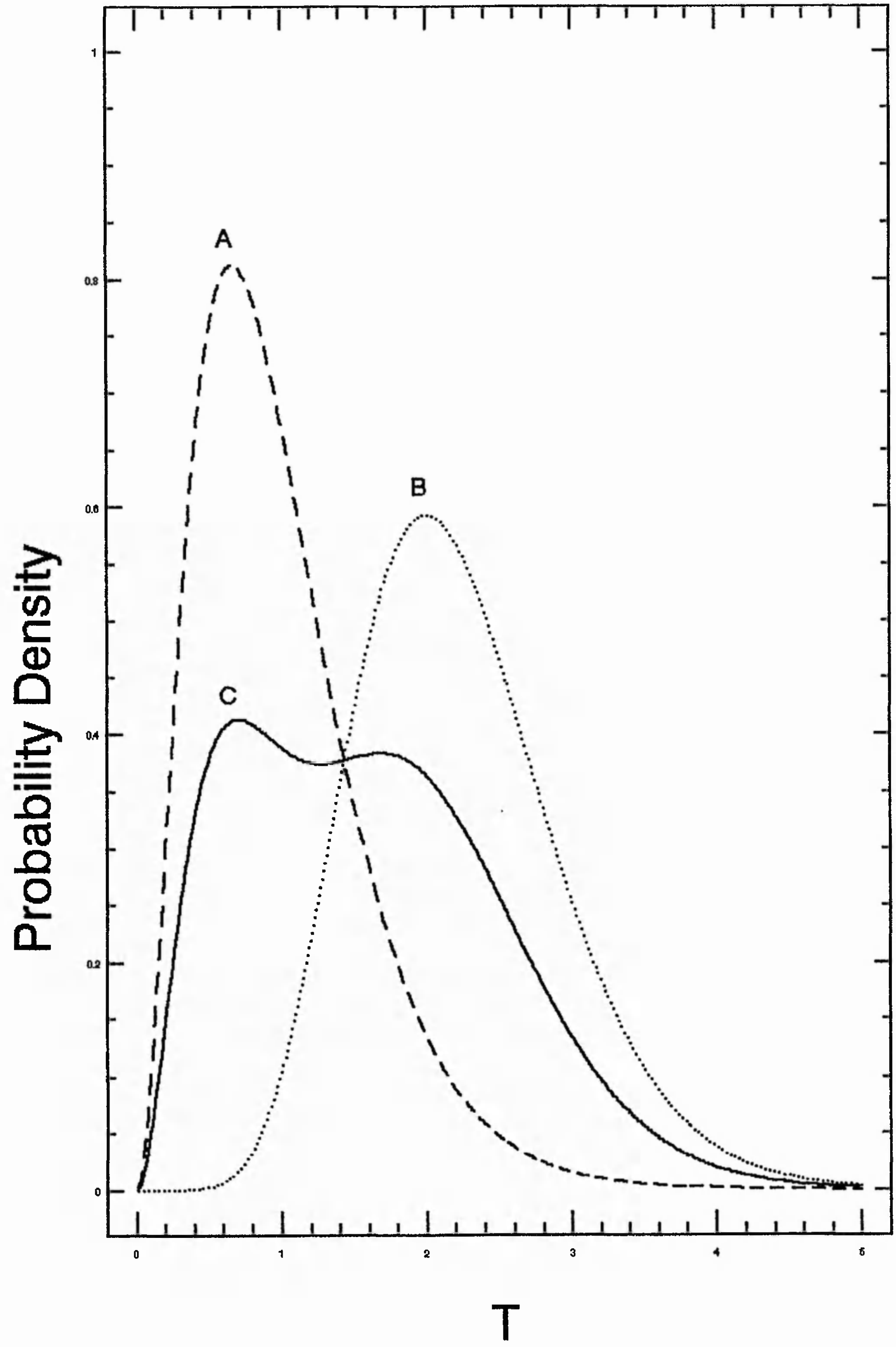


Figure 4.2 A diagram showing two bimodal activity times t_1 and t_2 in series.

Figure 4.2

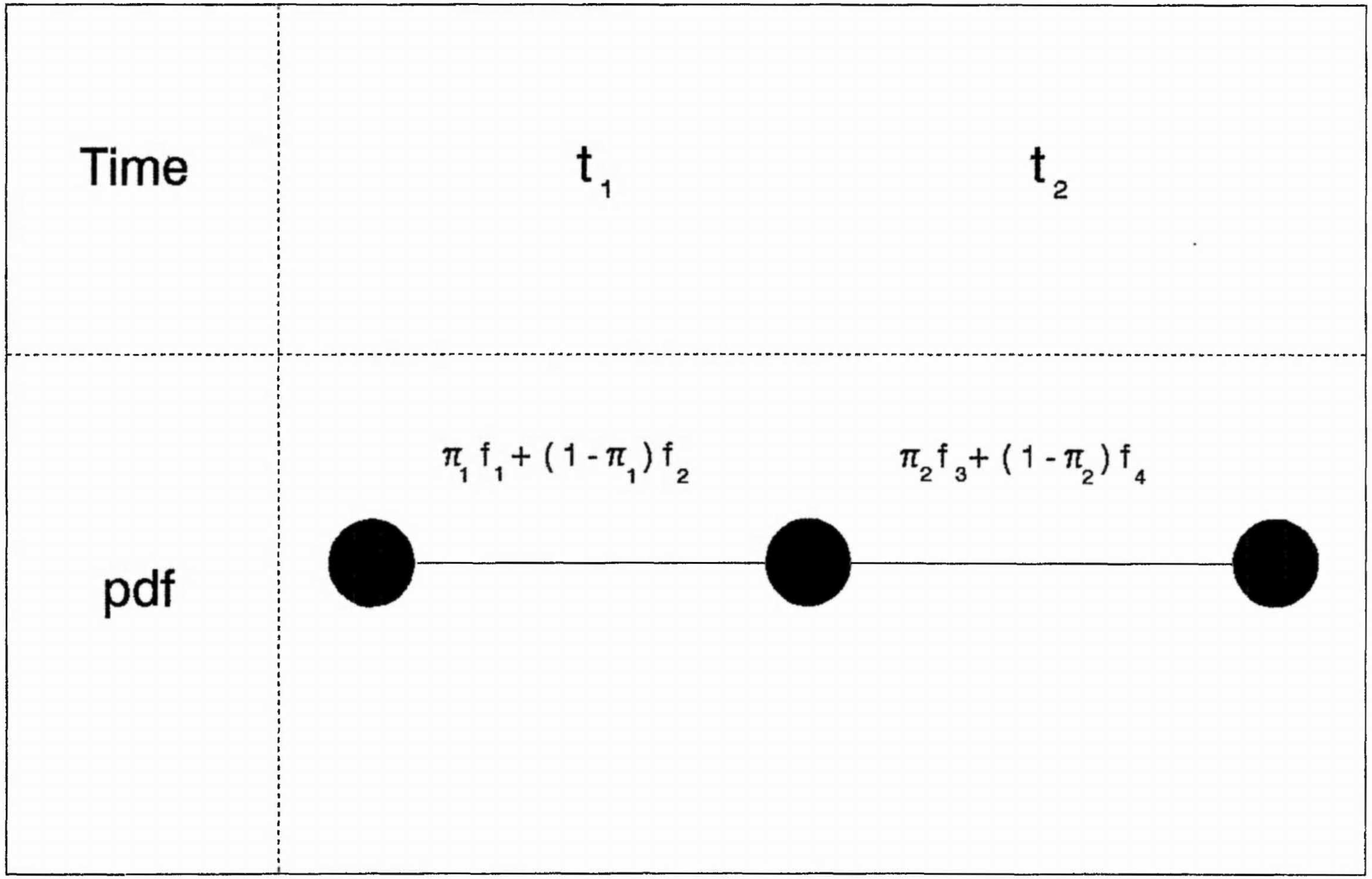


Figure 4.3 A diagram showing the weights and pfds of four unimodal activity times in series.

Figure 4.3

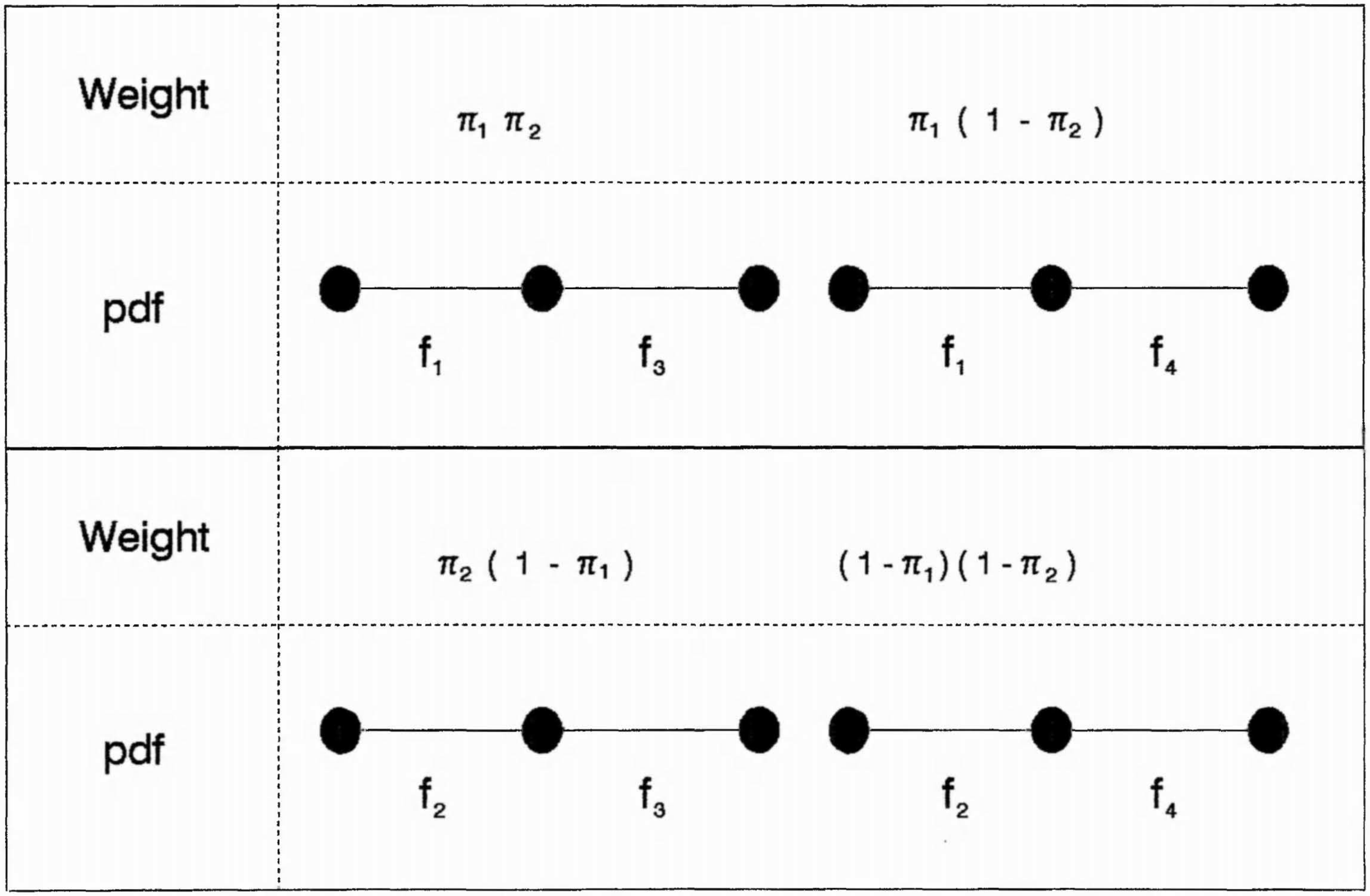

Weight
$\pi_1 \pi_2$
$\pi_1 (1 - \pi_2)$
pdf
f_1
f_3
f_1
f_4
Weight
$\pi_2 (1 - \pi_1)$
$(1 - \pi_1)(1 - \pi_2)$
pdf
f_2
f_3
f_2
f_4

Figure 4.4 A diagram showing two bimodal activity times in parallel.

Figure 4.4

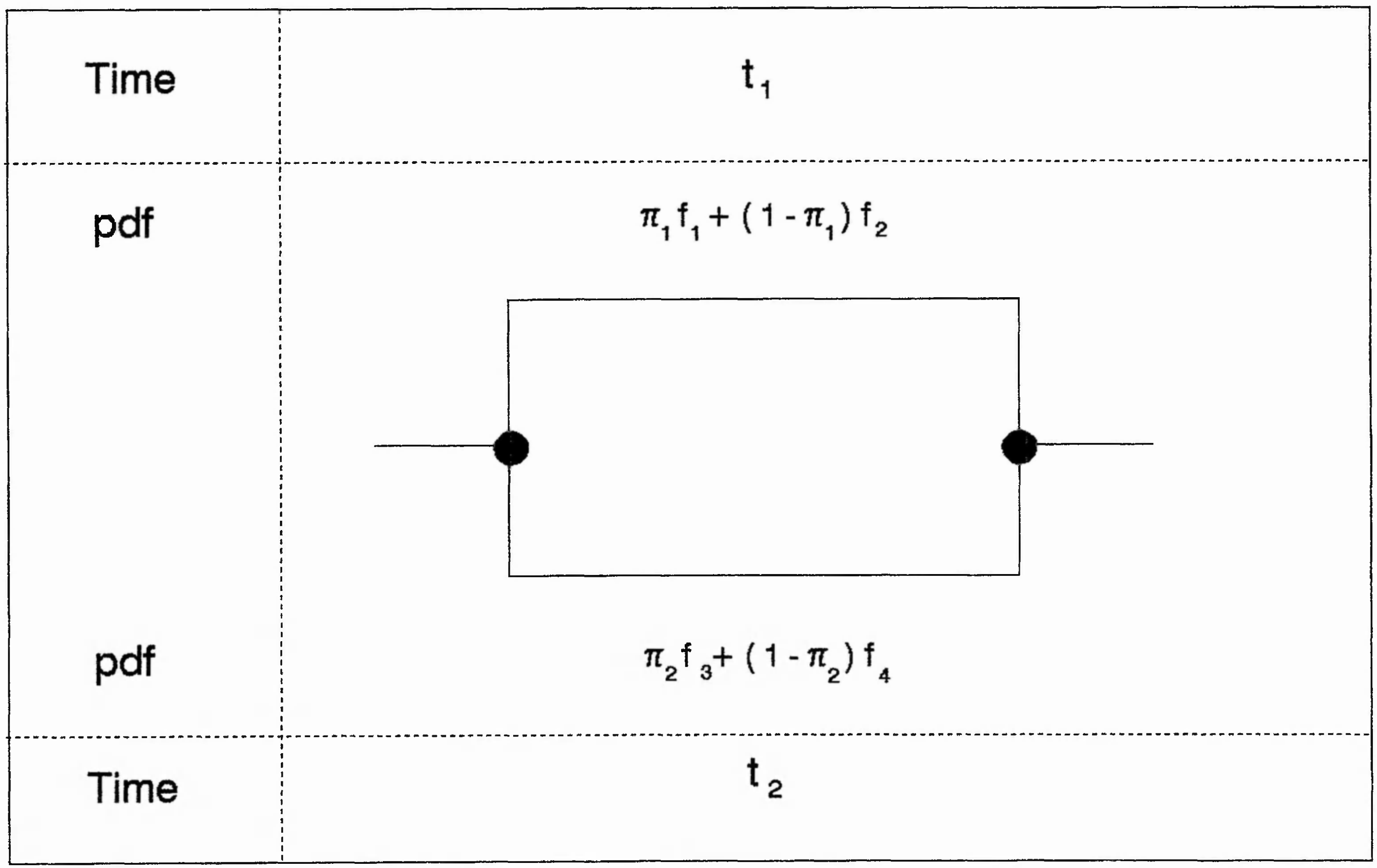
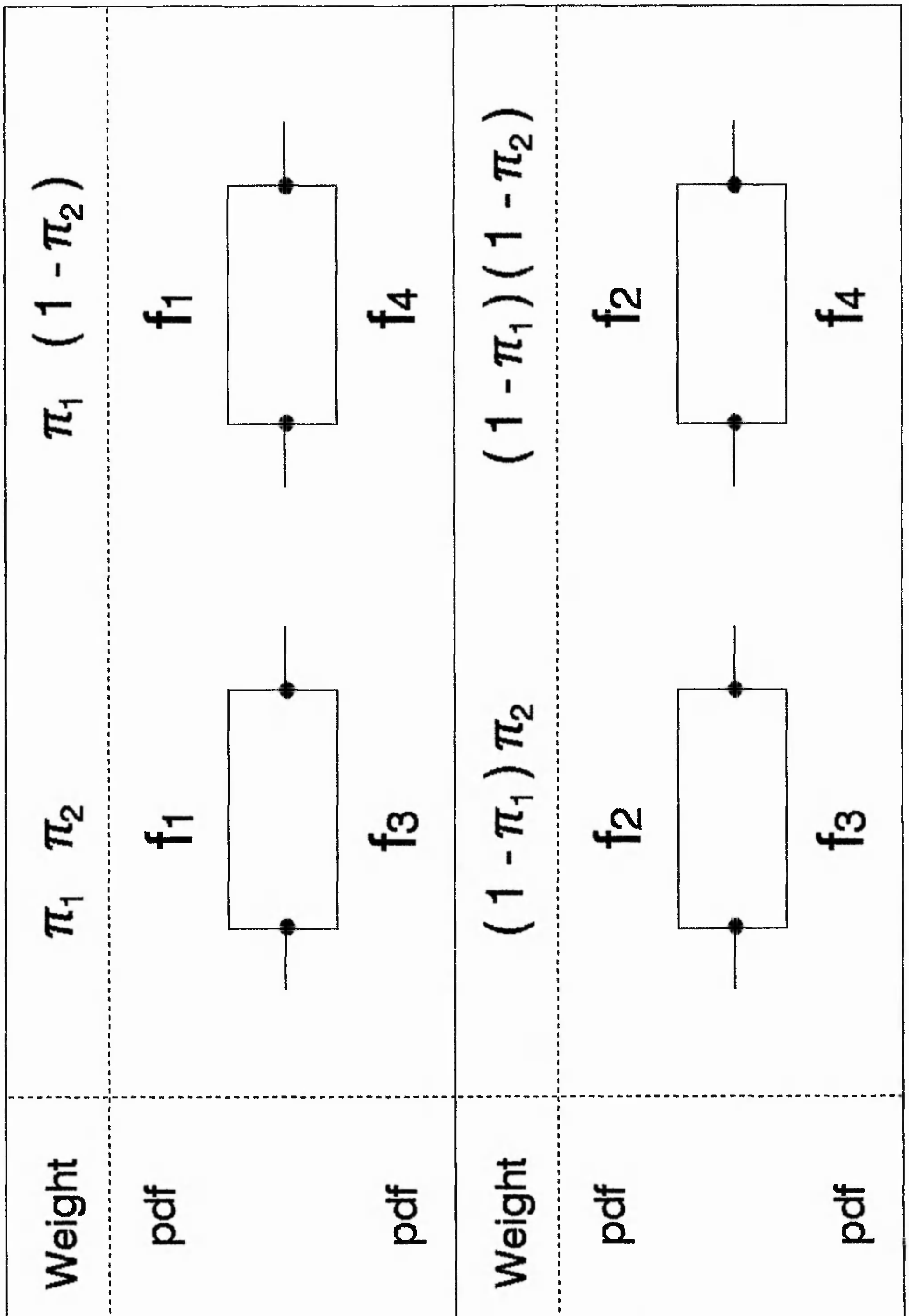


Figure 4.5 A diagram showing the weights and pdfs of four pairs of parallel Unimodal activity times.

Figure 4.5


Weight
pdf
pdf
$\pi_1 \quad \pi_2$
f_1
f_3
$\pi_1 \quad (1 - \pi_2)$
f_1
f_4
Weight
pdf
pdf
$(1 - \pi_1)\,\pi_2$
f_2
f_3
$(1 - \pi_1)(1 - \pi_2)$
f_2
f_4

CHAPTER FIVE

PROCEDURES FOR COLLAPSING NETWORKS

(A COMPARATIVE STUDY)

5.1 PROCEDURES FOR COLLAPSING NETWORKS

The determination of the moments of project completion time can be carried out by collapsing the network using one of two methods. In both methods, the activity times are assumed to be independent. The two methods are referred to as method (A) and method (B).

Method A: Networks are made up of a number of activities which are connected logically in series or in parallel. This method identifies all possible complete paths first, from start to finish, and secondly adds all activities on each of those paths, to reduce each path to a single activity. The network is then thought of as having (n) single activities in parallel. The last step in this method is the collapsing of those activities by computing their maximum activity time. The advantage of this method is that it is easy to apply and simple to program into a computer.

Method B: This method starts by adding all activities which are in series from the start of the network till a node is reached having more than one activity leading into it. All other series of activities leading into this node from start node are also summed. The maximum activity time at this node is then found. This process is continued progressively through the network until the end of the network is reached.

The two methods are compared using seven examples. Four of these examples have unimodal (Erlang) distributions, and the other three examples have bimodal (Weighted average of Erlang) input distributions. We examine whether one method of collapsing is considerably better than the other.

5.2 UNIMODAL INPUT DISTRIBUTIONS

The following four examples all have unimodal Erlang input distributions and the paths through each of the networks are all near critical. The total project time distribution is computed by using simulation, method (A), method (B) and PERT.

5.2.1 FIRST EXAMPLE

The network of this example is made up of three paths, see Figure 5.1 and Table 5.2. The total project time distribution is computed and results are displayed in Figure 5.2 and Table 5.1 . The graph shows that the moments method estimates the total project time very much better than PERT. Collapsing the network using method (B) gives better results than method (A). Percentage errors in the means and standard deviations are smaller using method (B) than method (A), see Table 5.1 .

5.2.2 SECOND EXAMPLE

This example has four paths in the network diagram and all are made critical initially, see Figure 5.3 and Table 5.3 . The results of the total project time distributions are displayed in Figure 5.4 . From the graph it is clear that the cdf curve of total project time obtained by using the moments method is closer to the simulation curve than the PERT method curve. Using method (B) to collapse the network gives a curve closer to simulation than using method (A), although the higher percentiles for method (A) are closer than those of method (B). Table 5.1 shows that the smallest percentage error in the mean is given by using method (B) and the smallest percentage error in the standard deviation is obtained by using the PERT method, even though the error in standard deviation by method (B) is less than five percent.

5.2.3 THIRD EXAMPLE

In this example the network is made up of five critical paths, see Figure 5.5 and Table 5.4 . The results of the distribution of total project times are displayed in Figure 5.6 . The graph shows that method (B) gives a curve very much closer to the simulation than using method (A) or the PERT approach. Comparing the percentage error in the mean, (see Table 5.1) indicates that the smallest percentage error is obtained by using method (B). The PERT method gives the smallest percentage error in the standard deviation.

5.2.4 FOURTH EXAMPLE

In this last example of the unimodal input distribution, the network is made up of nine independent paths, see Figure 5.7 and Table 5.5 . The results of total project time are displayed in Figure 5.8, which shows that method (B) is slightly better than method (A) and both are much better than PERT. Table 5.1 shows the percentage errors in the mean and standard deviation by using method (B) are the smallest.

5.3 BIMODAL INPUT DISTRIBUTIONS

In this section, three examples are tested where the input activity distributions in the network are bimodal Erlang(weighted average of Erlang). Initially all paths are made critical. The three methods used to collapse the network, are method (A), method (B) and PERT and the results are compared with simulation.

5.3.1 THE FIRST EXAMPLE

In this first example the network has six paths and all are made critical when mean times are actual times. All the activities of this network are bimodal Erlang activities, see Figure 5.9 and Table 5.6 . Results of collapsing this network are displayed in Figure 5.10 which shows that method (B) is the best in this case. Percentage errors in the mean and standard deviation are smallest when method (B) is used (see Table 5.1).

5.3.2 THE SECOND EXAMPLE

This network has six paths and all are critical when mean times are actual times, see Figure 5.11 and Table 5.7 . Method (A) shows a very slightly closer overall fit to simulation than method (B), see Figure 5.12 . The smallest percentage error in the means is given by using method (B) and the smallest percentage error in the standard deviations is by using method (A), see Table 5.1 .

5.3.3 THE THIRD EXAMPLE

In this last bimodal input activity times example the network has five paths which are all critical when mean times are actual times. Collapsing the network either by method (A) or method (B) does not make any difference, see Figure 5.13 and Table 5.8 . The result of collapsing this network is displayed in Figure 5.14 and shows that the moments method is closer to simulation than PERT. Percentage errors in the means and standard deviations are smaller by using the moments method than PERT, see Table 5.1.

5.4 CONCLUSIONS

Method (B), the progressive network reduction method appears to produce more accurate information on the project completion time distribution than that obtained by using method (A), the complete path method. Again, as for the examples of Chapter 3, it is clear from the examples that the shape parameters of completion times (C) are quite large, see Table 5.9 , which indicates that the shape of the distributions of total times is approximately normal probably as a result of the addition of a moderate number of activities in series.

In most of the cases, the errors in the mean and standard deviation are smaller when method (B) is used rather than method (A), this is because when using method (B) we do not violate the independence assumption as much if we would have with method (A). We shall therefore define the new unified methodology as one using method (B) to collapse the network and the 4-moments Pearson approach with Erlang input distributions to determine the distribution of project completion time. These conclusions determine the type of procedure to be adopted and used in the next Chapter, where realistic example from our industrial collaborator is given.

Table 5.1 The percentage errors in the mean and standard deviation of completion time using method (A), method (B) and PERT for the seven examples.

Table 5.1

Method	Moments	Unimodal example			
		One	Two	Three	Four
A	Mean	5.12	2.87	4.88	2.04
	Standard deviation	10.71	10.14	18.15	18.17
B	Mean	3.30	2.75	0.30	1.27
	Standard deviation	6.44	2.25	5.30	12.46
PERT	Mean	14.63	23.54	16.18	10.97
	Standard deviation	3.85	0.06	1.69	30.06

Method	Moments	Bimodal example		
		One	Two	Three
A	Mean	6.19	0.95	1.13
	Standard deviation	5.16	10.13	0.66
B	Mean	0.03	0.53	1.13
	Standard deviation	1.52	12.97	0.66
PERT	Mean	20.55	35.54	12.83
	Standard deviation	12.71	30.73	6.14

Table 5.2 The mean and variance of the nine activity times of Figure 5.1.

Table 5.2

Activity number	Mean	Variance
1	4.5	9.00000
2	1.5	1.41421
3	3.0	9.00000
4	10.0	21.40000
5	5.0	11.00000
6	3.5	3.40000
7	12.5	22.40000
8	4.0	8.00000
9	5.0	4.60000

Table 5.3 The mean and variance of the twelve activity times of Figure 5.3.

Table 5.3

Activity number	Mean	Variance
1	9.00	28.00
2	9.00	27.00
3	2.40	5.00
4	8.50	36.00
5	2.00	2.20
6	4.00	8.00
7	3.50	12.00
8	3.00	4.50
9	2.00	4.00
10	4.50	10.00
11	8.00	22.00
12	16.00	64.00

Table 5.4 The mean and variance of the twelve activity times of Figure 5.5.

Table 5.4

Activity number	Mean	Variance
1	10.00	25.00
2	3.50	12.00
3	8.00	16.00
4	10.00	20.00
5	7.00	7.00
6	5.00	23.00
7	5.50	10.00
8	6.00	9.00
9	8.50	12.00
10	6.00	4.00
11	7.50	22.50
12	14.50	21.00

Table 5.5 The mean and variance of the thirteen activity times of Figure 5.7.

Table 5.5

Activity number	Mean	Variance
1	9	1.8
2	6	1.2
3	8	1.6
4	6	1.2
5	9	1.8
6	12	2.4
7	5	1.0
8	3	0.6
9	3	0.6
10	7	1.4
11	7	1.4
12	4	0.8
13	11	2.2

Table 5.6 The mean, variance and the weight of the sixteen bimodal activity times of Figure 5.9.

Table 5.6

Activity time	π	f_1		f_2	
		Mean	Variance	Mean	Variance
1	0.45	19.05	72.58	9.54	30.34
2	0.30	36.00	216.00	25.00	125.00
3	0.25	2.98	4.44	2.01	4.08
4	0.80	4.88	11.91	18.00	81.00
5	0.50	25.00	125.00	42.00	252.00
6	0.75	20.00	80.00	40.00	80.00
7	0.45	6.00	12.00	7.02	16.43
8	0.65	4.70	11.05	6.42	14.69
9	0.55	6.14	18.85	4.90	12.01
10	0.35	12.00	24.00	30.00	150.00
11	0.60	13.50	60.75	8.92	19.89
12	0.50	4.92	6.05	6.00	12.00
13	0.60	6.15	12.61	5.0	8.37
14	0.20	48.00	192.00	20.00	100.00
15	0.55	24.00	96.00	35.00	175.00
16	0.50	32.00	256.00	60.00	200.00

Table 5.7 The mean, variance and the weight of the seventeen bimodal activity times of Figure 5.11.

Table 5.7

Activity time	π	f_1		f_2	
		Mean	Variance	Mean	Variance
1	0.50	10.00	25.00	6.00	18.00
2	0.20	20.00	25.00	3.00	4.50
3	0.70	4.00	8.00	4.50	10.00
4	0.60	2.00	2.00	3.50	6.00
5	0.30	9.00	13.50	10.50	22.00
6	0.60	5.50	10.00	8.00	10.50
7	0.50	5.00	12.50	8.50	14.50
8	0.35	10.00	16.50	3.00	8.50
9	0.55	15.00	25.00	9.50	22.50
10	0.85	11.50	22.00	14.00	24.00
11	0.50	7.50	18.50	6.00	18.50
12	0.70	5.50	7.50	9.00	20.00
13	0.40	7.00	24.00	2.50	6.00
14	0.60	2.00	4.00	2.50	6.00
15	0.60	2.00	1.00	2.00	4.00
16	0.65	4.00	8.00	4.00	16.00
17	0.25	3.50	6.00	1.50	2.00

Table 5.8 The mean, variance and the weight of the twelve bimodal activity times of Figure 5.13.

Table 5.8

Activity time	π	f_1		f_2	
		Mean	Variance	Mean	Variance
1	–	6.00	18.00	–	–
2	0.75	21.00	110.00	7.00	12.00
3	0.80	32.00	300.00	72.00	225.00
4	0.30	16.00	65.00	40.00	400.00
5	0.60	32.00	250.00	21.00	90.00
6	0.40	22.40	150.00	33.60	230.00
7	0.45	50.00	100.00	60.00	120.00
8	0.50	34.00	130.00	38.00	350.00
9	0.60	36.00	180.00	24.00	120.00
10	0.20	50.00	250.00	26.00	298.00
11	0.50	45.00	200.00	50.00	250.00
12	0.70	60.00	250.00	90.00	620.00

Table 5.9 The shape parameter of project completion time (C) for the Unimodal and the Bimodal examples, using method (A), method (B) and simulation.

Table 5.9

Method	Unimodal example			
	One	Two	Three	Four
A	28.48236	19.46277	47.39541	374.65070
B	25.05377	14.99550	32.00290	313.99188
Simulation	19.17050	14.85087	28.86760	234.61681
Method	Bimodal example			
	One	Two	Three	
A	35.90405	15.15160	28.84662	
B	27.80883	14.52184	28.84662	
Simulation	28.63894	18.73225	27.83688	

Figure 5.1 A network diagram of the nine Unimodal Erlang activity times for the first example.

Figure 5.1

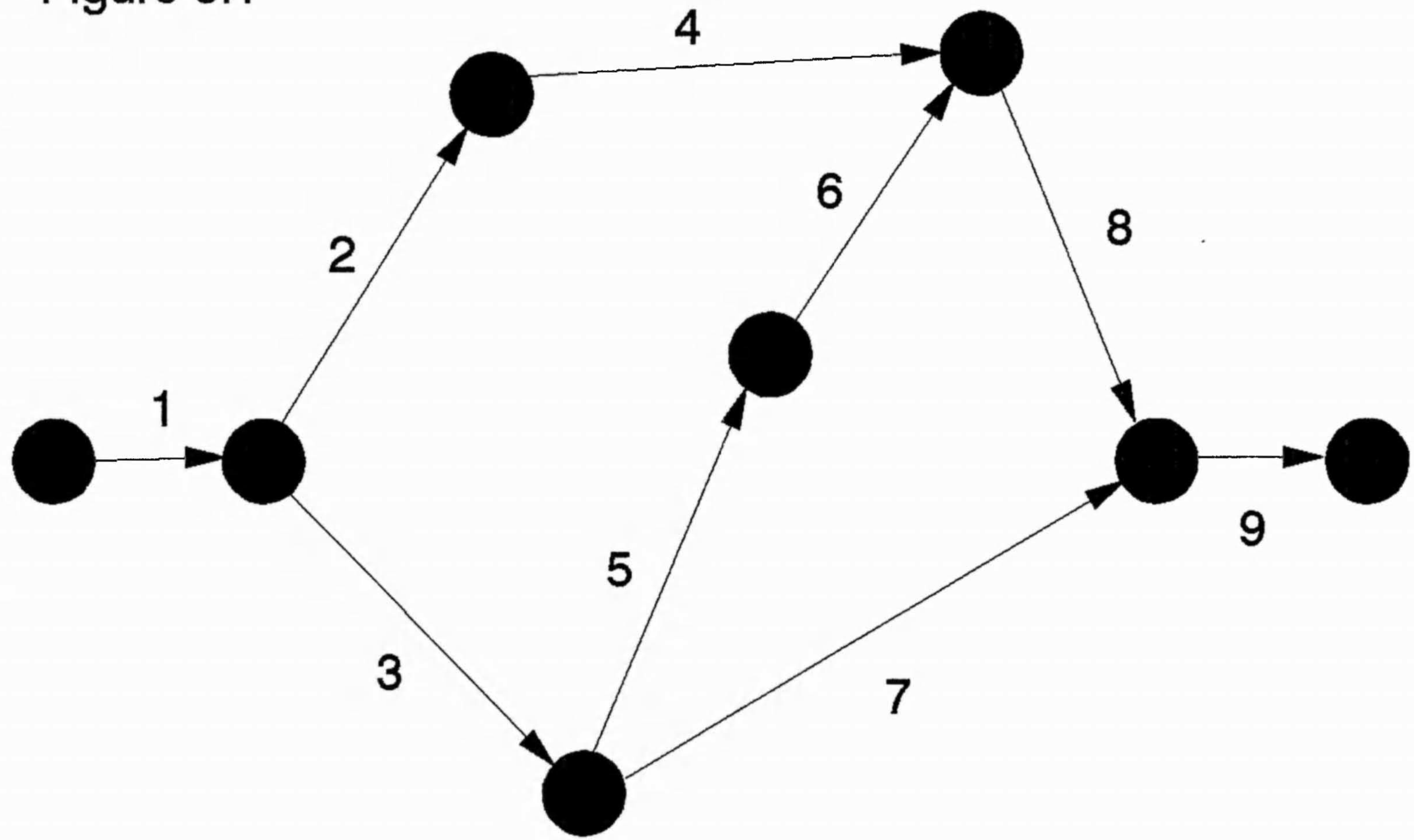


Figure 5.2 Cdf curves of completion time for unimodal input, first example using method (A), method (B), PERT and simulation.

Key:

-------- = Method A (The complete path method)

.... = Method B (The progressive network reduction method)

________ = Simulation

_.._.._ = PERT

Figre 5.2

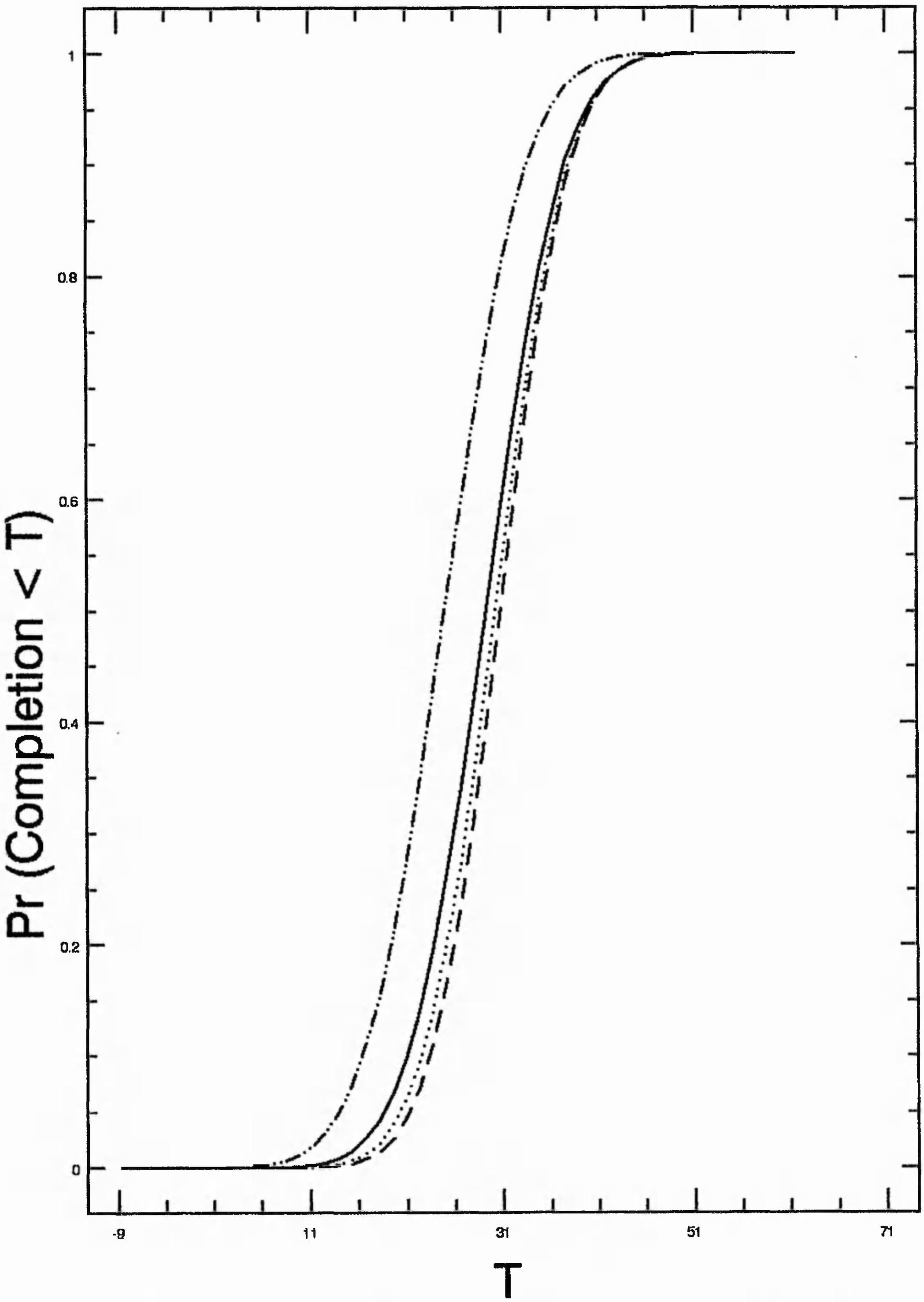


Figure 5.3 A network diagram of the twelve Unimodal Erlang activity times for the second example.

Figure 5.3

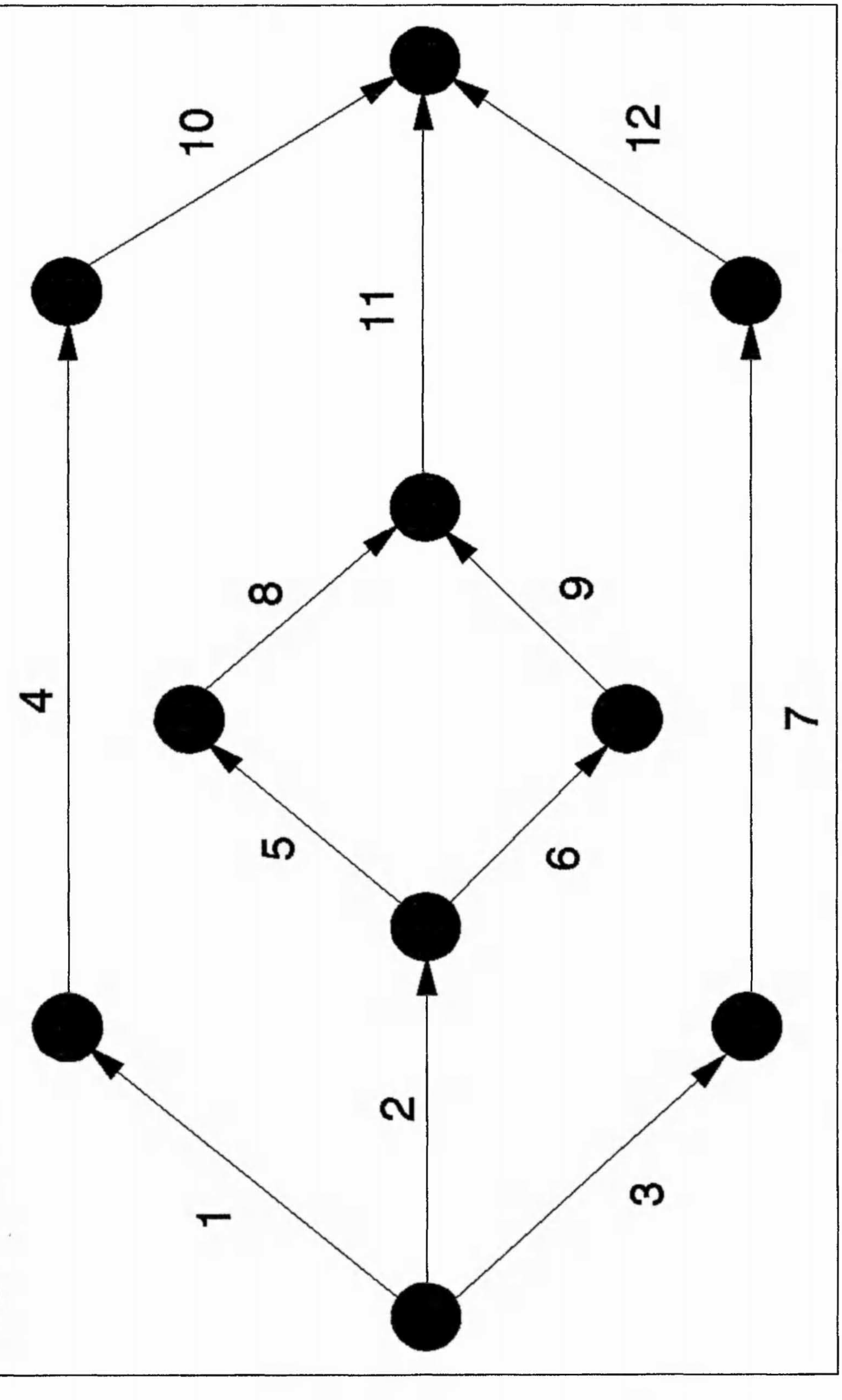


Figure 5.4 Cdf curves of completion time for unimodal input, second example using method (A), method (B), PERT and simulation.

Key:

------- = Method A (The complete path method)

.... = Method B (The progressive network reduction method)

________ = Simulation

_.._.._ = PERT

Figure 5.4

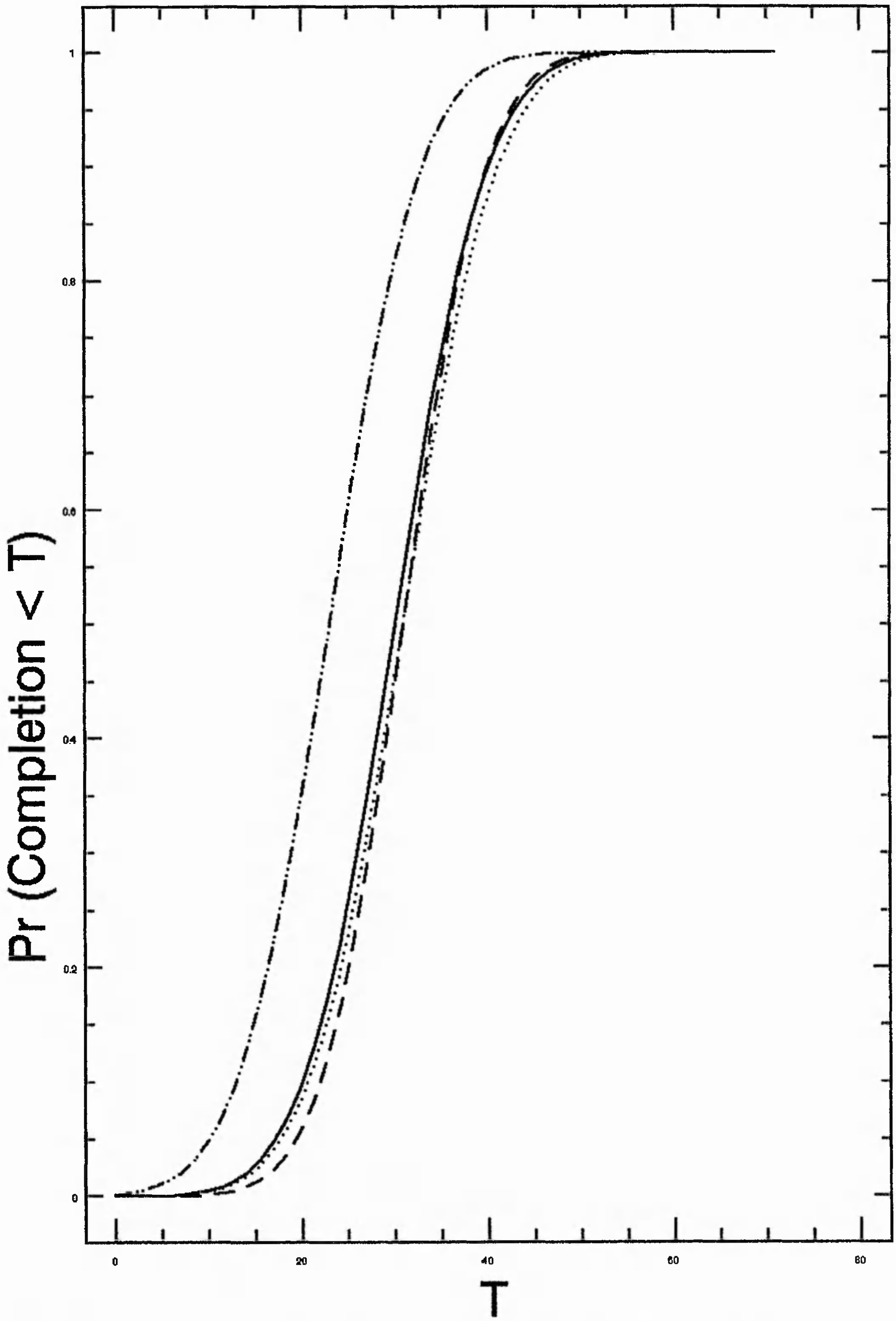


Figure 5.5 A network diagram of the twelve Unimodal Erlang activity times for the third example.

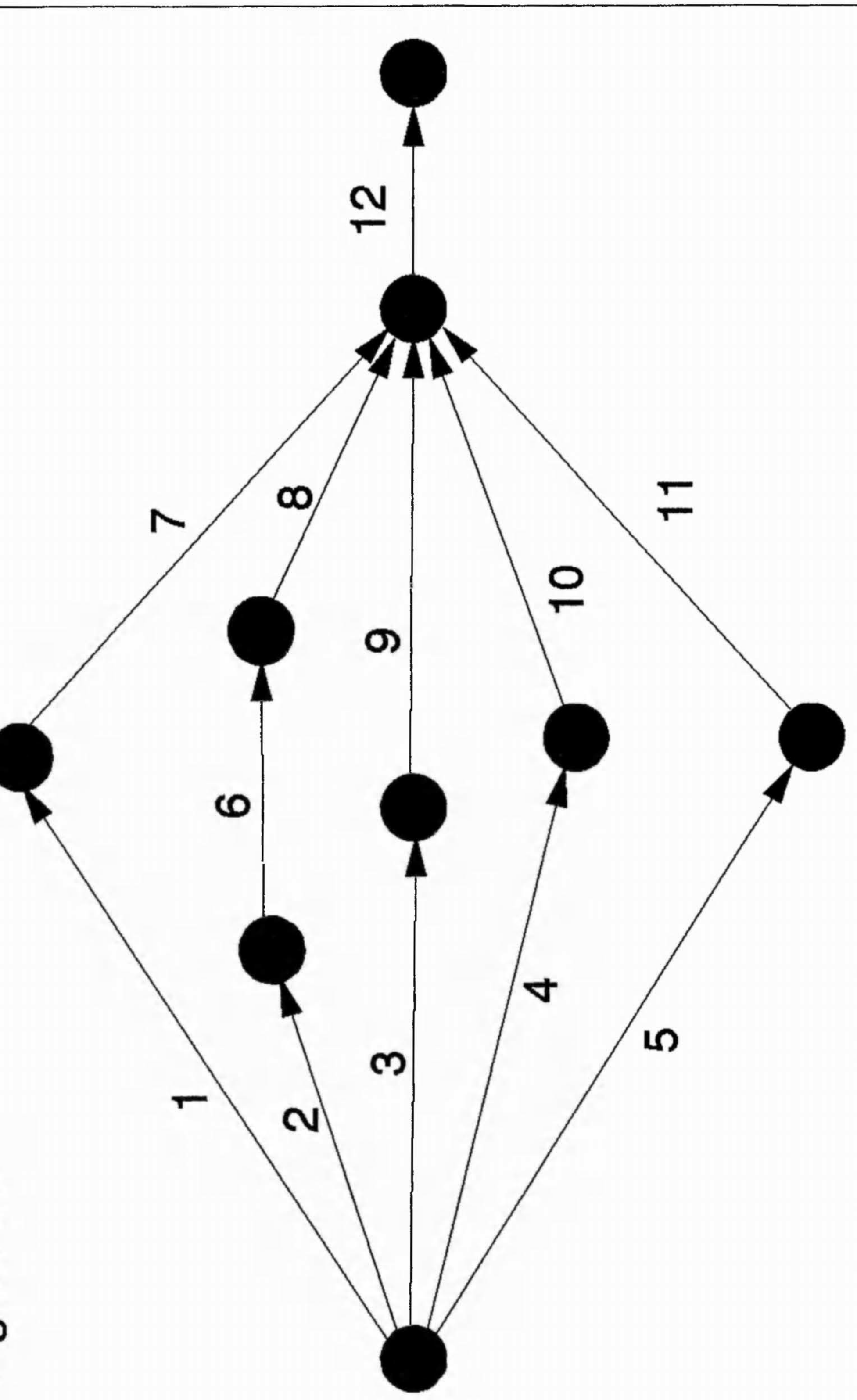


Figure 5.5

Figure 5.6 Cdf curves of completion time for unimodal input, third example using method (A), method (B), PERT and simulation.

Key:

------- = Method A (The complete path method)

.... = Method B (The progressive network reduction method)

_______ = Simulation

_.._.._ = PERT

Figure 5.6

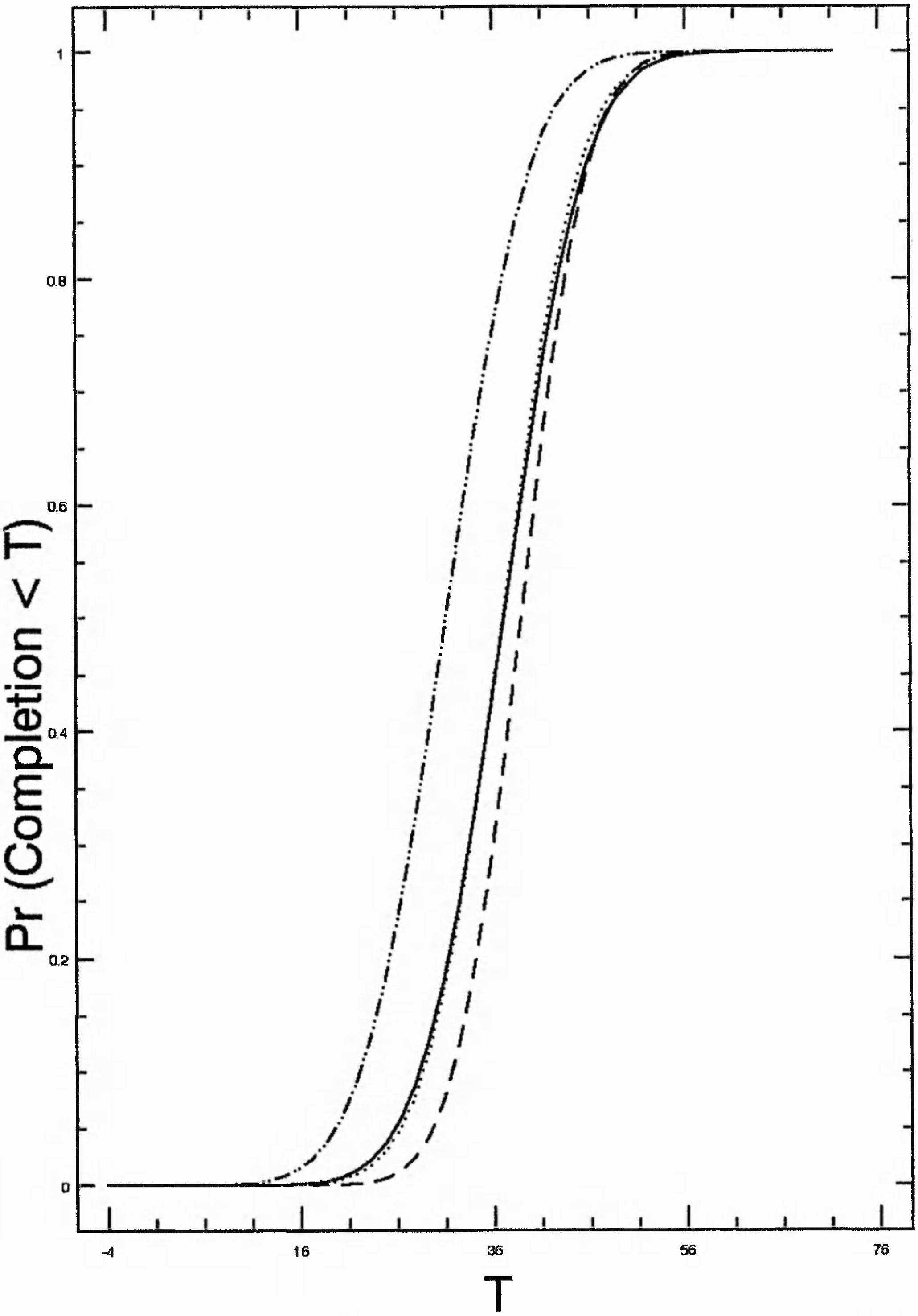


Figure 5.7 A network diagram of the thirteen Unimodal Erlang activity times for the fourth example.

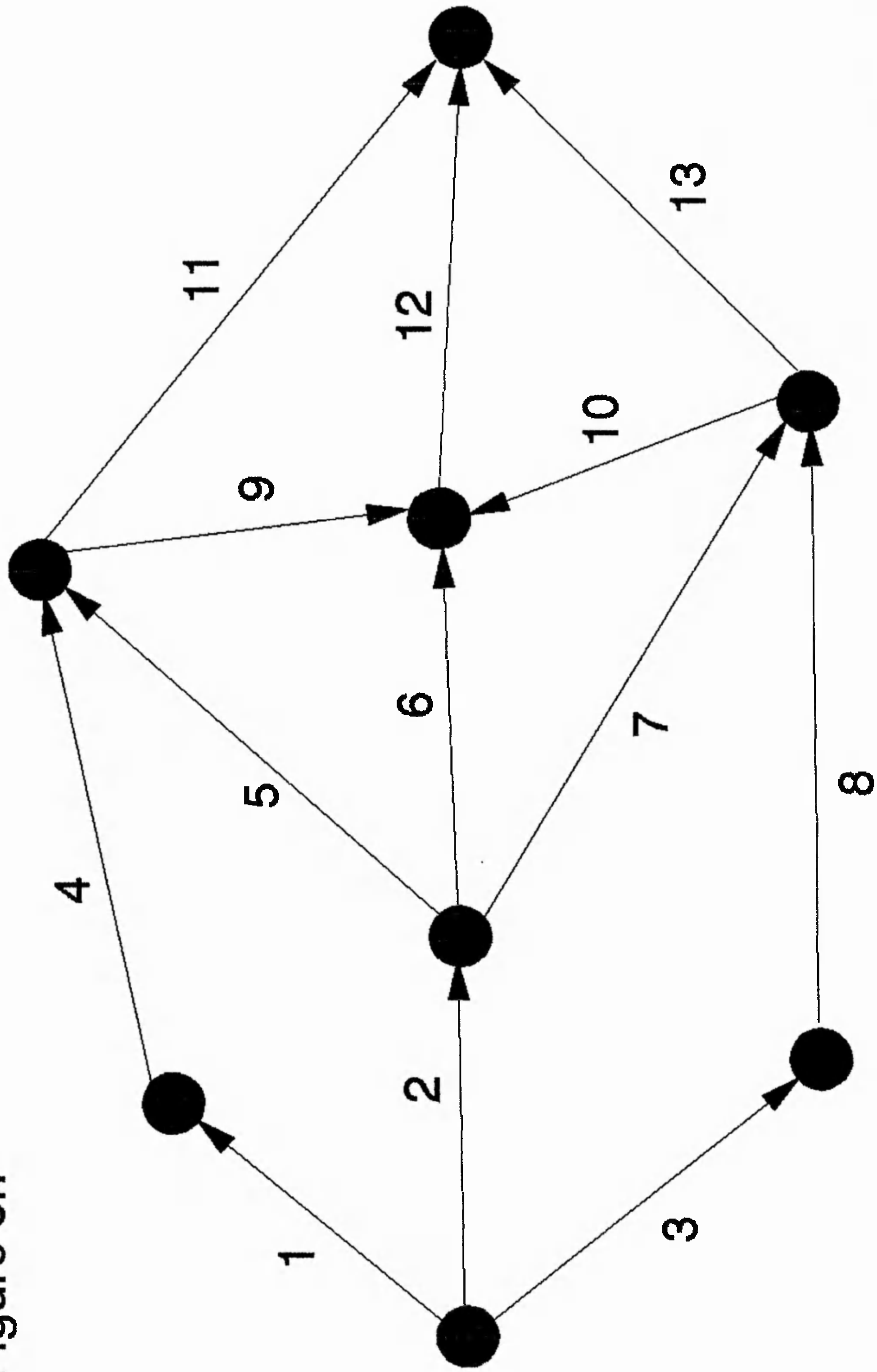


Figure 5.7

Figure 5.8 Cdf curves of completion time for unimodal input, fourth example using method (A), method (B), PERT and simulation.

Key:

-------- = Method A (The complete path method)

.... = Method B (The progressive network reduction method)

________ = Simulation

_.._.._ = PERT

Figre 5.8

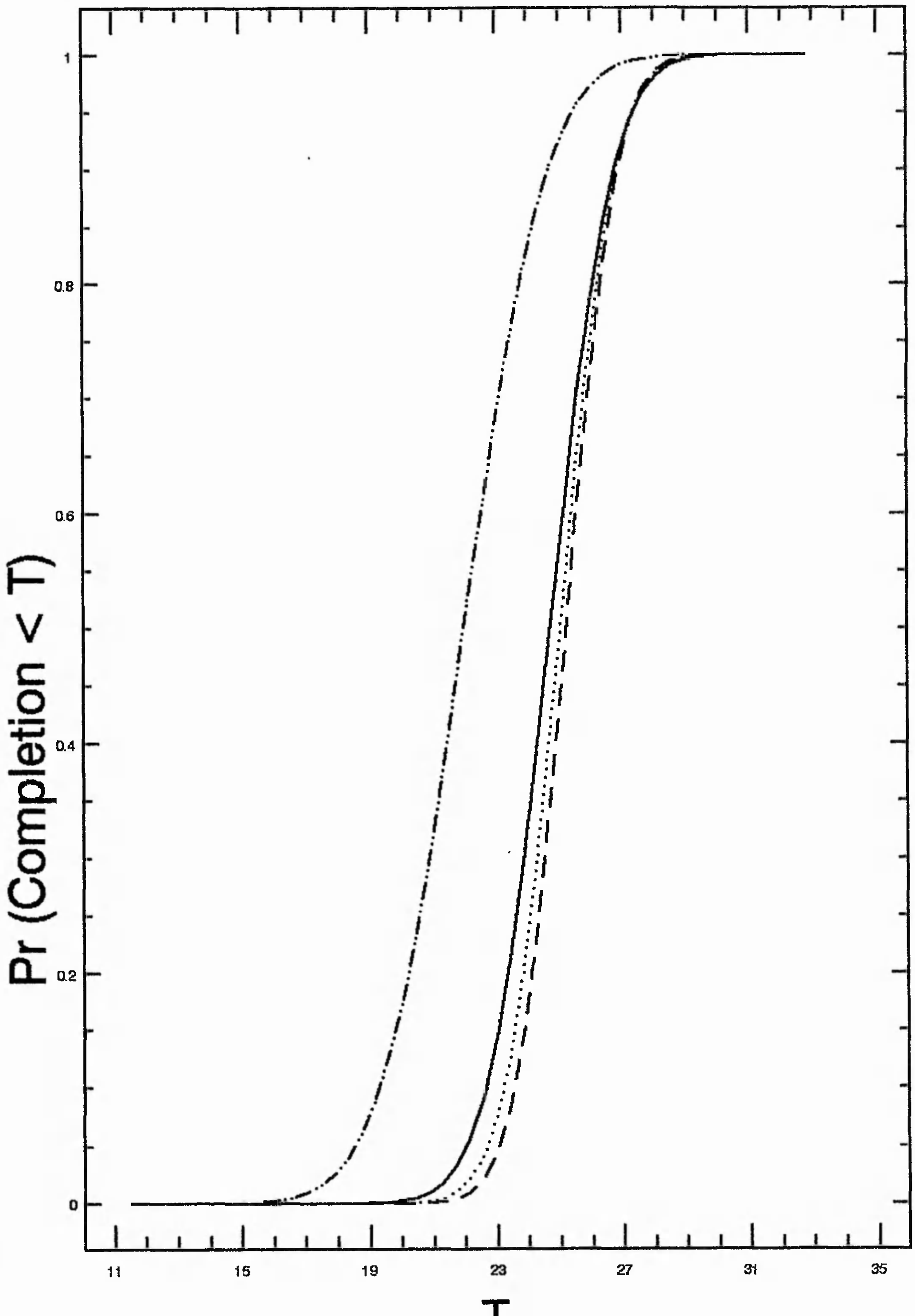


Figure 5.9 A network diagram of the sixteen Bimodal Erlang activity times for the first example.

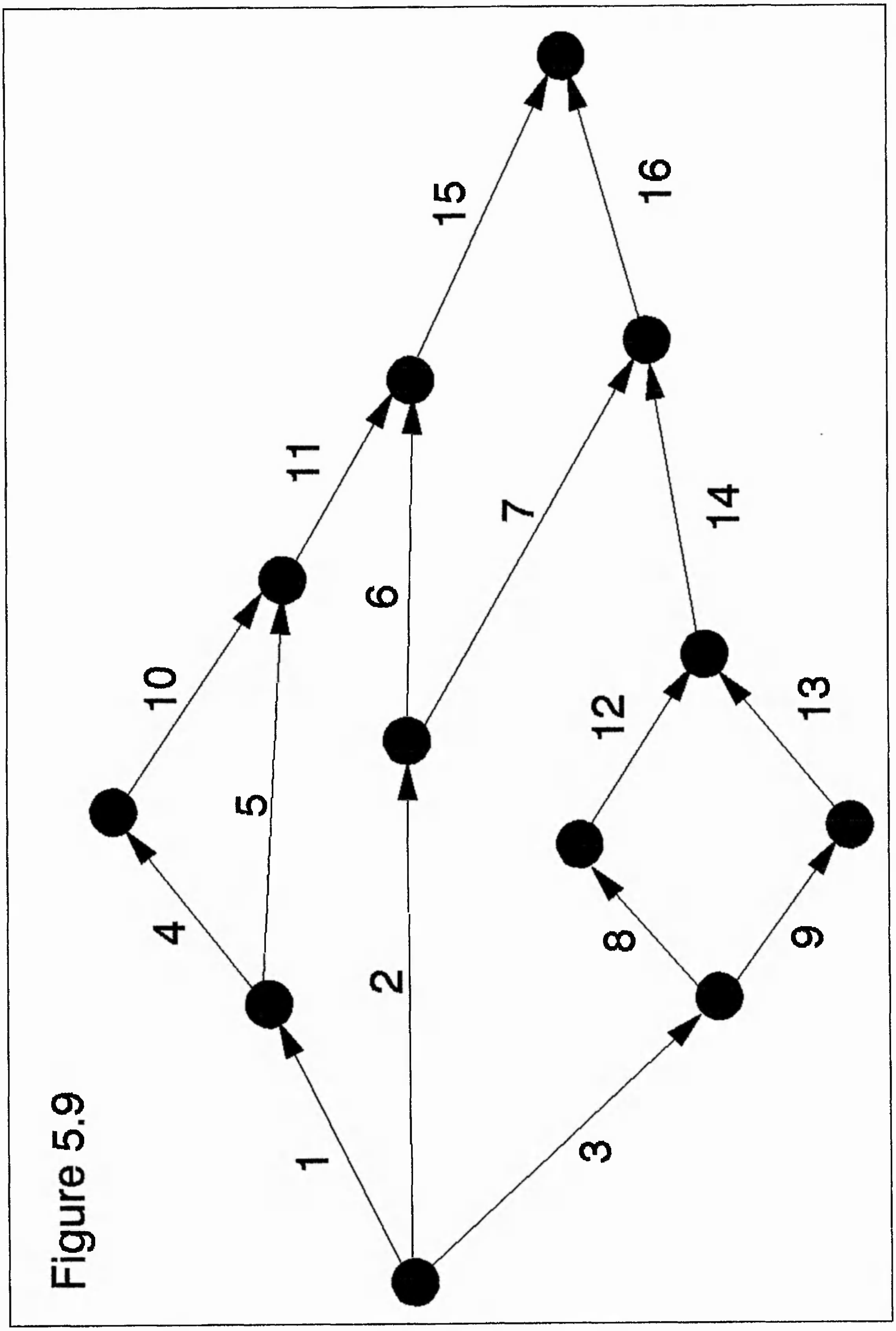


Figure 5.9

Figure 5.10 Cdf curves of completion time for bimodal input, first example using method (A), method (B), PERT and simulation.

Key:

------- = Method A (The complete path method)

.... = Method B (The progressive network reduction method)

________ = Simulation

_.._.._ = PERT

Figure 5.10

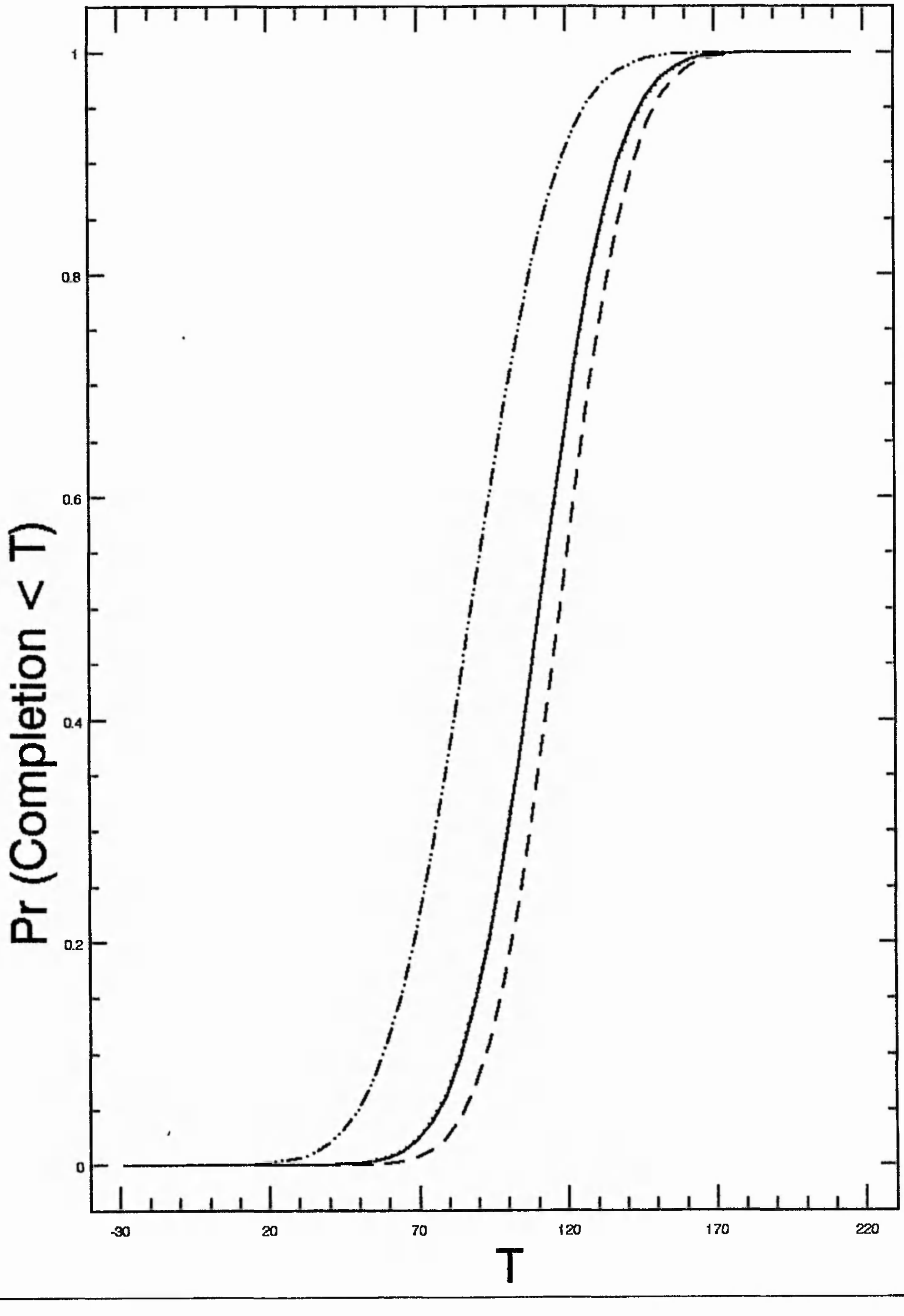


Figure 5.11 A network diagram of the seventeen Bimodal Erlang activity times for the second example.

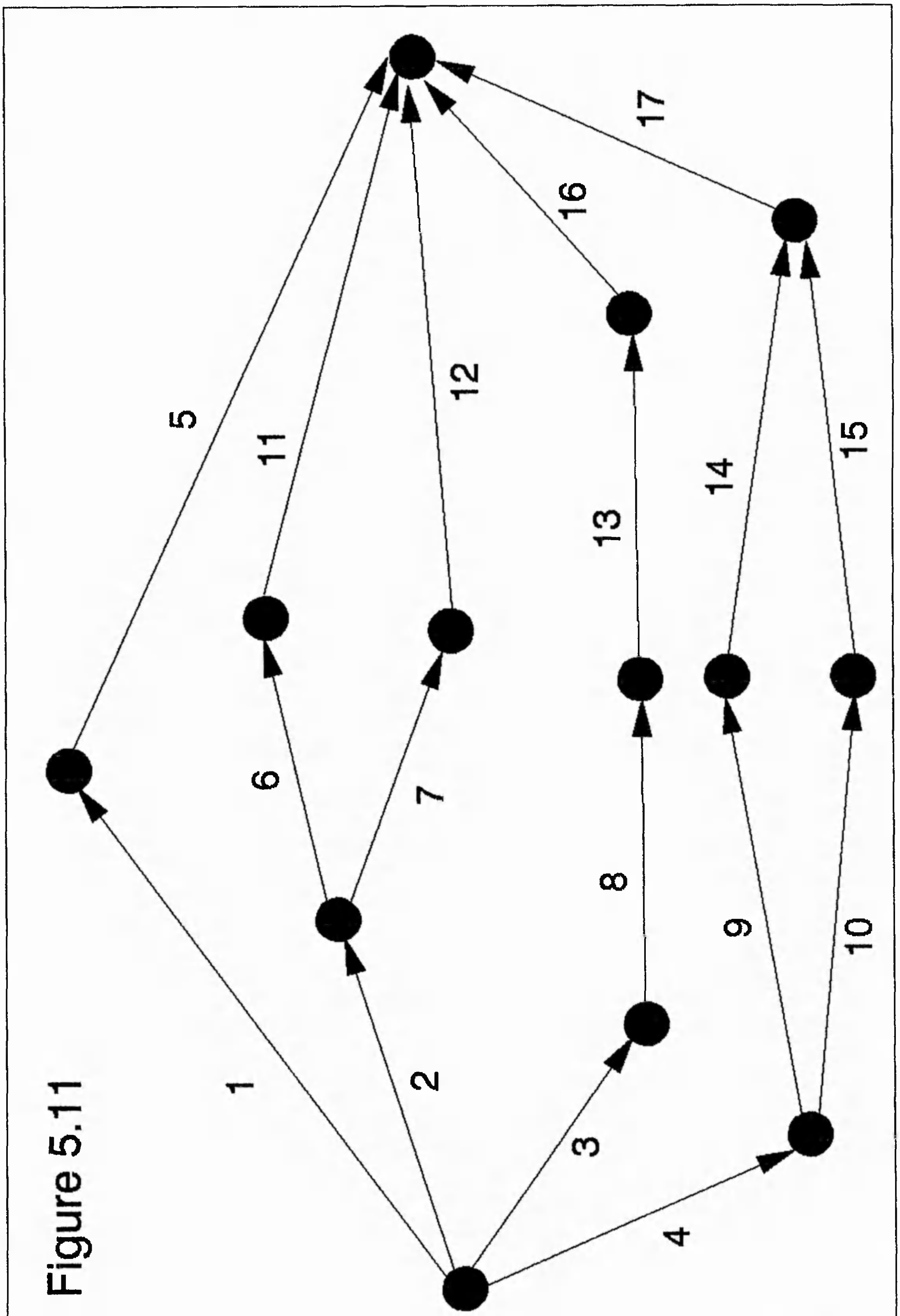


Figure 5.11

Figure 5.12 Cdf curves of completion time for bimodal input, second example using method (A), method (B), PERT and simulation.

Key:

-------- = Method A (The complete path method)

.... = Method B (The progressive network reduction method)

________ = Simulation

_.._.._ = PERT

Figure 5.12

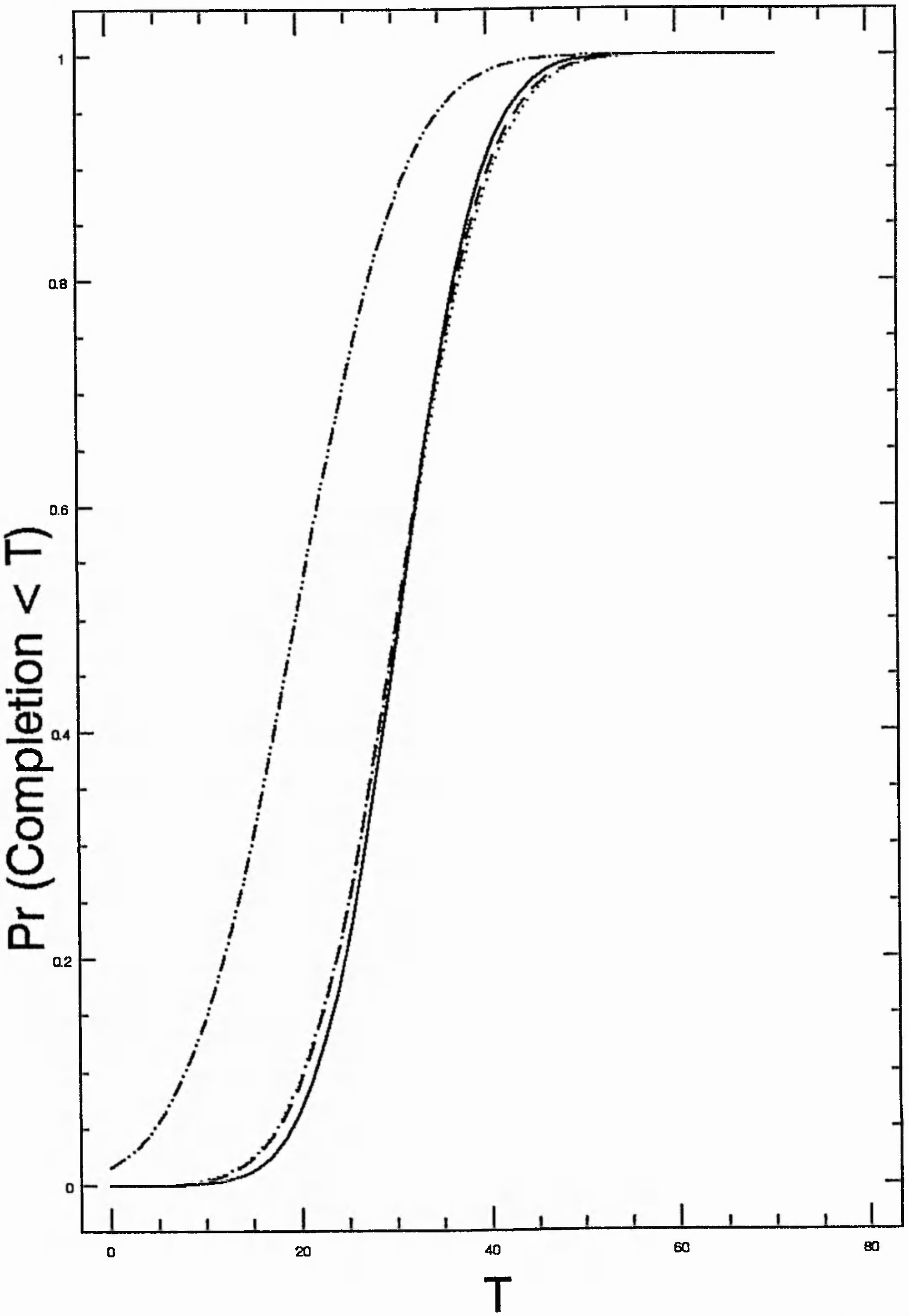


Figure 5.13 A network diagram of the twelve Bimodal Erlang activity times for the third example.

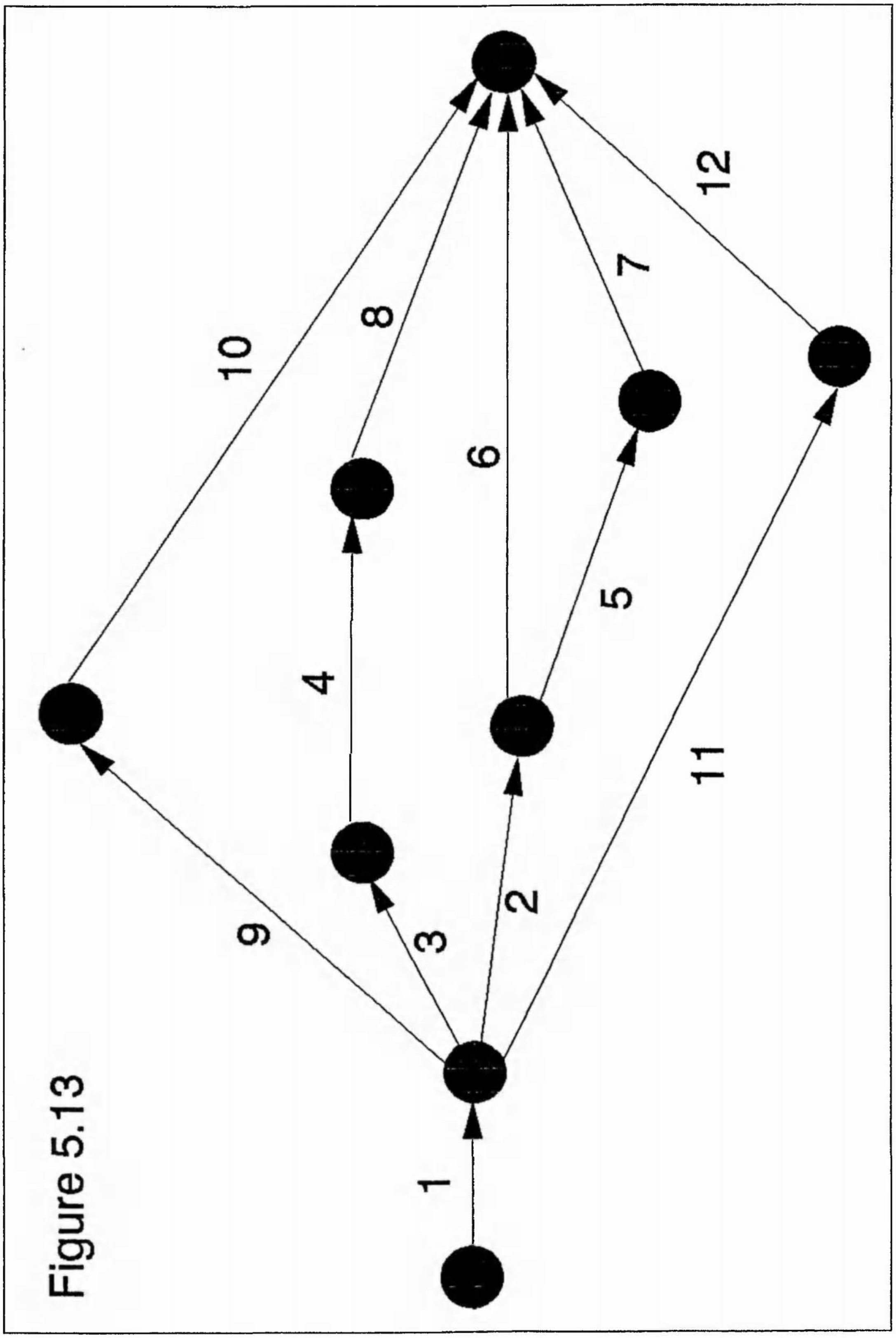


Figure 5.13

Figure 5.14 Cdf curves of completion time for bimodal input, third example using method (A), method (B), PERT and simulation.

Key:

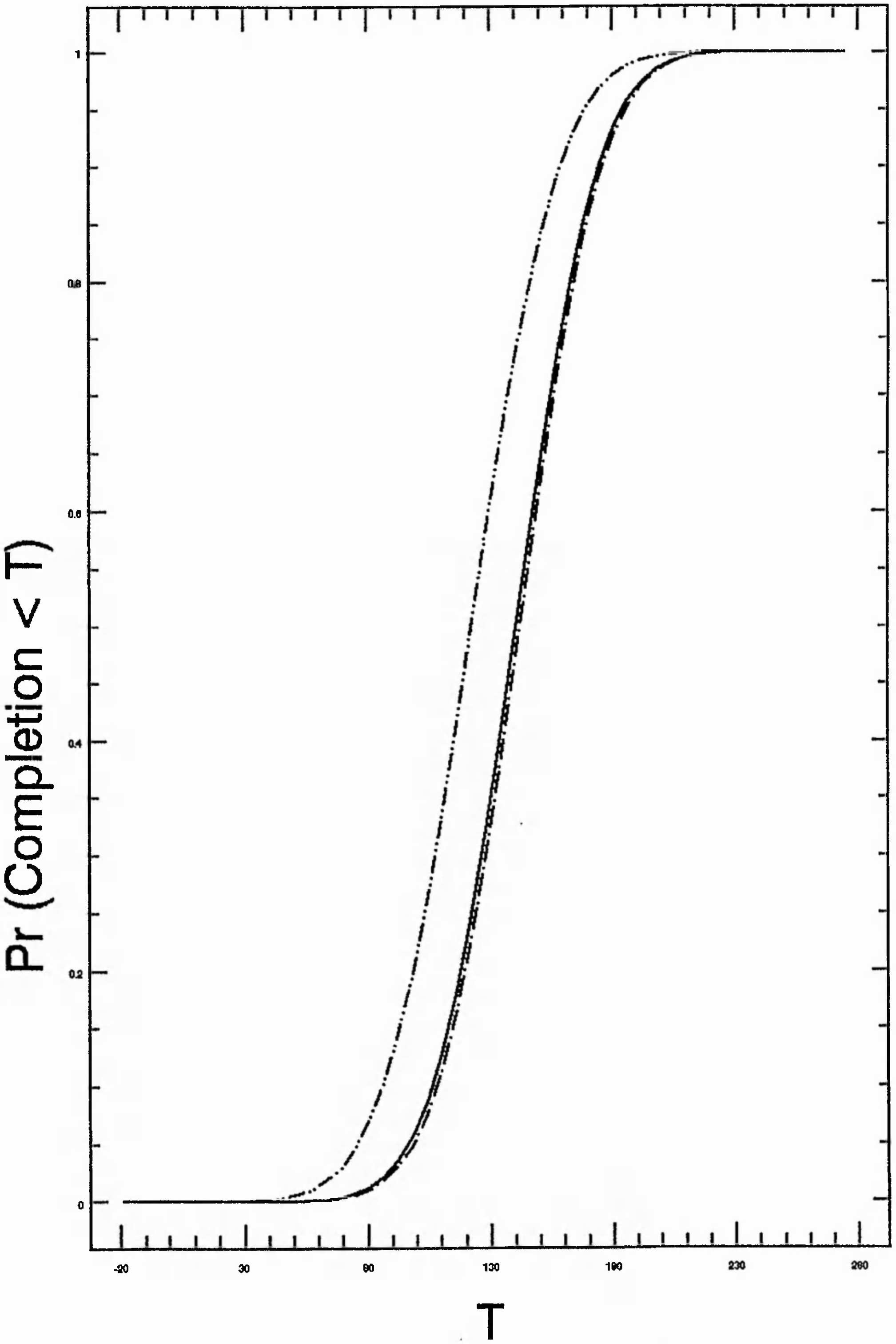
------- = Method A (The complete path method)

.... = Method B (The progressive network reduction method)

________ = Simulation

_.._.._ = PERT

Figure 5.14



CHAPTER SIX

APPLICATIONS OF RISK ANALYSIS MODELLING TO EMPIRICAL PROJECT MANAGEMENT DATA

6.1 INTRODUCTION

The availability of reliable and good quality data, is an important element in the assessment of risk. Previous practices and ideologies have created barriers between the construction industry and academics; but in recent years such barriers have been reduced or are in the process of being reduced and in many cases are now eliminated by the introduction of new programs and enterprises. One of these recent programs is the "Graduate Gateway", see Morgan, D. (1991), which has been developed and promoted by Universities and industrial institutions. Such a program has been organised, evolved and coordinated between the Nottingham Trent University and the Student Industrial Society and similar projects are being developed at Loughborough University. Other projects like the Shell Technology Enterprise Programme(STEP), see Shell (1993), have been promoted at other centres. These recent projects have contributed to easier access of data and other information in the field of Project Planning and Management.

The sets of data used and analysed in this Chapter have been made available to us by our industrial collaborator.

In this Chapter, two studies are discussed, the first is to analyse empirical data and apply the unified methodology to it, the second study is to examine the reliability of information from researchers and colleagues in the Department of Building and Environmental Health, at The Nottingham Trent University.

6.2 INDUSTRIAL PROJECT PLANNING - EMPIRICAL DATA

The current data has been collected with the assistance of our industrial collaborating establishment as part of a medium size construction project, see photos of the project building in Appendix 1 . The data covers the required activities from laying the foundations of the building, and ending with laying the roof slabs. This part of the project mainly consists of twenty two activities in total, see Figure 6.1 . Information regarding each activity is made up of three estimates, the optimistic, the most likely and the pessimistic times. It should be noted that these estimates are not exact; but subjective estimates. Each estimate could be thought of as a point estimate within a range, and the range would vary depending on many factors such as the type of project, the experience of the estimator and so on, see Table 6.1 .

To apply the four moments method, we need to estimate the first four central moments of each activity. The first and second moments for PERT analysis are:

$$\mu = \frac{1}{6}(a + 4m + b) \tag{6.1}$$

$$\sigma^2 = \frac{1}{36}(b-a)^2 \qquad (6.2)$$

where "**a**" is the optimistic time estimate, "**m**" is the most likely estimate and "**b**" is the pessimistic time estimate. Assuming that each activity could be fitted to an Erlang distribution, the above two equations are used to estimate the required two Erlang parameters C and B as:

$$B = \frac{(b-a)^2}{6(a+4m+b)} \qquad (6.3)$$

$$C = \left(\frac{a+4m+b}{b-a}\right)^2 \qquad (6.4)$$

The two estimated parameters for each activity of the project are computed, see Table 6.2 .

The results of collapsing the network by using method (B) -the progressive network reduction method- and applying the four moments Pearson approach, are in Figure 6.2 which shows the cdf curve of completion time. The simulation curve is shown for comparison. Percentage errors in the mean and standard deviation of the completion time are in table 6.3.

6.3 RELIABILITY STUDY OF PROJECT MANAGEMENT ESTIMATES

A study has been carried out between the Department of Mathematics, Statistics and Operational Research and the Department of Building and Environmental Health, to assess the reliability of estimates of activity times for a building project.

Questionnaires were designed, see Appendix 2, and distributed to twenty two members of staff and final year students. The questionnaires requested three time estimates, the optimistic, the most likely and the pessimistic of eight activities which represent part of a project. The three estimates, optimistic, the most likely and the pessimistic estimates were used because they are well understood by project management researchers. The questionnaires also requested a range for each of the three time estimates. The results of the requests are listed in Appendix 3.

Numerous plots of the data were examined for visual analysis, but to simplify the analysis it was decided to consider just the mean and standard deviations, using above formula, of each of the twenty two estimates of the optimistic the most likely and the pessimistic times, see Table 6.4.

Table 6.5 shows the coefficient of variation for each estimate, this being taken as a measure of its reliability. The table, shows no apparent relationship between the reliability of the estimates and their magnitude.

A mean difference paired t-test has been applied to the data of Table 6.5 . Since $t_{(0.025,7)}$ is 2.365, this shows no significant differences in the reliability of the three estimates, see Table 6.6 .

6.4 CONCLUSIONS

The first part of this Chapter shows that the new unified method can be applied to empirical project management data and produces reliable information on the distribution of total project time. Section 6.3 indicates that each of the three estimates of activity time, the pessimistic, optimistic and the most likely, have the same degree of reliability.

The assessment of risk made by our application of the unified methodology would help the project manager in the decision making process by which the implementation or otherwise of new projects could be put into affect.

Information about project completion time and cost could be made available to the manager on site in look-up tables or in graph form as means of solidifying decision making. The process would be even more advantageous if the unified methodology were programmed into a portable personnel computer which would give the project manager more flexibility in the range of input parameters and allow the percentage points of the project completion time to be speedily computed. The initial software to estimate the first four moments of the distribution of the sum or the maximum of two activities, has been written and documented, see Chapter seven.

Table 6.1 The optimistic, the most likely and the pessimistic estimates of the twenty two activity times of the empirical data.

Table 6.1

Activity number	Optimistic	Most likely	Pessimistic
1	0	0	0
2	12	15	20
3	8	10	20
4	4	6	15
5	1	2	5
6	9	10	25
7	4	4	10
8	2	3	10
9	2	2	5
10	7	7	7
11	4	5	12
12	2	2	4
13	1	2	3
14	5	6	13
15	1	1	3
16	7	7	7
17	1	1	2
18	5	6	9
19	5	7	12
20	7	7	7
21	9	12	18
22	0	0	0

Table 6.2 The mean, standard deviation, the estimated shape parameter and the estimated scale parameter of the twenty two activity times of the Industrial Project.

Table 6.2

Activity number	Mean	Standard deviation	C	B
1	0.00000	0.00000	0.00000	0.00000
2	15.33333	1.33333	132.25000	0.11594
3	11.33333	2.00000	32.11111	0.46499
4	7.16667	1.83333	15.28099	0.46899
5	2.33333	0.66667	12.25000	0.19048
6	12.33333	2.66667	21.39062	0.57658
7	5.00000	1.00000	25.00000	0.20000
8	4.00000	1.33333	9.00000	0.44444
9	2.50000	0.50000	25.00000	0.100000
10	7.00000	0.00000	–	–
11	6.00000	1.33333	20.25000	0.29630
12	2.33333	0.33333	49.00000	0.04762
13	2.00000	0.33333	36.00000	0.05556
14	7.00000	1.33333	27.56250	0.25397
15	1.33333	0.33333	16.00000	0.08333
16	7.00000	0.00000	–	–
17	1.16667	0.16667	49.00000	0.02381
18	6.33333	0.66667	90.25000	0.07018
19	7.50000	1.16667	41.32653	0.18148
20	7.00000	0.00000	–	–
21	12.50000	1.50000	69.44444	0.18000
22	0.00000	0.00000	0.00000	0.00000

Table 6.3 The percentage error in the mean and standard deviation of the project completion time using the unified methodology.

Table 6.3

Method	Mean	Standard deviation
Unified methodology	88.35959	3.83484
Simulation	86.33845	4.76851
%Error	2.34	19.58

Table 6.4 The mean and standard deviation of the optimistic (O), the most likely (M) and the pessimistic (P) activity times from the twenty two questionnaire respondents for the eight activities representing part of the total project.

Table 6.4

Activity number	O		M		P	
	μ	σ	μ	σ	μ	σ
1	9.5000	5.02138	13.4091	6.66661	22.0909	11.55487
2	18.2273	6.97569	23.7273	9.03001	35.3182	15.74617
3	13.2273	5.08904	18.5000	7.43063	27.3182	10.49438
4	19.0909	11.80762	24.0000	14.69047	36.8636	22.30800
5	12.8182	5.02031	17.4091	6.91210	24.9091	9.66047
6	18.3636	10.31664	23.5909	13.45418	36.5000	21.61735
7	11.8182	4.46827	16.3636	7.21470	25.0909	13.19780
8	13.5000	9.08033	17.8182	11.55785	28.0455	18.11466

Table 6.5 The coefficient of variation for the data of Table 6.4 .

Table 6.5

Activity	Most likely	Pessimistic	Optimistic
1	49.71706	52.30602	52.85663
2	38.05747	44.58373	38.27056
3	40.16557	38.41534	38.47760
4	61.21029	60.51498	61.84947
5	39.70395	38.78289	39.16548
6	57.03123	59.22562	56.17983
7	44.08993	52.59995	37.80838
8	64.86542	64.59026	67.26170

Table 6.6 Differences between the coefficients of variation of the three estimates, the optimistic, most likely and pessimistic for the eight activity times.

Table 6.6

Activity	(Optimistic-Most likely)	(Mostlikely-Pessimistic)	(Optimistic-Pessimistic)
1	3.13957	-2.58896	0.55061
2	0.21309	-6.52626	-6.31317
3	-1.69180	1.75023	0.05842
4	0.63918	0.69531	1.33449
5	-0.53846	0.92105	0.38259
6	-0.85139	-2.19439	-3.04578
7	-6.28155	-8.51002	-14.79157
8	2.39629	0.27516	2.67145
Mean	-0.37189	-2.02223	-2.39412
Standard deviation	2.88547	3.74711	5.75002
Paired t-statistic	-0.36454	-1.52644	1.17767

Figure 6.1 A network diagram of the twenty two activities of the Industrial Project.

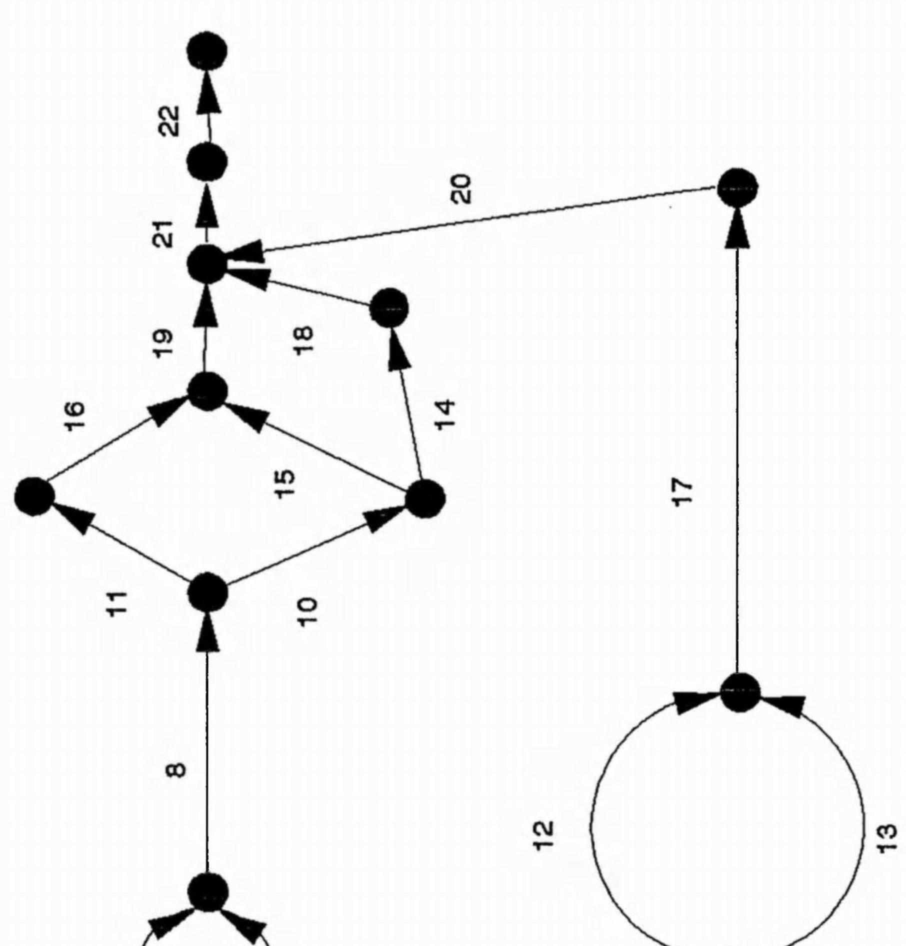


Figure 6.1

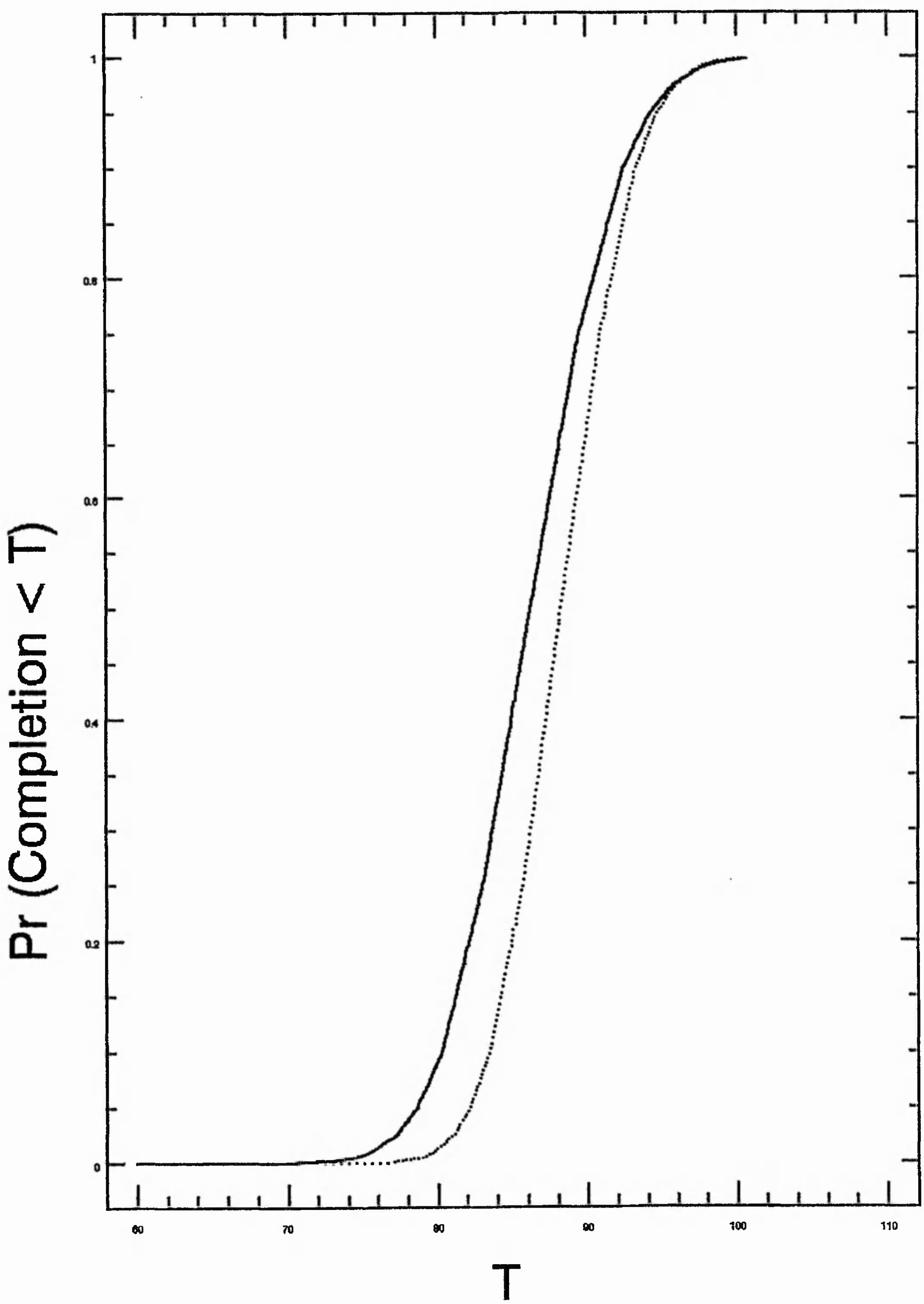
Figure 6.2 The Cdf curves of the completion time using the unified methodology and simulation for the Industrial project.

Key:

.... = Unified methodology.

_______ = Simulation

Figure 6.2



CHAPTER SEVEN

THE COMPUTER SOFTWARE USED FOR RISK ASSESSMENT

7.1 INTRODUCTION

We have explained and illustrated the procedure for computing the first four moments of the distribution of completion time, in section 2.4.2 for unimodal activities arranged in series, and in section 3.2 for activities arranged in parallel. If the input activity times are bimodal, this requires more computations and repetitive use of the mentioned equations has been described in Chapter four.

It is feasible to deal with very small simple networks by hand; but for large size networks involving activities arranged in parallel, this would require computer software to perform the computations so as to reduce the chance of miscalculations and also to reduce the amount of computation time.

We have written Fortran subroutines, see Appendix 4, to compute the first four moments of the distribution of the sum or the maximum of two unimodal or bimodal activity times, or mixture of the two. The software code is written in a sequential form and for use with a "**FORTRAN 77**" or similar compiler. The code does not require external libraries or routines so it can be easily transformed into other high or low level computer languages. The next three sections contain descriptions of the required input parameters, functionality of the software and what the user would expect from the software code.

7.2 PROCEDURE INPUT REQUIREMENTS

When invoking the executable image of the code, the user is asked to enter a name for the output file so that all the information given by the user and all the output from the code can be written to the file for inspection and future reference.

The two activity times can be either unimodal or bimodal. If the weighting factor, (π) is less than 1.0 and greater than 0.0 then the activity time is considered to be a bimodal, and if (π) is exactly 1.0, then the activity time is considered to be a unimodal. For unimodal activity times, the first four moments of the two input activity times need to be entered. If the two input activity times are bimodal with pdfs, $\pi_1 f_1 + (1-\pi_1)f_2$ and $\pi_2 f_3 + (1-\pi_2)f_4$, then the user has to enter the values of (π_1 , π_2) and the first four moments of each of f_1, f_2, f_3 and f_4 .

7.3 SOFTWARE DESCRIPTION AND FUNCTIONALITY

The software code is written in a sequential form and it consists of a collection of modules(Fortran Procedures), where each performs a specific task in the assessment of the distribution of the sum or the maximum.

The first task the code performs, is to write to the output file the information and parameters that are fed to it by the user during the input phase. Next the procedure "**INITIAL1**" is called by the main body of the code to initialize the output variables so to prevent accumulation of previous information in the case of repetitive use of the code. At this point the code would call one of two main procedures, depending on whether the activities are in parallel or in series.

To compute the first four moments of the distribution of the sum, procedure "**SUBX1**" is called to perform the computations for the moments, using the results of section (2.4.2.1) for unimodal activity times or section (4.2 onwards) for bimodal activity times. The output is written to the output file by calling procedure "**OUTX1**".

Computation of the first four moments of the distribution of the maximum is performed by calling procedure "**SUBX2**". This procedure makes a number of calls to procedure "**MAXP1**", depending on the type of input activity times whether they are unimodal or bimodal, see Figure 4.5 for the arrangement of the pairs of the activities. The computations performed in procedure "**MAXP1**" produce the first four moments of the distribution of the maximum of two unimodal activity times as explained and discussed in section (3.2) . The results are then written to the output file by calling procedure "**OUTX2**".

7.4 PROCEDURE OUTPUT EXPECTATIONS

The user of this software code, probably the project manager, should expect to get in the output file all the input information that was fed to the program during the execution phase and also the first four moments of the distribution of the sum or the maximum. These output moments can be used as input for more calculations as the network is collapsed as explained in section (2.4.2.1). Finally, the moments could be used to fit Pearson curves to the distribution of completion time from which the percentiles of the distribution of completion time could be found.

On the other hand, if we assume that the distribution of completion time is Erlang, then only the first two moments need to be considered in the input and output phase of the procedure.

CHAPTER EIGHT

SUMMARY AND CONCLUSIONS

8.1 SUMMARY

This Thesis is concerned with the methodologies of assessing Project Planning Risk and developing a new unified methodology. The new unified methodology is based on the moments method and is the first to use Erlang input distributions. It is unified in the sense that it is capable of dealing with a wide variety of networks and input distributions (unimodal and multimodal). The methodology has been tested with a good number of examples and shown to be robust with a good degree of accuracy and few limitations.

In Chapter 1 various definitions of risk were discussed and also the type of risk to be considered in relation to time overrun and cost variation. The uses and features of risk are explained. The Chapter also contains a brief discussion of the current methodologies, an outline of the problem of risk assessment and the aim of the research.

Chapter 2 gives a review of the most well known methodologies of risk assessment and a description of the advantages and the disadvantages of each. The Chapter has a good number of illustrative examples on each methodology.

A unified methodology is considered in Chapter 3, where the appropriateness of the Erlang model is explained. The Chapter concentrates on unimodal activity times, the activity networks being made up of series and parallel elements. Analytical results giving the first four central moments of parallel and series activity time distributions are derived and discussed fully. The completion time distribution is estimated by using either a 4-moments Pearson curves method or a method based on 2-moments and an Erlang completion time.

The unified methodology is taken further to cover bimodal activity times in Chapter 4 . The first four central moments for bimodal activities arranged in series and in parallel are derived and explained.

In Chapter 5 a comparison was made between two methods of collapsing networks. One method is based on identifying complete paths, the other on a sequential approach similar to that used in calculating early event times. The latter approach was found to give better results in most cases.

Application of the unified methodology is applied to project management data in Chapter 6 . A study was arranged with our industrial collaborating establishment. A review and analysis of the empirical data is carried out.

8.2 A UNIFIED METHODOLOGY

Chapter 3 highlights a method of risk assessment that can be termed a unified methodology since it can be applied, with an acceptable degree of accuracy to a wide range of situations in which an assessment of risk is required in project planning. The method assumes that the Erlang distribution is the most suitable model to represent the input distribution of activity times and also that the activity times are independent. The paths of networks are assumed to be independent also. The Erlang distribution is assumed to be stable under maximization, i.e. the maximum of two Erlang variables is approximately Erlang.

Although the Erlang distribution is the chosen activity time distribution, the method is capable of handling activity times with a multimodal distribution, since a procedure is developed to reduce such activity networks to equivalent Erlang cases.

The new unified method is based upon using the moments of the input distributions to estimate the first four moments of the completion time, then using either Pearson curves or the assumption of an Erlang completion time to evaluate probabilities of project completion time by some specified date. These probabilities are regarded as an assessment of the risk involved in going ahead with the project.

8.3 CONCLUSIONS

The proposed unified methodology is found to be acceptable for general use with a good degree of accuracy and robust even when some of the assumptions are violated. The requirement of only two input parameters for each activity makes it an attractive methodology to be adopted in real life applications.

The characteristics of the distribution of project completion times in all the examples tend to be those of the normal distribution. This may be partly due to the fact that the distribution of the sum of a number of activity times arranged in series, is close to the normal distribution when the Central Limit Theorem conditions are valid. When the shape parameter (C) of the Erlang distribution is large, (i.e. $C > 10$), the normal distribution would make a good approximation to the Erlang distribution. While the affect on the completion time distribution of parallel activities may work against a normally distributed end time, this effect seems to be relatively insignificant.

8.4 LIMITATIONS AND FURTHER DEVELOPMENTS

The use of the New Unified Methodology presented in this Thesis, is limited to the applicability conditions specified in section (2.4.5). Further enhancements and program developments may be as follows:

(1) Where only the mean activity time is known, an estimate of the variance is needed. Such an estimate might be made possible by investigating the ratio of the mean to the variance in similar activities.

(2) An extension of the programming code to cover the computation of the moments of the distribution of completion time rather than considering two activity times at a time. This would be possible if all the required input information came from a batch file.

(3) The development of the Unified Methodology to adopt similar ideas of the "Path Deletion Approach", to reduce the amount of unnecessary computation.

(4) The enhancement of the programming procedure to allow for "What If" scenarios.

(5) Allow for dependency and conditional input activity times, as for example in the "Voting Systems" networks.

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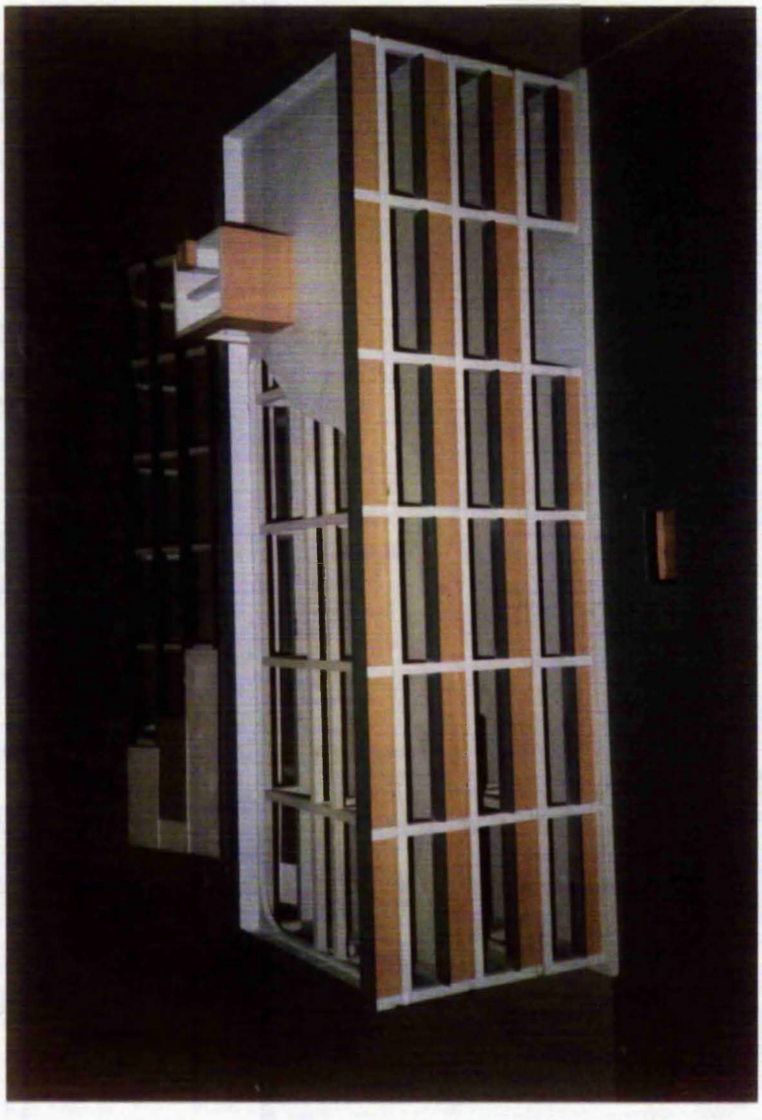
APPENDICES

APPENDIX 1

Two photos show the construction site of the project which is analysed in this Thesis.



APPENDIX 2

A questionnaire was used in the Department of Building and Environment Health Project.

NOTTINGHAM POLYTECHNIC
DEPARTMENT OF BUILDING AND ENVIRONMENTAL HEALTH

Tuesday, 19th November, 1991

Dear Colleagues,

Re : Operations Research Enquiry

Some of youn may know that I represent the department as a "collaborator" in a research project currently being undertaken by the operations research unit in the department of mathematics.

This research, if successful when completed, could have an immediate input into some of our own teaching and could form a useful basis from which to prepare some bids for funding research in our department or both departments at a future date.
Therefore it would be very much appreciated if you could find a little time to complete the attached questionnaire.

The current study concerns a unified approach to risk and the construction industry project has been selected by the mathematicians as the industrial applications area for testing and proving their work.

Yours Sincerely

BRIAN GEORGE

NOTTINGHAM POLYTECHNIC
DEPARTMENT OF BUILDING AND ENVIRONMENTAL HEALTH

OPERATIONS RESEARCH QUESTIONNAIRE

Please consider the modle of the office building on the mezzanine floor. [Do not consider the factory]

Please assume that only the reinforced concrete in situ frame is to be constructed

The frame is of simple construction ie isolated pad foundations, ground beams and solid floors

All reduced level excavation has been completed

Please accept that the follwing list denotes those activities that are required to be executed

Excavation
Substructure including ground floor slab
Ground storey columns and staircase and liftshaft
First floor beamsand and slab
First storey columns and staircase and liftshaft
Second floor beams and slab
Second storey columns and staircase and liftshaft
Roof beams and slab and parapet beams

Please accept that the excavation work involves direct work only ie excavation and backfill

Please accept that the substructure work involves direct work only ie reinforcement, formwork and concrete

Please accept that the superstructure work involves direct work only ie reinforcement, formwork, and concrete

For each of the activities in the list please provide:

1 An anticipated most likely duration for completion
2 An anticipated pessimistic duration for completion
3 An anticipation optimistic duration for completion

NB: Optimistic time is the shortest possible time in which the activity could be completed, assuming that everything gose well

NB: Pessimistic time is the longest time the activity could take, assuming that everything goes badly

NB: Most likely time is the time that the manager would probably give if asked for a single time estimate

Please also indicate the degree of confidence that you hold concerning the estimates which you provide eg optimistic time equals 6 weeks plus or minus 5 percent

NOTTINGHAM POLYTECHNIC

DEPARTMENT OF BUILDING AND ENVIRONMENTAL HEALTH

OPERATIONS RESEARCH QUESTIONNAIRE

ACTIVITY LIST	+	To	−	+	Tp	−	+	Tm	−
EXCAVATION									
SUBSTRUCTURE									
GROUND STOREY									
FIRST FLOOR SLAB									
FIRST STOREY									
SECOND FLOOR SLAB									
SECOND STOREY									
ROOF SLAB									

APPENDIX 3

Results of the twenty two questionnaires. Time estimates for the first eight activities.

Abbreviations are used to represent:

Q.N = Questionnaire respondent number

A.T = Activity time

To = Time(Optimistic)

Tm = Time(Most likely)

Tp = Time(Pessimistic)

– = Lower time limit

\+ = Upper time limit

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
1	1	14.25	15	16.50	19.00	20	22.00	31.50	35	42.00
1	2	28.50	30	33.00	38.00	40	44.00	72.00	80	96.00
1	3	19.00	20	22.00	28.50	30	33.00	45.00	50	60.00
1	4	28.50	30	33.00	38.00	40	44.00	63.00	70	84.00
1	5	19.00	20	22.00	28.50	30	33.00	45.00	50	60.00
1	6	28.50	30	33.00	38.00	40	44.00	72.00	80	96.00
1	7	19.00	20	22.00	28.50	30	33.00	63.00	70	84.00
1	8	14.25	15	16.50	28.50	30	33.00	53.00	60	72.00

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
2	1	2.40	3	3.60	4.75	5	5.25	9.00	10	11.00
2	2	8.00	10	12.00	14.25	15	15.75	22.50	25	27.50
2	3	8.00	10	12.00	12.35	13	13.65	16.20	18	19.80
2	4	12.00	15	18.00	19.00	20	21.00	27.00	30	33.00
2	5	6.40	8	9.60	9.50	10	10.50	18.00	20	22.00
2	6	12.00	15	18.00	19.00	20	21.00	27.00	30	33.00
2	7	6.40	8	9.60	9.50	10	10.50	18.00	20	22.00
2	8	16.00	20	24.00	23.75	25	26.25	31.50	35	38.50

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
3	1	1.80	2	2.20	3.60	4	4.40	5.40	6	6.60
3	2	9.00	10	11.00	10.80	12	13.20	13.50	15	16.50
3	3	9.00	10	11.00	13.50	15	16.50	18.00	20	22.00
3	4	7.20	8	8.80	10.80	12	13.20	13.50	15	16.50
3	5	9.00	10	11.00	13.50	15	16.50	18.00	20	22.00
3	6	7.20	8	8.80	10.80	12	13.20	13.50	15	16.50
3	7	9.00	10	11.00	13.50	15	16.50	18.00	20	22.00
3	8	7.20	8	8.80	10.80	12	13.20	13.50	15	16.50

Q.N	A.T	−	To	+	−	Tm	+	−	Tp	+
4	1	3.80	4	4.20	6.93	7	7.07	13.86	14	14.14
4	2	11.20	14	16.80	15.20	16	16.80	29.45	31	32.55
4	3	12.75	17	21.25	14.40	18	21.60	25.65	27	28.35
4	4	38.00	40	42.00	44.55	45	45.45	57.00	60	63.00
4	5	14.25	15	15.75	15.84	16	16.16	23.75	25	26.25
4	6	33.25	35	36.75	39.60	40	40.40	54.45	55	55.55
4	7	14.85	15	15.15	14.25	15	15.75	19.80	20	20.20
4	8	36.00	45	54.00	52.25	55	57.75	63.00	70	77.00

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
5	1	9.00	10	11.00	14.25	15	15.75	26.25	35	43.75
5	2	13.50	15	16.50	17.10	18	18.90	37.50	50	62.50
5	3	9.50	10	10.50	14.50	15	15.75	18.75	25	31.25
5	4	14.25	15	15.75	18.00	20	22.00	37.50	50	62.50
5	5	9.50	10	10.50	14.25	15	15.75	15.00	20	25.00
5	6	12.35	13	13.65	13.50	15	16.50	30.00	40	50.00
5	7	9.50	10	10.50	14.25	15	15.75	15.00	20	25.00
5	8	12.35	13	13.65	18.00	20	22.00	37.50	50	62.50

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
6	1	14.25	15	22.50	19.00	20	28	19.40	40	46.00
6	2	19.00	20	24.00	27.00	30	33	42.50	50	55.00
6	3	19.00	20	26.00	23.75	25	30	36.00	40	44.00
6	4	19.60	20	28.00	28.50	30	39	42.50	50	52.50
6	5	19.60	20	30.00	24.50	25	35	36.00	40	46.00
6	6	24.50	25	32.50	31.50	35	42	46.75	55	56.10
6	7	19.60	20	29.00	29.40	30	42	40.50	45	51.75
6	8	24.50	25	32.50	33.25	35	42	51.00	60	63.00

Q.N	A.T	−	To	+	−	Tm	+	−	Tp	+
7	1	19.00	20	22.00	27.00	30	31.50	45.00	50	55.00
7	2	19.00	20	22.00	27.00	30	31.50	45.00	50	55.00
7	3	14.25	15	16.50	18.00	20	21.00	27.00	30	33.00
7	4	9.50	10	11.00	13.50	15	15.75	22.50	25	27.50
7	5	9.50	10	11.00	13.50	15	15.75	27.00	30	33.00
7	6	9.50	10	11.00	13.50	15	15.75	22.50	25	27.50
7	7	9.50	10	11.00	13.50	15	15.75	27.00	30	33.00
7	8	9.50	10	11.00	13.50	15	15.75	22.50	25	27.50

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
8	1	14.50	15	15.75	19.00	20	21.00	28.50	30	31.50
8	2	19.00	20	21.00	23.75	25	26.25	28.50	30	31.50
8	3	17.00	20	23.00	28.50	30	31.50	36.00	40	44.00
8	4	36.00	40	44.00	45.00	50	55.00	67.50	75	82.50
8	5	18.00	20	22.00	23.75	25	26.25	28.50	30	31.50
8	6	33.25	35	36.75	42.75	45	47.25	66.50	70	73.50
8	7	14.25	15	15.75	19.00	20	21.00	28.50	30	31.50
8	8	11.70	13	14.50	13.50	15	16.50	18.00	20	22.00

Q.N	A.T	−	To	+	−	Tm	+	−	Tp	+
9	1	14.25	15	15.75	19.00	20	21.00	28.50	30	31.50
9	2	19.00	20	21.00	23.75	25	26.25	28.50	30	31.50
9	3	18.00	20	22.00	27.00	30	33.00	36.00	40	44.00
9	4	36.00	40	44.00	45.00	50	55.00	67.50	75	82.50
9	5	19.00	20	21.00	23.75	25	26.25	28.50	30	31.50
9	6	33.25	35	36.75	42.75	45	47.25	66.50	70	73.50
9	7	14.25	15	15.75	19.00	20	21.00	28.50	30	31.50
9	8	11.70	13	14.30	13.50	15	16.50	18.00	20	22.00

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
10	1	14.25	15	15.75	19.00	20	21.00	28.50	30	31.50
10	2	19.00	20	21.00	23.75	25	26.25	28.50	30	31.50
10	3	18.00	20	22.00	27.00	30	33.00	36.00	40	44.00
10	4	36.00	40	44.00	45.00	50	55.00	67.50	75	82.50
10	5	19.00	20	21.00	23.75	25	26.25	28.50	30	31.50
10	6	33.25	35	36.75	42.75	45	47.25	66.50	70	73.50
10	7	14.25	15	15.75	19.00	20	21.00	28.50	30	31.50
10	8	11.70	13	14.30	13.50	15	16.50	18.00	20	22.00

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
11	1	7.20	8	8.80	9.00	10	11.00	12.00	15	18.00
11	2	20.00	25	31.25	25.50	30	34.50	31.50	45	56.25
11	3	7.60	8	8.40	13.50	15	16.50	24.00	30	36.00
11	4	18.00	20	23.00	19.00	20	21.00	25.50	30	36.00
11	5	7.60	8	8.00	12.00	15	15.00	18.00	30	34.50
11	6	17.00	20	25.00	22.50	25	27.50	26.00	40	42.00
11	7	3.75	5	6.50	4.50	5	5.50	9.75	13	16.90
11	8	2.55	3	3.75	2.70	3	3.45	5.60	8	10.00

Q.N	A.T	−	To	+	−	Tm	+	−	Tp	+
12	1	2.00	4	4.00	4.00	8	8.00	4.00	20	20.00
12	2	18.75	25	26.25	19.25	35	35.00	10.00	50	50.00
12	3	6.00	10	10.50	6.75	15	15.00	7.50	25	25.00
12	4	10.00	20	21.00	10.00	25	25.00	9.00	45	45.00
12	5	6.00	10	10.50	6.75	15	15.00	6.00	20	20.00
12	6	10.00	20	21.00	12.00	30	30.00	8.00	40	40.00
12	7	6.00	10	10.50	6.75	15	15.00	6.00	20	20.00
12	8	2.75	5	5.25	5.00	10	10.00	4.50	15	15.00

Q.N	A.T	−	To	+	−	Tm	+	−	Tp	+
13	1	4.00	5	6.00	5.00	10	11.00	0.00	20	22.00
13	2	22.50	25	27.50	28.00	35	42.00	30.00	40	48.00
13	3	9.00	10	10.00	16.00	20	22.00	22.50	30	33.00
13	4	20.00	25	27.50	26.50	35	38.50	28.00	40	44.00
13	5	13.50	15	15.00	20.00	25	27.50	24.00	30	33.00
13	6	22.50	25	27.50	28.00	35	42.00	32.00	40	44.00
13	7	14.25	15	15.00	20.00	25	27.50	24.00	30	33.00
13	8	22.50	25	31.25	25.50	30	33.00	42.50	50	57.50

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
14	1	4.90	5	5.50	7.60	8	8.40	9.00	10	11.00
14	2	29.40	30	31.50	36.00	40	44.00	47.50	50	55.00
14	3	9.50	10	10.50	14.25	15	15.75	19.00	20	21.00
14	4	9.50	10	10.50	14.25	15	15.75	19.00	20	21.00
14	5	9.50	10	10.50	12.35	13	13.65	17.10	18	18.90
14	6	9.50	10	10.50	14.25	15	15.75	19.00	20	21.00
14	7	9.50	10	10.50	12.35	13	13.65	17.10	18	18.90
14	8	9.50	10	10.50	14.25	15	15.75	19.00	20	21.00

Q.N	A.T	−	To	+	−	Tm	+	−	Tp	+
15	1	3.00	10	17.00	11.20	14	16.80	16.00	20	24.00
15	2	8.50	10	11.50	10.20	12	13.80	12.75	15	17.25
15	3	5.70	6	6.30	7.60	8	8.40	11.40	12	12.60
15	4	7.60	8	8.40	9.50	10	10.50	15.20	16	16.80
15	5	5.60	6	6.30	7.60	8	8.40	11.40	12	12.60
15	6	7.60	8	8.40	9.50	10	10.50	15.20	16	16.80
15	7	5.70	6	6.30	7.60	8	8.40	11.40	12	12.60
15	8	13.30	14	14.70	17.10	18	18.90	24.70	26	27.30

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
16	1	8.82	9	9.45	9.70	10	11.00	12.00	15	12.96
16	2	6.86	7	7.70	7.60	8	8.80	10.20	12	12.84
16	3	12.00	12	12.84	11.88	12	12.60	12.60	14	14.70
16	4	4.00	4	4.20	3.80	4	4.08	5.70	6	6.12
16	5	12.00	12	12.84	11.88	12	12.60	12.60	14	14.70
16	6	4.00	4	4.20	3.80	4	4.08	5.70	6	6.12
16	7	11.00	11	11.77	10.89	11	11.55	13.50	15	15.75
16	8	4.00	4	4.20	4.75	5	5.10	6.65	7	7.14

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
17	1	14.70	15	15.30	19.60	20	20.40	22.50	25	28.75
17	2	19.60	20	20.20	19.40	20	21.00	31.50	35	38.50
17	3	14.70	15	15.30	14.55	15	15.60	20.00	25	30.00
17	4	9.50	10	10.50	9.60	10	10.40	18.00	20	22.00
17	5	9.50	10	11.00	14.40	15	15.45	18.00	20	22.00
17	6	9.50	10	10.50	9.30	10	10.50	12.75	15	17.25
17	7	9.80	10	10.20	14.40	15	15.75	17.60	20	22.00
17	8	9.70	10	10.30	9.60	10	10.50	12.00	15	17.55

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
18	1	9.50	10	10.50	14.25	15	15.75	23.75	25	26.25
18	2	18.00	20	22.00	22.50	25	27.5	31.50	35	38.50
18	3	9.50	10	10.50	14.25	15	15.75	23.75	25	26.25
18	4	14.25	15	15.75	14.25	15	15.75	19.00	20	21.00
18	5	9.50	10	10.50	14.25	15	15.75	19.00	20	21.00
18	6	14.25	15	15.75	14.25	15	15.75	23.75	25	26.25
18	7	9.50	10	10.50	14.25	15	15.75	19.00	20	21.00
18	8	9.50	10	10.50	14.25	10	15.75	19.00	20	21.00

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
19	1	5.70	6	6.30	7.60	8	8.40	9.50	10	10.50
19	2	18.50	19	19.95	19.95	21	22.05	26.60	28	29.40
19	3	5.70	6	6.30	7.60	8	8.40	11.40	12	12.60
19	4	8.55	9	9.45	9.50	10	10.50	15.20	16	16.80
19	5	5.70	6	6.30	6.65	7	7.35	9.50	10	10.50
19	6	8.55	9	9.45	9.50	10	10.50	15.20	16	16.80
19	7	5.70	6	6.30	6.65	7	7.35	9.50	10	10.50
19	8	9.50	10	10.50	11.40	12	12.60	15.20	16	16.80

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
20	1	7.60	8	8.40	9.50	10	10.50	12.35	13	13.65
20	2	20.90	22	23.10	23.75	25	26.25	30.40	32	33.60
20	3	6.65	7	7.35	9.50	10	10.50	12.35	13	13.65
20	4	10.45	11	11.55	14.55	15	15.45	17.10	18	18.90
20	5	8.55	9	9.45	9.00	10	10.30	13.30	14	14.70
20	6	13.30	14	14.70	16.66	17	17.85	19.00	20	21.00
20	7	7.60	8	8.40	9.90	10	10.10	13.30	14	14.70
20	8	8.55	9	9.45	11.40	12	12.60	14.25	15	15.75

Q.N	A.T	–	To	+	–	Tm	+	–	Tp	+
21	1	4.75	5	5.25	5.70	6	6.30	7.60	8	8.40
21	2	12.60	14	15.40	13.50	15	16.50	17.10	19	20.90
21	3	14.25	15	16.50	16.20	18	19.80	27.00	30	31.50
21	4	9.50	10	11.00	10.80	12	13.20	18.00	20	21.00
21	5	12.35	13	14.30	15.30	17	18.70	27.00	30	31.50
21	6	7.60	8	8.80	9.90	11	12.10	18.00	20	21.00
21	7	10.45	11	12.10	14.40	16	17.60	27.00	30	31.50
21	8	6.65	7	7.70	9.00	10	11.00	18.0	20	21.00

Q.N	A.T	−	To	+	−	Tm	+	−	Tp	+
22	1	9.00	10	11.00	13.50	15	16.50	22.50	25	27.50
22	2	14.25	15	15.75	19.00	20	21.00	23.75	25	26.25
22	3	19.60	20	20.40	29.40	30	30.60	34.30	35	35.70
22	4	19.00	20	21.00	23.75	25	26.25	33.25	35	36.75
22	5	19.60	20	20.40	29.40	30	30.60	34.30	35	35.70
22	6	19.00	20	21.00	23.75	25	26.25	33.25	35	36.75
22	7	19.60	20	20.40	29.40	30	30.60	34.30	35	35.70
22	8	14.25	15	15.75	19.00	20	21.00	28.50	30	31.50

APPENDIX 4

The computer software used for Risk Assessment.

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C     C:\CODE.FOR
C
C     A procedure for computing the first four central moments of
C     the sum or maximum of two unimodal or bi-modal input activity
C     times.
C
C
C     If bi-modal input activity times:
C
C
C   Sum  { [π1f1+(1.0-π1)f2]  &  [π2f3+(1.0-π2)f4] }
C
C     OR:
C
C   Max  { [π1f1+(1.0-π1)f2]  ,  [π2f3+(1.0-π2)f4] }
C
C
C     Possible input parameters and moments of t(1):
C
C
C     (1) First weight (π1)
C     (2) First  central moment of activity (1)
C     (3) Second central moment of activity (1)
C     (4) Third  central moment of activity (1)
C     (5) Fourth central moment of activity (1)
C     (6) First  central moment of activity (2)
C     (7) Second central moment of activity (2)
C     (8) Third  central moment of activity (2)
C     (9) Fourth central moment of activity (2)
C
C     Possible input parameters and moments of t(2):
C
C
C     (10) Second weight (π2)
C     (11) First  central moment of activity (3)
C     (12) Second central moment of activity (3)
C     (13) Third  central moment of activity (3)
C     (14) Fourth central moment of activity (3)
C     (15) First  central moment of activity (4)
C     (16) Second central moment of activity (4)
C     (17) Third  central moment of activity (4)
C     (18) Fourth central moment of activity (4)
C
C     Note: If the input activity time is a unimodal,
C           the weight of the activity time must be 1.0.
```

```
C
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      PROGRAM CODE
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT DOUBLE PRECISION (K-Z)
      COMMON /COMO/ PI1,MU1,VAR1,SK1,KR1,MU2,VAR2,SK2,KR2,
     1              PI2,MU3,VAR3,SK3,KR3,MU4,VAR4,SK4,KR4
      COMMON /COM1/ MUX,VARX,SKX,KRX
      DOUBLE PRECISION DUM1,DUM2,DUM3,C1,C2
C----------------------------------------------------------------------
      CHARACTER FILEOUT*20
      INTEGER NUMC,ICOUNT
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C     INITIALISE THE INPUT PARAMETERS AND THE MOMENTS
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      PI1 =0.0D0
      MU1 =0.0D0
      VAR1=0.0D0
      SK1 =0.0D0
      KR1 =0.0D0
      MU2 =0.0D0
      VAR2=0.0D0
      SK2 =0.0D0
      KR2 =0.0D0
      PI2 =0.0D0
      MU3 =0.0D0
      VAR3=0.0D0
      SK3 =0.0D0
      KR3 =0.0D0
      MU4 =0.0D0
      VAR4=0.0D0
      SK4 =0.0D0
      KR4 =0.0D0
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      DO 100 I=1,15
      WRITE(6,200)
200   FORMAT(///'')
100   CONTINUE
      WRITE(6,300)
300   FORMAT(' *** ( C:\CODE.FOR ) *** '//,
     1 '-------------------------------------------------'/,
     2 ' A procedure for computing the first four '/,
     3 ' central moments of the sum or maximum of '/,
     4 ' two unimodal or bi-modal input activity times'/,
     5 '-------------------------------------------------'//,
     6 ' ENTER NAME OF OUTPUT FILE=')
```

```
      READ(*,400)FILEOUT
400   FORMAT(A20)
      OPEN(UNIT=9,FILE=FILEOUT,STATUS='UNKNOWN')
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      ICOUNT=0
500   WRITE(6,1000)
1000  FORMAT(/' ------------------------'//,
     1         ' Enter value of the first weight =')
      READ(*,*)PI1
      WRITE(6,1050)
1050  FORMAT(' ------------------------')
      IF (PI1.EQ.0.0D0) THEN
      WRITE(6,1055)
1055  FORMAT(//'***** ( Worning: First weight must be > 0.0 ) *****'/)
      GOTO 500
      ENDIF
      ICOUNT=ICOUNT+1
      WRITE(6,1100)ICOUNT
1100  FORMAT(' Enter first central moment of activity(',I5,')=')
      READ(*,*)MU1
      WRITE(6,1200)ICOUNT
1200  FORMAT(' Enter second central moment of activity(',I5,')=')
      READ(*,*)VAR1
      WRITE(6,1210)ICOUNT
1210  FORMAT(' Enter third central moment of activity(',I5,')=')
      READ(*,*)SK1
      WRITE(6,1220)ICOUNT
1220  FORMAT(' Enter fourth central moment of activity(',I5,')=')
      READ(*,*)KR1
      IF (PI1.EQ.1.0D0) GOTO 1450
      ICOUNT=ICOUNT+1
      WRITE(6,1300)ICOUNT
1300  FORMAT(' Enter first central moment of activity(',I5,')=')
      READ(*,*)MU2
      WRITE(6,1400)ICOUNT
1400  FORMAT(' Enter second central moment of activity(',I5,')=')
      READ(*,*)VAR2
      WRITE(6,1410)ICOUNT
1410  FORMAT(' Enter third central moment of activity(',I5,')=')
      READ(*,*)SK2
      WRITE(6,1420)ICOUNT
1420  FORMAT(' Enter fourth central moment of activity(',I5,')=')
      READ(*,*)KR2
1450  CONTINUE
C-------------------
1460  WRITE(6,1500)
1500  FORMAT(/' ------------------------'/,
```

```
     1         '  Enter value of the second weight =')
      READ(*,*)PI2
      WRITE(6,1550)
1550  FORMAT(' -----------------------')
      IF (PI2.EQ.0.0D0) THEN
      WRITE(6,1555)
1555  FORMAT(//'***** ( Worning: Second weight must be > 0.0 ) *****'/)
      GOTO 1460
      ENDIF
      ICOUNT=ICOUNT+1
      WRITE(6,1600)ICOUNT
1600  FORMAT(' Enter first central moment of activity(',I5,')=')
      READ(*,*)MU3
      WRITE(6,1700)ICOUNT
1700  FORMAT(' Enter second central moment of activity(',I5,')=')
      READ(*,*)VAR3
      WRITE(6,1710)ICOUNT
1710  FORMAT(' Enter third central moment of activity(',I5,')=')
      READ(*,*)SK3
      WRITE(6,1720)ICOUNT
1720  FORMAT(' Enter fourth central moment of activity(',I5,')=')
      READ(*,*)KR3
      IF (PI2.EQ.1.0D0) GOTO 1950
      ICOUNT=ICOUNT+1
      WRITE(6,1800)ICOUNT
1800  FORMAT(' Enter first central moment of activity(',I5,')=')
      READ(*,*)MU4
      WRITE(6,1900)ICOUNT
1900  FORMAT(' Enter second central moment of activity(',I5,')=')
      READ(*,*)VAR4
      WRITE(6,1910)ICOUNT
1910  FORMAT(' Enter third central moment of activity(',I5,')=')
      READ(*,*)SK4
      WRITE(6,1920)ICOUNT
1920  FORMAT(' Enter fourth central moment of activity(',I5,')=')
      READ(*,*)KR4
1950  CONTINUE
2000  CONTINUE
C2050  FORMAT(D15.5)
      WRITE(6,2100)
2100  FORMAT(///'------------------------------------------- '/,
     1 ' (1) The two activities are arranged in series '/,
     2 ' (2) The two activities are arranged in parallel '///,
     3 ' Please enter ( 1 OR 2 ) =')
      READ(*,*)NUMC
      IF ((NUMC.NE.1).OR.(NUMC.NE.2)) GOTO 2000
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
```

```
C     Output information to the dump file
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      WRITE(9,3000)FILEOUT
3000  FORMAT(//'     RESULTS OF PROGRAM C:\CODE.FOR'/,
     1 '---------------------------------------------------'/,
     2 ' A procedure for computing the first four '/,
     3 ' central moments of the sum or maximum of '/,
     4 ' two unimodal or bi-modal input activity times'/,
     5 '---------------------------------------------------'//,
     6 ' Name of output file= ',A20///,
     7 ' Input parameters and moments:'/,
     8 ' ----------------------------------------'//)
      IF (NUMC.EQ.1) THEN
      WRITE(9,3010)
3010  FORMAT(' THE TWO ACTIVITIES ARE IN SERIES')
      ELSE
      WRITE(9,3020)
3020  FORMAT(' THE TWO ACTIVITIES ARE IN PARALEL')
      ENDIF
      WRITE(9,3100)PI1,MU1,VAR1,SK1,KR1,MU2,VAR2,SK2,KR2,
     1             PI2,MU3,VAR3,SK3,KR3,MU4,VAR4,SK4,KR4
3100  FORMAT(' '/,
     1 ' First weight  = ',F15.5/,
     2 ' First  central moment of activity (1) = ',F15.5/,
     3 ' Second central moment of activity (1) = ',F15.5/,
     4 ' Third  central moment of activity (1) = ',F15.5/,
     5 ' Fourth central moment of activity (1) = ',F15.5/,
     6 ' First  central moment of activity (2) = ',F15.5/,
     7 ' Second central moment of activity (2) = ',F15.5/,
     8 ' Third  central moment of activity (2) = ',F15.5/,
     9 ' Fourth central moment of activity (2) = ',F15.5//,
     1 ' Second weight  = ',F15.5/,
     2 ' First  central moment of activity (3) = ',F15.5/,
     3 ' Second central moment of activity (3) = ',F15.5/,
     4 ' Third  central moment of activity (3) = ',F15.5/,
     5 ' Fourth central moment of activity (3) = ',F15.5/,
     6 ' First  central moment of activity (4) = ',F15.5/,
     7 ' Second central moment of activity (4) = ',F15.5/,
     8 ' Third  central moment of activity (4) = ',F15.5/,
     9 ' Fourth central moment of activity (4) = ',F15.5//)
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C     Calling a procedure of the sum or the maximum
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IF (NUMC.EQ.1) THEN
      CALL SUMACT
      ELSE
      CALL PARACT
```

```
      ENDIF
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      WRITE(9,4000)MUX,VARX,SKX,KRX
4000  FORMAT(' OUTPUT INFORMATION'/
     1 '____________________________________________________________'/,
     2 ' THE FIRST  CENTRAL MOMENT ABOUT THE MEAN = ',F30.5/,
     3 ' THE SECOND CENTRAL MOMENT ABOUT THE MEAN = ',F30.5/,
     4 ' THE THIRD  CENTRAL MOMENT ABOUT THE MEAN = ',F30.5/,
     5 ' THE FOURTH CENTRAL MOMENT ABOUT THE MEAN = ',F30.5/,
     6 '____________________________________________________________')
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      CLOSE(9)
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      STOP
      END
CXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
CXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
C-----------------------------------------------------------------------
C-----------------------------------------------------------------------
C
C       SUMACT
C
C      A procedure for computing the first four central moments of
C      two activities are arranged in series.
C
C-----------------------------------------------------------------------
C-----------------------------------------------------------------------
      SUBROUTINE SUMACT
      IMPLICIT DOUBLE PRECISION (K-Z)
      COMMON /COM0/ PI1,MU1,VAR1,SK1,KR1,MU2,VAR2,SK2,KR2,
     1              PI2,MU3,VAR3,SK3,KR3,MU4,VAR4,SK4,KR4
      COMMON /COM1/ MUX,VARX,SKX,KRX
C--------------------------------------------------------------
C      Initialise the output moments
C--------------------------------------------------------------
      MUX  = 0.0D0
      VARX = 0.0D0
      SKX  = 0.0D0
      KRX  = 0.0D0
C--------------------------------------------------------------
C Computing the first four moments around the origin
C--------------------------------------------------------------
      MUP21 = VAR1 + MU1**2
      MUP22 = VAR2 + MU2**2
      MUP23 = VAR3 + MU3**2
      MUP24 = VAR4 + MU4**2
C------------
```

```
      MUP31 = SK1+3.0D0*MUP21*MU1-2.0D0*MU1**3
      MUP32 = SK2+3.0D0*MUP22*MU2-2.0D0*MU2**3
      MUP33 = SK3+3.0D0*MUP23*MU3-2.0D0*MU3**3
      MUP34 = SK4+3.0D0*MUP24*MU4-2.0D0*MU4**3
C-----------
      MUP41 = KR1+4.0D0*MUP31*MU1-6.0D0*MUP21*(MU1**2)+3.0D0*(MU1**4)
      MUP42 = KR2+4.0D0*MUP32*MU2-6.0D0*MUP22*(MU2**2)+3.0D0*(MU2**4)
      MUP43 = KR3+4.0D0*MUP33*MU3-6.0D0*MUP23*(MU3**2)+3.0D0*(MU3**4)
      MUP44 = KR4+4.0D0*MUP34*MU4-6.0D0*MUP24*(MU4**2)+3.0D0*(MU4**4)
C------------------------------------------------------------
C Computing the weighted moments
C------------------------------------------------------------
      MU1T1 = PI1*MU1+(1.0D0-PI1)*MU2
      MU2T1 = PI1*MUP21+(1.0D0-PI1)*MUP22-(MU1T1**2)
      MU3T1 = PI1*MUP31+(1.0D0-PI1)*MUP32-3.0D0*MU1T1*MU2T1-(MU1T1**3)
      MU4T1 = PI1*MUP41+(1.0D0-PI1)*MUP42-4.0D0*MU1T1*MU3T1
      MU4T1 = MU4T1-6.0D0*(MU1T1**2)*MU2T1-(MU1T1**4)
C-----------
      MU1T2 = PI2*MU3+(1.0D0-PI2)*MU4
      MU2T2 = PI2*MUP23+(1.0D0-PI2)*MUP24-(MU1T2**2)
      MU3T2 = PI2*MUP33+(1.0D0-PI2)*MUP34-3.0D0*MU1T2*MU2T2-(MU1T2**3)
      MU4T2 = PI2*MUP43+(1.0D0-PI2)*MUP44-4.0D0*MU1T2*MU3T2
      MU4T2 = MU4T2-6.0D0*(MU1T2**2)*MU2T2-(MU1T2**4)
C------------------------------------------------------------------
C     Computing the first four central moments of the distribution
C     of the sum.
C------------------------------------------------------------------
      MUX  = MU1T1 + MU1T2
      VARX = MU2T1 + MU2T2
      SKX  = MU3T1 + MU3T2
      KRX  = MU4T1 + MU4T2 + 6.0D0*MU2T1*MU2T2
C-----------------------------------------------
      RETURN
      END
CXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
CXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
C--------------------------------------------------------------------
C
C       PARACT
C
C     A procedure for computing the first four central moments of
C     two activities are arranged in parallel.
C
C--------------------------------------------------------------------
C--------------------------------------------------------------------
      SUBROUTINE PARACT
      IMPLICIT DOUBLE PRECISION (K-Z)
```

```
      COMMON /COM0/ PI1,MU1,VAR1,SK1,KR1,MU2,VAR2,SK2,KR2,
     1              PI2,MU3,VAR3,SK3,KR3,MU4,VAR4,SK4,KR4
      COMMON /COM1/ MUX,VARX,SKX,KRX
C-----------------------------------------------------------
C     INITIALIZING THE OUTPUT MOMENTS
C-----------------------------------------------------------
      MUX  = 0.0D0
      VARX = 0.0D0
      SKX  = 0.0D0
      KRX  = 0.0D0
C...................
      MU13  = 0.0D0
      VAR13 = 0.0D0
      SK13  = 0.0D0
      KR13  = 0.0D0
      MU14  = 0.0D0
      VAR14 = 0.0D0
      SK14  = 0.0D0
      KR14  = 0.0D0
      MU23  = 0.0D0
      VAR23 = 0.0D0
      SK23  = 0.0D0
      KR23  = 0.0D0
      MU24  = 0.0D0
      VAR24 = 0.0D0
      SK24  = 0.0D0
      KR24  = 0.0D0
C...................
      MU2P13 = 0.0D0
      MU2P14 = 0.0D0
      MU2P23 = 0.0D0
      MU2P24 = 0.0D0
      MU3P13 = 0.0D0
      MU3P14 = 0.0D0
      MU3P23 = 0.0D0
      MU3P24 = 0.0D0
      MU4P13 = 0.0D0
      MU4P14 = 0.0D0
      MU4P23 = 0.0D0
      MU4P24 = 0.0D0
C-----------------------------------------------------------
C     Possibility 1: Max [ UNIMODAL , UNIMODAL ]
C-----------------------------------------------------------
      IF ((PI1.EQ.1.0D0).AND.(PI2.EQ.1.0D0)) THEN
      CALL MAXP1(MU1,MU3,VAR1,VAR3,SK1,SK3,KR1,KR3,MU13,VAR13,SK13,KR13)
      GOTO 1000
      ENDIF
```

```
C-------------------------------------------------------------
C      Possibility 2: Max [ BIMODAL , UNIMODAL ]
C-------------------------------------------------------------
       IF ((PI1.NE.1.0D0).AND.(PI2.EQ.1.0D0)) THEN
       CALL MAXP1(MU1,MU3,VAR1,VAR3,SK1,SK3,KR1,KR3,MU13,VAR13,SK13,KR13)
       CALL MAXP1(MU2,MU3,VAR2,VAR3,SK2,SK3,KR2,KR3,MU23,VAR23,SK23,KR23)
       GOTO 1000
       ENDIF
C-------------------------------------------------------------
C      Possibility 3: Max [ UNIMODAL , BIMODAL ]
C-------------------------------------------------------------
       IF ((PI1.EQ.1.0D0).AND.(PI2.NE.1.0D0)) THEN
       CALL MAXP1(MU1,MU3,VAR1,VAR3,SK1,SK3,KR1,KR3,MU13,VAR13,SK13,KR13)
       CALL MAXP1(MU1,MU4,VAR1,VAR4,SK1,SK4,KR1,KR4,MU14,VAR14,SK14,KR14)
       GOTO 1000
       ENDIF
C-------------------------------------------------------------
C      Possibility 4: Max [ BIMODAL , BIMODAL ]
C-------------------------------------------------------------
       IF ((PI1.NE.1.0D0).AND.(PI2.NE.1.0D0)) THEN
       CALL MAXP1(MU1,MU3,VAR1,VAR3,SK1,SK3,KR1,KR3,MU13,VAR13,SK13,KR13)
       CALL MAXP1(MU1,MU4,VAR1,VAR4,SK1,SK4,KR1,KR4,MU14,VAR14,SK14,KR14)
       CALL MAXP1(MU2,MU3,VAR2,VAR3,SK2,SK3,KR2,KR3,MU23,VAR23,SK23,KR23)
       CALL MAXP1(MU2,MU4,VAR2,VAR4,SK2,SK4,KR2,KR4,MU24,VAR24,SK24,KR24)
       ENDIF
1000   CONTINUE
C-------------------------------------------------------------
C Computing the first four moments around the origin
C-------------------------------------------------------------
       MU2P13 = VAR13+(MU13**2)
       MU2P14 = VAR14+(MU14**2)
       MU2P23 = VAR23+(MU23**2)
       MU2P24 = VAR24+(MU24**2)
C...................
       MU3P13 = SK13+3.0*MU2P13*MU13-2.0D0*(MU13**3)
       MU3P14 = SK14+3.0*MU2P14*MU14-2.0D0*(MU14**3)
       MU3P23 = SK23+3.0*MU2P23*MU23-2.0D0*(MU23**3)
       MU3P24 = SK24+3.0*MU2P24*MU24-2.0D0*(MU24**3)
C...................
       MU4P13 = KR13+4.0D0*MU3P13*MU13-6.0D0*MU2P13*(MU13**2)
       MU4P13 = MU4P13+3.0D0*(MU13**4)
       MU4P14 = KR14+4.0D0*MU3P14*MU14-6.0D0*MU2P14*(MU14**2)
       MU4P14 = MU4P14+3.0D0*(MU14**4)
       MU4P23 = KR23+4.0D0*MU3P23*MU23-6.0D0*MU2P23*(MU23**2)
       MU4P23 = MU4P23+3.0D0*(MU23**4)
       MU4P24 = KR24+4.0D0*MU3P24*MU24-6.0D0*MU2P24*(MU24**2)
       MU4P24 = MU4P24+3.0D0*(MU24**4)
```

```
C-------------------------------------------------------------
C Computing the weighted moments
C-------------------------------------------------------------
      MUX  = PI1*PI2*MU13+PI1*(1.0D0-PI2)*MU14
      MUX  = MUX+PI2*(1.0D0-PI1)*MU23+(1.0D0-PI1)*(1.0D0-PI2)*MU24
C..........
      VARX = PI1*PI2*MU2P13+PI1*(1.0D0-PI2)*MU2P14
      VARX = VARX+PI2*(1.0D0-PI1)*MU2P23
      VARX = VARX+(1.0D0-PI1)*(1.0D0-PI2)*MU2P24-(MUX**2)
C..........
      SKX  = PI1*PI2*MU3P13+PI1*(1.0D0-PI2)*MU3P14
      SKX  = SKX+PI2*(1.0D0-PI1)*MU3P23
      SKX  = SKX+(1.0D0-PI1)*(1.0D0-PI2)*MU3P24
      SKX  = SKX-3.0D0*MUX*VARX-(MUX**3)
C..........
      KRX  = PI1*PI2*MU4P13+PI1*(1.0D0-PI2)*MU4P14
      KRX  = KRX+PI2*(1.0D0-PI1)*MU4P23
      KRX  = KRX+(1.0D0-PI1)*(1.0D0-PI2)*MU4P24
      KRX  = KRX-4.0D0*MUX*SKX-6.0D0*(MUX**2)*VARX-(MUX**4)
C..........
      RETURN
      END
CXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
CXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
C---------------------------------------------------------------------
C---------------------------------------------------------------------
C
C        MAXP1
C
C      A procedure for computing the first four central moments of
C      two unimodal activities are arranged in parallel.
C
C
C---------------------------------------------------------------------
C---------------------------------------------------------------------
      SUBROUTINE MAXP1(M1IN,M2IN,V1IN,V2IN,SK1IN,SK2IN,KR1IN,KR2IN,
     1                 MOUT,VOUT,SKOUT,KROUT)
      IMPLICIT DOUBLE PRECISION (K-Z)
      INTEGER I,J,IC1,IC2
      DOUBLE PRECISION C1,C2
C---------------------------------------------
C      COMPUTING C1 & C2
C---------------------------------------------
      C1=(M1IN**2)/V1IN
      C2=(M2IN**2)/V2IN
      IC1=IDINT(C1)
      IC2=IDINT(C2)
```

```
      IF (IC1.LT.1) THEN
      IC1=1
      ENDIF
      IC2=IDINT(C2)
      IF (IC2.LT.1) THEN
      IC2=1
      ENDIF
C     WRITE(9,100)C1,C2,IC1,IC2
C100  FORMAT(' C1   = ',F15.10,'  C2 = ',F15.10/,
C     1        ' IC1 = ',I10,' IC2 = ',I10)
C--------------------------------------------
C     COMPUTING P AND Q
C--------------------------------------------
      P = (M2IN*C1) / (M2IN*C1+M1IN*C2)
      Q = 1.0D0 - P
C     WRITE(9,200)P,Q
C200  FORMAT(' P = ',F15.10,'   Q = ',F15.10)
C--------------------------------------------
C     COMPUTING THE FIRST MOMENT
C--------------------------------------------
      PART1X=M1IN
C     WRITE(9,300)PART1X
C300  FORMAT(' PART1X=',F20.10)
C.........
      PART2X=M2IN
C     WRITE(9,400)PART2X
C400  FORMAT(' PART2X=',F20.10)
C.........
      PART3A=M1IN*(P**(IC1+1))
      PART3C=1.0D0
      DO 1000 I=1,IC2-1
      PART3B=1.0D0
      DO 1100 J=1,I
      PART3B=PART3B*Q*((DBLE(IC1)+DBLE(J))/DBLE(J))
1100  CONTINUE
      PART3C=PART3C+PART3B
1000  CONTINUE
      PART3X=PART3A*PART3C
C     WRITE(9,1150)PART3X
C1150  FORMAT(' PART3X=',F20.10)
C.........
      PART4A=M2IN*(Q**(IC2+1))
      PART4C=1.0D0
      DO 1200 I=1,IC1-1
      PART4B=1.0D0
      DO 1300 J=1,I
      PART4B=PART4B*P*((DBLE(IC2)+DBLE(J))/DBLE(J))
```

```
1300   CONTINUE
       PART4C=PART4C+PART4B
1200   CONTINUE
       PART4X=PART4A*PART4C
C      WRITE(9,1350)PART4X
C1350   FORMAT(' PART4X=',F20.10)
C..........
       MOUT=PART1X+PART2X-PART3X-PART4X
C      WRITE(9,1360)MOUT
C1360   FORMAT(' MOUT =',F20.10///)
C-------------------------------------------------------------------
C      COMPUTING THE SECOND MOMENT
C-------------------------------------------------------------------
       PART1X=((DBLE(IC1)+1.0D0)/DBLE(IC1))*M1IN**2
C      WRITE(9,1370)PART1X
C1370   FORMAT(' PART1X=',D30.10)
C..........
       PART2X=((DBLE(IC2)+1.0D0)/DBLE(IC2))*M2IN**2
C      WRITE(9,1375)PART2X
C1375   FORMAT(' PART2X=',F20.10)
C..........
       PART3A=((DBLE(IC1)+1.0D0)/DBLE(IC1))*(M1IN**2)*P**(IC1+2)
       PART3C=1.0D0
       DO 1400 I=1,IC2-1
       PART3B=1.0D0
       DO 1500 J=1,I
       PART3B=PART3B*Q*((DBLE(IC1)+DBLE(J)+1.0D0)/DBLE(J))
1500   CONTINUE
       PART3C=PART3C+PART3B
1400   CONTINUE
       PART3X=PART3A*PART3C
C      WRITE(9,1550)PART3X
C1550   FORMAT(' PART3X=',F20.10)
C..........
       PART4A=((DBLE(IC2)+1.0D0)/DBLE(IC2))*(M2IN**2)*(Q**(IC2+2))
       PART4C=1.0D0
       DO 1600 I=1,IC1-1
       PART4B=1.0D0
       DO 1700 J=1,I
       PART4B=PART4B*P*((DBLE(IC2)+DBLE(J)+1.0D0)/DBLE(J))
1700   CONTINUE
       PART4C=PART4C+PART4B
1600   CONTINUE
       PART4X=PART4A*PART4C
C      WRITE(9,1800)PART4X
C1800   FORMAT(' PART4X=',F20.10)
C..........
```

```
      VOUT=PART1X+PART2X-PART3X-PART4X
      VOUT=VOUT-(MOUT**2)
C     WRITE(9,1900)VOUT
C1900  FORMAT(' VOUT=',F20.10)
C-----------------------------------------------------------------
C-----------------------------------------------------------------
C-----------------------------------------------------------------
C     COMPUTING THE THIRD MOMENT
C-----------------------------------------------------------------
      PART1X=(DBLE(IC1)+1.0D0)*(DBLE(IC1)+2.0D0)/(DBLE(IC1)**2)
      PART1X=PART1X*(M1IN**3)
C     WRITE(9,3000)PART1X
C3000  FORMAT(' PART1X=',D30.10)
C..........
      PART2X=(DBLE(IC2)+1.0D0)*(DBLE(IC2)+2.0D0)/(DBLE(IC2)**2)
      PART2X=PART2X*(M2IN**3)
C     WRITE(9,3100)PART2X
C3100  FORMAT(' PART2X=',F20.10)
C..........
      PART3A=PART1X*P**(IC1+3)
      PART3C=1.0D0
      DO 3200 I=1,IC2-1
      PART3B=1.0D0
      DO 3300 J=1,I
      PART3B=PART3B*Q*((DBLE(IC1)+DBLE(J)+2.0D0)/DBLE(J))
3300  CONTINUE
      PART3C=PART3C+PART3B
3200  CONTINUE
      PART3X=PART3A*PART3C
C     WRITE(9,3400)PART3X
C3400  FORMAT(' PART3X=',F20.10)
C..........
      PART4A=PART2X*Q**(IC2+3)
      PART4C=1.0D0
      DO 3500 I=1,IC1-1
      PART4B=1.0D0
      DO 3600 J=1,I
      PART4B=PART4B*P*((DBLE(IC2)+DBLE(J)+2.0D0)/DBLE(J))
3600  CONTINUE
      PART4C=PART4C+PART4B
3500  CONTINUE
      PART4X=PART4A*PART4C
C     WRITE(9,3700)PART4X
C3700  FORMAT(' PART4X=',F20.10)
C..........
      PART5X = PART1X+PART2X-PART3X-PART4X
      PART6X = (3.0D0*MOUT*VOUT)+(MOUT**3)
```

```
C      WRITE(9,3750)PART5X,PART6X
C3750  FORMAT(' PART5X=',F20.10/,' PART6X=',F20.10)
       SKOUT  = PART5X - PART6X
C      WRITE(9,3800)SKOUT
C3800  FORMAT(' SKOUT=',F20.10)
C-----------------------------------------------------------------
C-----------------------------------------------------------------
C      COMPUTING THE FOURTH MOMENT
C-----------------------------------------------------------------
       PART1X=(DBLE(IC1)+1.0D0)*(DBLE(IC1)+2.0D0)*(DBLE(IC1)+3.0D0)
       PART1X=PART1X/(DBLE(IC1)**3)
       PART1X=PART1X*(M1IN**4)
C      WRITE(9,4000)PART1X
C4000  FORMAT(' PART1X=',D30.10)
C.........
       PART2X=(DBLE(IC2)+1.0D0)*(DBLE(IC2)+2.0D0)*(DBLE(IC2)+3.0D0)
       PART2X=PART2X/(DBLE(IC2)**3)
       PART2X=PART2X*(M2IN**4)
C      WRITE(9,4100)PART2X
C4100  FORMAT(' PART2X=',F20.10)
C.........
       PART3A=PART1X*P**(IC1+4)
       PART3C=1.0D0
       DO 4200 I=1,IC2-1
       PART3B=1.0D0
       DO 4300 J=1,I
       PART3B=PART3B*Q*((DBLE(IC1)+DBLE(J)+3.0D0)/DBLE(J))
4300   CONTINUE
       PART3C=PART3C+PART3B
4200   CONTINUE
       PART3X=PART3A*PART3C
C      WRITE(9,4400)PART3X
C4400  FORMAT(' PART3X=',F20.10)
C.........
       PART4A=PART2X*Q**(IC2+4)
       PART4C=1.0D0
       DO 4500 I=1,IC1-1
       PART4B=1.0D0
       DO 4600 J=1,I
       PART4B=PART4B*P*((DBLE(IC2)+DBLE(J)+3.0D0)/DBLE(J))
4600   CONTINUE
       PART4C=PART4C+PART4B
4500   CONTINUE
       PART4X=PART4A*PART4C
C      WRITE(9,4700)PART4X
C4700  FORMAT(' PART4X=',F20.10)
C.........
```

```
      PART5X = PART1X+PART2X-PART3X-PART4X
      PART6X = 4.0D0*MOUT*SKOUT+6.0D0*(MOUT**2)*VOUT+(MOUT**4)
C     WRITE(9,4750)PART5X,PART6X
C4750  FORMAT(' PART5X=',F20.10/,' PART6X=',F20.10)
      KROUT  = PART5X - PART6X
C     WRITE(9,4800)KROUT
C4800  FORMAT(' KROUT=',F20.10)
C-------------------------------------------------------------
      RETURN
      END
C-------------------------------------------------------------
CXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
CXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
```