Stochastic and Asymptotic Analysis Applied to the Study of Stochastic Models of Classical and Quantum Mechanics

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Preface

Applying WKB method we obtain multiplicative small time and semiclassical asymptotics for Green functions (fundamental solutions) and for the solutions of Cauchy problem for the stochastic heat equation driven by a Lévy noise. The relevant theory of stochastic Hamilton systems and Hamilton-Jacobi equations is developed.

We also give conditions for non-explosion of solutions of Newton systems driven by a Lévy noise and conditions for transience of solutions of such systems driven by α-stable noise. As a solution of particular Newton system we consider α-stable Ornstein-Uhlenbeck process for which we estimate the rate of escape. The connections between the objects studied in this theses are shown on the scheme at page V.

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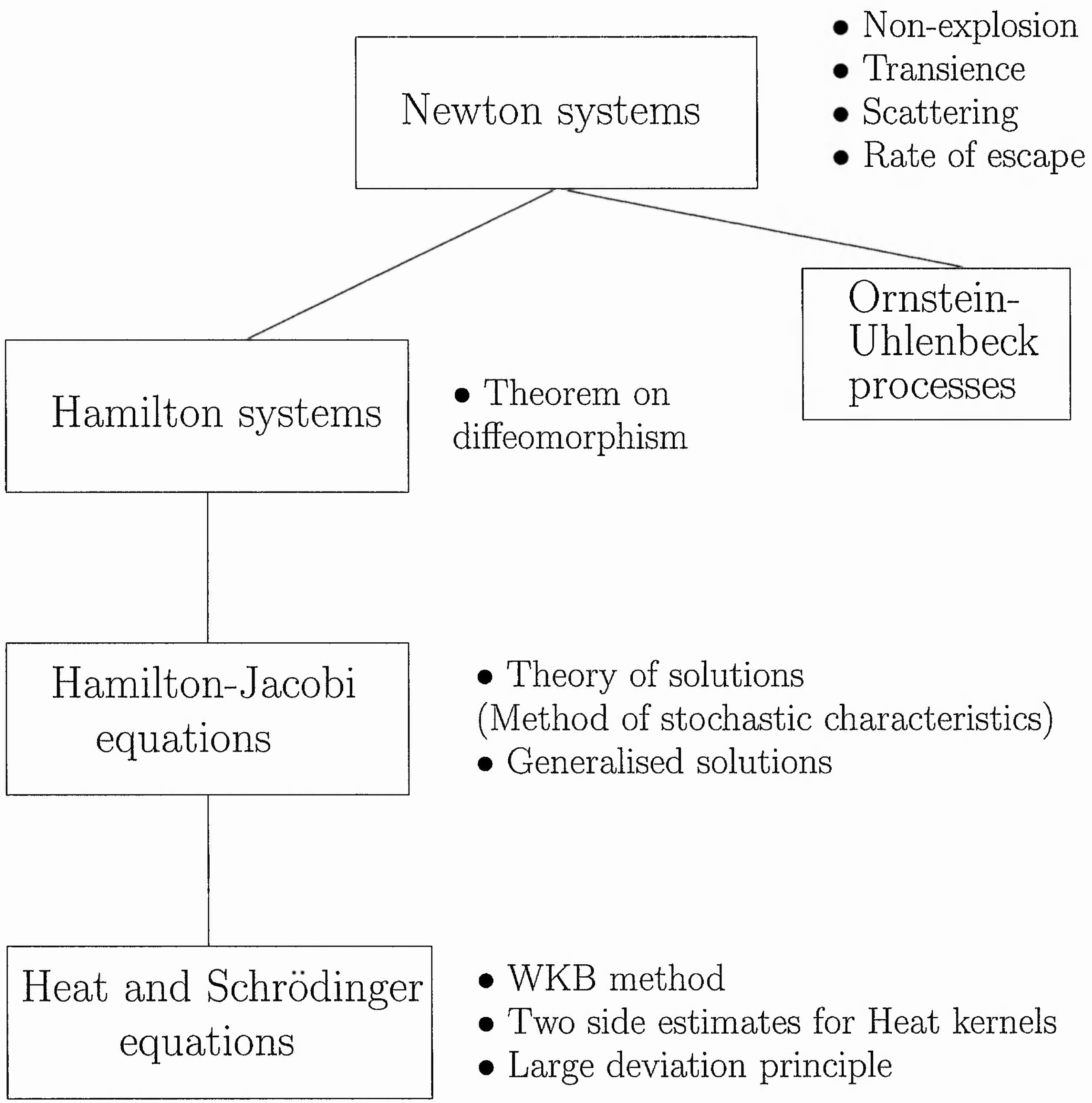
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Introduction

Recent years saw the series of papers [K1], [K2], [AK], [AHK1], [AHK2] devoted to the qualitative study of the Newton equations driven by random noise (see also [AHZ], [MW], [No1], [No2] [AKl], [KuMar] and references therein for related results). On one hand, these equations are interesting on its own, for example, as models for dynamics of particles moving in random media (see e.g. [Ne]), in the theory of interacting particles (see e.g. [OV], [OVY]) and in the theory of random matrices (see e.g. [Me]). On the other hand, the study of this equations serve as an important tool for studying partial differential equations, in particular Hamilton-Jacobi, Heat, Schrödinger equations, driven by random noise (see [TrZ1], [TrZ2], [K4], [K5], [K6], [K7]).

The papers mentioned above were mostly concerned with the case where the driving noise was the standard Wiener process. It is known, however, that exponentially decreasing tails of normal distribution are not adequate for describing a variety of processes appearing in science, engineering and economics. The natural generalisation of normal distribution which also appear as the limits of sum of i.i.d. random variables but at the same time have fat tails (decreasing polynomially and not exponentially at infinity) is given by the class of stable laws. This leads to the study of random models given by a stable or even more general Lévy noises. At the same time the Lévy process having both a rich probabilistic structure and a clear analytic representation constitute a natural intermediate class of processes between Wiener processes and general semimartingale. The latter are also relevant for physical applications, see e.g. [BaH], where a general class of linear stochastic second order equations driven by semimartingales was found that preserves a.s. the L^2-norm of a solution. This class describes general stochastic models of continuous quantum measurement.

In this thesis we give the conditions for non-explosion of the solutions of the Newton system

$$\begin{cases} dx &= p\,dt \\ dp &= -\frac{\partial V}{\partial x}\,dt - \frac{\partial c}{\partial x}\,d\xi_t, \end{cases} \tag{0.1}$$

where $\xi_t = (\xi_t^1, \dots, \xi_t^d)$ is a Lévy process, $d \geqslant 1$, $c \in C^2(\mathbb{R}^d, \mathbb{R}^d)$, $\partial c/\partial x$ is uniformly bounded, $V \in C^2(\mathbb{R}^d)$, $V \geqslant 0$. We also give conditions for transience of solutions of (0.1) when $d \geqslant 3$, $\xi_t = w_\alpha(t)$ is a general α-stable noise.

We proceed with one particular but important case of Newton system. Position $x(t)$ of a Newtonian particle driven by the white noise force is described by the system of stochastic equations

$$\begin{cases} dx &= v\,dt \\ dv &= dw(t), \end{cases} \tag{0.2}$$

where $x \in \mathbb{R}^d$ is the position of the particle, $v \in \mathbb{R}^d$ is its velocity and w is the standard d-dimensional Wiener process. Allowing of a linear friction force in this model leads to the equation

$$\begin{cases} dx &= v\,dt \\ dv &= -\beta v\,dt + dw(t), \end{cases} \tag{0.3}$$

where $\beta > 0$ is some constant. Processes $x(t)$ and $v(t)$ satisfying (0.3) are called the position Ornstein-Uhlenbeck (OU) process and the velocity OU process respectively. Notice that though the pair $(x(t), v(t))$ is a Markov process, the position of the particle $x(t)$ is already a non Markovian process. In [AK], [K1], [K2] the rate of escape of the position process $x(t)$ described by (0.2) or (0.3) was estimated. For the case $\beta = 0$ these estimates later on were essentially improved in [KhSh]. Here we generalise these results to the case of the general stable noise $w_\alpha, \alpha \in (0, 2)$, instead of the normal Gaussian noise $w = w_2$ above. In particular, we prove that for an increasing positive function $f(t)$ such that $f(t) = o(t^{1+\frac{1}{\alpha}})$ and $t/f(t) = o(1)$ as $t \to \infty$ and

$$\int_1^\infty (f(t)t^{-(1+\frac{1}{\alpha})})\,dt < \infty$$

one has

$$\lim_{t\to\infty} \frac{|x(t)|}{f(t)} = \infty \quad \text{a.s},$$

where $(x(t), v(t))$ is a solution of (0.2) with $w = w_\alpha$. The analogous result holds for the solutions of (0.3) with $w = w_\alpha$. These results are used to construct the scattering theory for the system

$$\begin{cases} dx &= v\,dt \\ dv &= (K(x) - \beta v)\,dt + dw_\alpha(t), \end{cases} \tag{0.4}$$

where the deterministic force K is considered as giving a perturbation of the free motion described by equation (0.2) or (0.3). We prove, in particular, the existence of random wave operator $\Omega_{w_\alpha} : (x(0), v(0)) \to (\widetilde{x}(0), \widetilde{v}(0))$, which assign to the initial conditions $(x(0), v(0))$ of any solution (x, v) of equation (0.3) with $w = w_\alpha$ the initial conditions $(\widetilde{x}(0), \widetilde{v}(0))$ of some solution $(\widetilde{x}, \widetilde{v})$ of equation (0.4) such that $||(x, v) - (\widetilde{x}, \widetilde{v})|| \to 0$ as $t \to \infty$.

Coming back to Newton systems driven by Lévy noise we study the existence of solutions for boundary value problem for the system

$$\begin{cases} dx &= \frac{\partial H}{\partial p}\,dt \\ dp &= -\frac{\partial H}{\partial x}\,dt - \frac{\partial c}{\partial x}\,d\xi_t. \end{cases} \tag{0.5}$$

with $H = (p^2/2) - V(x)$. Observe that well-posedness of boundary value problem is equivalent to the statement (which we call theorem on diffeomorphism) that the map $p_0 \to X(t, t_0, x_0, p_0)$ (where $X(t, t_0, x_0, p_0)$ is a solution of (0.5) with initial conditions (x_0, p_0) at time $t = t_0$) is a diffeomorphism. Boundary value problems for Hamilton systems of type (0.5) with $\{\xi_t\}_{t\geqslant 0}$ being a Wiener process and their connections with the calculus of variations were investigated in [K4]. However, the proof of the existence and uniqueness of the solution of the boundary value problem was only sketched in [K4]. In this thesis we give complete proofs of the corresponding results for Hamilton systems driven by Lévy noise without a Brownian part.

An important tool for the analysis of the behaviour of the solutions for Hamilton systems is the study of their linearised approximations (equation in variations). These linearised approximations turn out to be linear non-homogeneous Hamilton systems. Using perturbation theory we can derive a representation of the solutions of such linear systems as series of multiple stochastic integrals. In order to prove the convergence of these series, we are led to obtaining estimates for multiple stochastic integrals. We use these estimates as auxiliary tools for the study of linear stochastic Hamilton systems. However we believe that they are of independent value. Let us mention here the paper [Ta], where a rather general linear system driven by Brownian motion was considered, convergence of the series from perturbations theory proved, and necessary estimates for multiple integrals obtained. Multiple stochastic integrals with respect to general semimartingales or infinitely divisible processes were also considered, see e.g. [KwW], [Sz] and references given there.

The solutions of the boundary value problem for equation (0.5) is closely connected to the solutions of the Cauchy problem for Hamilton-Jacobi-Bellman (HJB for short) equations. Over the last few years interest in stochastic HJB equations has increased, see e.g. the papers [R], [So], [DaPDe] and references given there. The HJB equations are important as they describe the evolution of optimally controlled systems with random dynamics, but they are also useful tools when studying various classes of stochastic models in probability theory and mathematical physics. Presently, the notion of *stochastic HJB equation* is used in two different contexts: firstly, for classical differential equations with a random Hamiltonian and, secondly, for truly stochastic differential equations where the Hamiltonian includes a non-homogeneous semimartingale term which does not allow to write down the corresponding equation in classical form.

In the sequel we will consider the second type of HJB equations, that is to say equations of the form

$$dS + H\left(x, \frac{\partial S}{\partial x}\right) dt + c(x)\, d\xi_t = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \tag{0.6}$$

where $H : \mathbb{R}^{2d} \to \mathbb{R}$ and $c : \mathbb{R}^d \to \mathbb{R}^d$ are smooth functions and ξ_t is a stochastic process (driving noise) in $\mathbb{R}^d$. The equation (0.6) with $\{\xi_t\}_{t\geqslant 0}$ being a Wiener process was considered in [K2], [K4], and [TrZ1], [TrZ2] for various classes of real H and c. The corresponding case of complex valued H and c was taken up in [K5]. Our objective is to study the case of equation (0.6) with $\{\xi_t\}_{t\geqslant 0}$ being a Lévy noise without Brownian part and to develop a stochastic analogue of the theory of classical (i.e. smooth in x) solutions of the Cauchy problem for equation (0.6). Generalised solutions can then be constructed in the same way as they are constructed for the case of a Wiener process $\{\xi_t\}_{t\geqslant 0}$ in [K2], [K4] (see also [KMa]).

We next apply theory of stochastic Hamilton systems and stochastic Hamilton-Jacobi equations developed above to the study of stochastic heat equations. More precisely, we consider the equation

$$h\, d_t\psi(t,x) = \tag{0.7}$$
$$= -\left(-\frac{h^2}{2}\mathrm{tr}\frac{\partial^2}{\partial x^2} + V(x) + ha(x)\right)\psi(t,x)\, dt + h\psi(t-,x)c(x)\, d\xi_t,$$

where $a, V : \mathbb{R}^d \to \mathbb{R}$, $c = (c_1, \dots, c_d) : \mathbb{R}^d \to \mathbb{R}^d$, h is a positive parameter, and assume that $c(x)y \geqslant 0$ for all $x \in \mathbb{R}^d$, $y \in \operatorname{supp}\nu$, where ν is a Lévy measure of process $\{\xi_t\}_{t\geqslant 0}$. Applying WKB method we obtain multiplicative small time and semiclassical asymptotics for the Green function and for the solutions of the Cauchy problem for equation (0.7). The first step in this construction consists in solving the corresponding stochastic Hamilton-Jacobi equation

$$dS + \frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 dt - V\, dt - ha\, dt + hc\, d\xi_t = 0, \tag{0.8}$$

which constitutes the "classical part" of the semiclassical approximation.

In deterministic case asymptotics for the Green function of heat and Schrödinger equation is well known (see e.g. [K5], [Ma], [MaF] and references given there).

Stochastic Schrödinger and heat equations appear naturally in stochastic filtering [Za], quantum stochastic filtering, quantum measurement and more generally in the theory of open quantum systems (see e.g. [Bel1], [Bel2], [BelHiHu], [Di], [Q]). Here we consider only heat equations. The application of the methods developed here to the case of stochastic Schrödinger equations will be considered elsewhere. It seems also possible to apply the methods developed in this paper to the construction of asymptotics for the Burgers equation driven by Lévy noise, since, as is well known, the (nonlinear) Burgers equation can be reduced to a standard heat equation by simple change of the variables. The case of the Burgers equation driven by Wiener process was considered in [TrZ3]. The case of heat equation driven by Wiener process was studied in [K2], [K4], [TrZ1], [TrZ2]. We generalise the known results on stochastic heat equations driven by Wiener noise to the case of Lévy processes. Some statements of this thesis are valid also for general semimartingale noises.

One of the central features of Lévy processes that distinguish them from diffusion processes is the possibility of jumps of their trajectories. These jumps complicate the analysis essentially. The formulae for the leading term of the asymptotics in Lévy case will contain an infinite product over the process of jumps that must be controlled when doing the relevant estimate. This is the reason why we can not find explicit solution even for a vanishing potential unlike the case of the heat equation driven by a standard Brownian motion (see [BelK], [K2], [TrZ1]).

Let us give a brief outline how this thesis is organised. In Chapter 1 we estimate the rate of escape of α-stable Ornstein-Uhlenbeck process, construct the scattering theory for perturbations and discuss properties of random wave operators. Chapter 2 is concerned with obtaining the conditions for non-explosion and transience for the solutions of system (0.1). In Chapter 3 we obtain estimates for stochastic multiple integrals. In Chapter 4 we study well-posedness of the boundary value problem for system (0.5) and construct the solutions of Cauchy problem for equation (0.6). The final Chapter focuses on constructing of small time and semiclassical asymptotics for Green function and for the solutions of Cauchy problem for equation (0.7).

Preliminaries

Our standard references for Lévy processes are the monographs by Bertoin [Ber1] and Sato [Sa]. For Lévy processes and stochastic calculus with jumps we use the books by Jacod and Shiryaev [JSh] and Protter [Pro]. We will collect a few definitions and results from these books.

A *Lévy process* (on $\mathbb{R}^d$) is a stochastic process $\{\xi_t\}_{t\geqslant 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with stationary and independent increments which is also stochastically continuous. We will assume that $\xi_0 = 0$ a.s. The state space will always be $\mathbb{R}^d$. We can (and will) choose a version that has right-continuous sample paths with everywhere finite left-hand limits (càdlàg, for short); if not otherwise mentioned, we will use the augmented canonical filtration of $\{\xi_t\}_{t\geqslant 0}$. The process $\{\xi_t\}_{t\geqslant 0}$ is uniquely (up to stochastic equivalence) determined through its Fourier transform,

$$\mathbb{E}e^{i\eta\xi_t} = e^{-t\psi(\eta)}, \qquad t > 0,\ \eta \in \mathbb{R}^d,$$

where the characteristic exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ is given by the Lévy-Khinchine representation

$$\psi(\eta) = i\ell\cdot\eta + \eta\cdot Q\eta + \int_{y\neq 0}\left(1 - e^{iy\cdot\eta} + \frac{iy\cdot\eta}{1+|y|^2}\right)\nu(dy). \tag{0.9}$$

Here, ℓ is some vector in $\mathbb{R}^d$, $Q \in \mathbb{R}^{d\times d}$ is a positive semi-definite matrix and ν is the Lévy or jump measure with support in $\mathbb{R}^d \setminus \{0\}$ such that $\int_{y\neq 0}|y|^2 \wedge 1\,\nu(dy) < \infty$. The Lévy-Khinchine formula is actually a one-to-one correspondence between the function ψ and the *Lévy triplet* (ℓ, Q, ν).

Stochastically, the Lévy-Khinchine representation translates into a path decomposition of the process $\{\xi_t\}_{t\geqslant 0}$. Fix some Borel set $A \subset \mathbb{R}^d \setminus \{0\}$, and write $N_t(\omega, A)$ for the Poisson point process with intensity measure $\nu(A)$. It is known that $N_t(\omega, A)$ describes jumps of ξ_t with sizes contained in A and we get

$$\xi_t(\omega) = \alpha t + B_t(\omega) + M_t(\omega) + J_t(\omega), \tag{0.10}$$

where $\alpha = \mathbb{E}\left(\xi_1 - \int_{|y|\geqslant 1} x\, N_1(\omega, dy)\right)$ is the drift coefficient, B_t is a d-dimensional Wiener process with (possibly degenerate) covariance matrix Q,

$$M_t(\omega) = \int_{|y|<1} y\,(N_t(\omega, dy) - t\nu(dy))$$

is a martingale which is the compensated sum of all small jumps (modulus less than 1), and

$$J_t(\omega) = \sum_{0<s\leqslant t} \Delta\xi_s\, \mathbf{1}_{\{|\Delta\xi_s|\geqslant 1\}}$$

is the sum of all big jumps (modulus greater than 1). As usual, we write $\Delta\xi_s = \xi_s - \xi_{s-} = \xi_s - \lim_{r\uparrow s}\xi_r$ for the jump at time $s > 0$. Note that J_t is a process of bounded variation on compact time-intervals. This is the case since càdlàg paths can have only finitely many jumps of size $\geqslant 1$ on any finite time interval. The above decomposition of ξ_t shows that Lévy processes are semimartingales and, therefore, good stochastic integrators.

The following two formulae for point processes hold whenever the right-hand side is finite:

$$\mathbb{E}\left(\int_A f(y)\, N_t(\bullet, dy)\right) = t\int_A f(y)\,\nu(dy)$$

and

$$\mathbb{E}\left(\left\{\int_A f(y)(N_t(\bullet, dy) - t\nu(dy))\right\}^2\right) = t\int_A f(y)^2\,\nu(dy). \tag{0.11}$$

In particular, we get

$$\mathbb{E}\left(\sum_{s\leqslant t} f(\Delta\xi_s)\right) = t\int f(y)\,\nu(dy) \tag{0.12}$$

for finite right-hand sides. It is not hard to see that

$$t \mapsto \xi_t \quad \text{has a.s. finite variation if and only if} \quad \int_{0<|y|<1} |y|\,\nu(dy) < \infty$$

and that

$$\mathbb{E}|\xi_t| < \infty \quad \text{if and only if} \quad \int_{|y|\geqslant 1} |y|\,\nu(dy) < \infty.$$

If ξ_t has a.s. bounded jumps, i.e., if the support of ν is a bounded set, ξ_t has absolute moments of any order.

Most of our notation should be standard or self-explanatory. All stochastic integrals are Itô-integrals and our main reference texts for stochastic integrals with jumps are Jacod and Shiryaev [JSh] and Protter [Pro].

We will also need the following simple Lemma. Since we could not find a precise reference for it, we include a short proof.

Lemma 0.0.1. *Let $\{\xi_t\}_{t\geqslant 0}$, $\xi = (\xi_1, \ldots, \xi_d)$ be a Lévy process with $Q = 0$ and Lévy measure ν satisfying $\int\limits_{|y|>1} |y|^2\,\nu(dy) < \infty$. For any $0 < \varepsilon < \frac{1}{2}$ we find a stopping time $\mathcal{R}_\varepsilon(\omega) < 1$ such that*

$$\vartheta_t = 2\sum_{i=1}^{d}\left(\sup_{\tau\in[0,t]} |\xi_{i,\tau}| + ([\xi_i,\xi_i]_t)^{\frac{1}{2}}\right) < t^{\frac{1}{2}-\varepsilon}$$

holds for all $t < \mathcal{R}_\varepsilon$, where $\mathbb{P}(\mathcal{R}_\varepsilon > 0) = 1$. In particular, one can find a stopping time $\mathcal{R} > 0$ a.s. such that for all $t < \mathcal{R}$

$$\vartheta_t = 2\sum_{i=1}^{d}\left(\sup_{\tau\in[0,t]} |\xi_{i,\tau}| + ([\xi_i,\xi_i]_t)^{\frac{1}{2}}\right) < 1. \tag{0.13}$$

Remark. Lemma 0.0.1 remains valid if $Q \neq 0$. Since we do not need this result, we settle for the case $Q = 0$ and the somewhat simpler proof.

Proof. As usual we write $\xi_t^* = \sup_{\tau\in[0,t]} |\xi_\tau|$. Since $Q = 0$, we get from (0.10)

$$\mathbb{E}\left(\{\xi_t^*\}^2\right) \leqslant 3\left[|\alpha|^2 t^2 + \mathbb{E}\left(\{M_t^*\}^2\right) + \mathbb{E}\left(\{J_t^*\}^2\right)\right].$$

From (0.12) we see that

$$\mathbb{E}\left(\{J_t^*\}^2\right) \leqslant t \int\limits_{|x|\geqslant 1} |x|^2\,\nu(dx)$$

and Doob's martingale inequality and (0.11) give

$$\mathbb{E}\left(\{M_t^*\}^2\right) \leqslant 4\mathbb{E}\left(M_t\right)^2 = 4t \int\limits_{|x|<1} |x|^2\,\nu(dx).$$

Formula (0.12) implies that

$$\mathbb{E}([\xi,\xi]_t) = t \int\limits_{|x|>0} |x|^2 \, \nu(dx).$$

Thus, the process

$$\zeta_t = \{\xi_t^*\}^2 + [\xi,\xi]_t$$

satisfies $\mathbb{E}\zeta_t < Ct$, where $C = C(\nu) > 0$ is a constant and $t < 1$. By Chebyshev's inequality

$$\mathbb{P}(\zeta_t > R) \leqslant \frac{\mathbb{E}(\zeta_t)}{R} \leqslant \frac{Ct}{R}.$$

Choosing $t = 2^{-k}$ and $R = (8d)^{-1}2^{-(1-\varepsilon)k}$ we find

$$\sum_{k=1}^{\infty} \mathbb{P}\left\{\zeta_{2^{-k}} > (8d)^{-1}2^{-(1-\varepsilon)k}\right\} \leqslant 8dC \sum_{k=1}^{\infty} 2^{-\varepsilon k} < \infty.$$

The Borel-Cantelli Lemma implies that

$$\zeta_{2^{-k}} \leqslant (8d)^{-1}2^{-(1-\varepsilon)k} \quad \text{for} \quad k > k_0(\omega) \quad \text{for some} \quad k_0(\omega) \in \mathbb{N}.$$

Set $k_1(\omega) = k_0(\omega) \vee \left[\frac{(1-2\varepsilon)}{\varepsilon} + 1\right]$. Then $(1-2\varepsilon)\frac{k+1}{k} \leqslant 1-\varepsilon$ for $k > k_1(\omega)$. If $2^{-(k+1)} \leqslant t < 2^{-k}$ for some $k > k_1(\omega)$ we find, as $\tau \to \zeta_\tau$ is an increasing function,

$$(8d)\zeta_t \leqslant (8d)\zeta_{2^{-k}} \leqslant 2^{-(1-\varepsilon)k} \leqslant \left(2^{-(k+1)}\right)^{\frac{(1-\varepsilon)k}{k+1}} \leqslant \left(2^{-(k+1)}\right)^{(1-2\varepsilon)} \leqslant t^{(1-2\varepsilon)}.$$

Using the elementary inequality $(a_1 + \ldots + a_{2d})^2 \leqslant (2d)(a_1^2 + \ldots + a_{2d}^2)$ we get with $\mathcal{R}_\varepsilon = 2^{-k_1(\omega)}$

$$\vartheta_t^2 \leqslant (8d)\zeta_t \leqslant t^{1-2\varepsilon}, \qquad \forall\, t < \mathcal{R}_\varepsilon,$$

and the lemma follows. □

Chapter 1

The rate of escape of stable Ornstein-Uhlenbeck processes and the scattering theory for their perturbations

1.1 Stable O.-U. processes and the scattering theory for their perturbations

General symmetric stable Lévy motion with the index of stability $\alpha \in (0,2)$ can be defined as the time homogeneous and space homogeneous stochastic process w_α with the transition probability density $p_{w_\alpha}(x,t)$, whose characteristic function has the form

$$\widetilde{p}_{w_\alpha}(q,t) = \exp\left\{-t\Lambda^\alpha(q)\right\}, \tag{1.1}$$

where

$$\Lambda^\alpha(q) = \int\limits_{S^{d-1}} |(q,n)|^\alpha \mu(dn), \tag{1.2}$$

μ is a finite symmetric Borelian measure on S^{d-1} (see e.g. [ST]). We shall further assume that

$$C_1 \leqslant \Lambda(q/|q|) \leqslant C_2 \quad \text{for some} \quad C_1, C_2 > 0 \quad \text{and} \quad \Lambda(\cdot) \in C^{2d+3}(\mathbb{R}^d \setminus 0). \tag{1.3}$$

In order to ensure the last condition it is sufficient to suppose that μ has a smooth density with respect to Lebesgue measure. For example the case of the uniform spectral measure μ satisfies all the assumptions. The stable Ornstein – Uhlenbeck process (x, v) is the solution to the system

$$\begin{cases} dx &= v\,dt \\ dv &= -\beta v\,dt + dw_\alpha(t), \end{cases} \tag{1.4}$$

where $\beta \geqslant 0$ is a constant. In other words, v and x can be expressed as the integrals of the stable Lévy motion w_α by the formulae

$$v(t) = v_0 \exp\{-\beta t\} + \int_0^t \exp\{-\beta(t-\tau)\}\,dw_\alpha(\tau) \tag{1.5}$$

and

$$x(t) = x_0 + v_0 \int_0^t \exp\{-\beta\tau\}\,d\tau + \int_0^t \int_0^\tau \exp\{-\beta(\tau-\tau_1)\}\,dw_\alpha(\tau_1)d\tau. \tag{1.6}$$

We shall prove that for $d \geqslant 3$ and for $d = 2$ (for some α) almost surely $|x(t)| \to \infty$ as $t \to \infty$ and obtain the estimate of growth of $x(t)$.

Theorem 1.1.1. *Suppose $d \geqslant 3$, $0 < \alpha < 2$ or $d = 2$, $0 < \alpha < 3/2$. Let $\beta > 0$, $f(t)$ be an increasing positive function such that $f(t) = o(t^{1/\alpha})$ as $t \to \infty$ and $\int_1^\infty (f(t)^d t^{-d/\alpha}) t^\omega\,dt < \infty$, where*

$$\omega > \max\left\{1 - \frac{1}{\alpha}, 0\right\}. \tag{1.7}$$

Then

$$\lim_{t\to\infty} \frac{|x(t)|}{f(t)} = \infty$$

almost surely.

Theorem 1.1.2. *Let $d \geqslant 2$, $0 < \alpha < 2$, $\beta = 0$, $f(t)$ be an increasing positive function such that $f(t) = o(t^{(1+1/\alpha)})$, $t/f(t) = o(1)$ as $t \to \infty$ and*

$$\int_1^\infty \left(f(t)t^{-(1+1/\alpha)}\right)^d dt < \infty.$$

Then

$$\lim_{t\to\infty}\frac{|x(t)|}{f(t)}=\infty$$

almost surely.

The proofs of these theorems follow from the following technical results. Observe that deterministic part of (1.6) is bounded for $\beta > 0$ and

$$\lim_{t\to\infty}\frac{|x_0+v_0t|}{f(t)}=0$$

for any function $f(t)$ from Theorem 1.1.2 and so it is sufficient to prove Theorems 1.1.1, 1.1.2 for processes $(x(t), v(t))$ with $x(0) = 0, v(0) = 0$. Let $B^{t,l}_{A,f}$ be the event which consists of all trajectories $x(\cdot)$ such that the set $\{x(s) : s \in [t, t + l]\}$ has a nonempty intersection with the ball $\{x \in \mathbb{R}^d : |x| \leqslant Af(t)\}$ for some constant A and function $f(t)$.

Proposition 1.1.1. *Let A be a positive constant, $d \geqslant 2$, $0 < l(t) \leqslant 1$ and let $f(t)$ be an increasing positive function on $\mathbb{R}_+$ such that $f(t) = o(t^{1/\alpha})$ as $t \to \infty$. Then*

$$\mathsf{P}\{B^{t,l}_{A,f}\} = O(f(t)^d t^{-d/\alpha}) + O(t^{-1}l^{\alpha+1}) + O(t^{-(1/\alpha)}l^2) + O(t^{-d/\alpha}l^d).$$

Proposition 1.1.2. *Let A be a positive constant, $\beta = 0$, $d \geqslant 2$, and let $f(t)$ be an increasing positive function on $\mathbb{R}_+$ such that $f(t) = o(t^{(1+1/\alpha)})$ as $t \to \infty$. Then*

$$\mathsf{P}\{B^{t,1}_{A,f}\} = O(f(t)t^{-(1+1/\alpha)})^d + O(t^{-(1+\alpha)}) + O(t^{-(1+1/\alpha)}) + O(t^{-d}).$$

The proofs of Propositions 1.1.1 and 1.1.2 are given in Section 1.3.

of Theorem 1.1.1. Let us take $\omega' \leqslant \omega$ such that it still satisfies condition (1.7) and $\gamma = \frac{\omega'}{1+\omega'}$ satisfies the inequality

$$\gamma\Big(2-\frac{1}{\alpha}\Big) > 1-\frac{1}{\alpha}. \tag{1.8}$$

In particular if $\alpha \geqslant 1$, then one can take $\omega' = \omega$. Denote $c_n = \sum_{k=2}^{n-1} k^{-\gamma}$. By Proposition 1.1.1

$$\begin{aligned}\sum_{n=2}^{\infty}\mathsf{P}\{B^{c_n,n^{-\gamma}}_{A,f}\} &= O(1)\sum_{n=2}^{\infty}\Big[f(c_n)^d c_n^{-d/\alpha} + \frac{c_n^{-1}}{n^{(\alpha+1)\gamma}} + \frac{c_n^{-(1/\alpha)}}{n^{2\gamma}} + \frac{c_n^{-d\alpha}}{n^{\gamma d}}\Big]\\ &= O(1)\,[\mathbf{I}+\mathbf{II}+\mathbf{III}+\mathbf{IV}].\end{aligned}$$

We are going to show that

$$\sum_{n=2}^{\infty} \mathsf{P}\{B_{A,f}^{c_n,n^{-\gamma}}\} < \infty. \tag{1.9}$$

Since $c_n = O(n^{1-\gamma})$ and $1/(c_n - c_{n-1}) = (n-1)^\gamma = O(1)c_n^{\gamma/(1-\gamma)} = O(1)c_n^{\omega'}$, it follows that

$$\begin{aligned} \mathbf{I} &= \sum_{n=2}^{\infty} f(c_n)^d c_n^{-d/\alpha} = \sum_{n=2}^{\infty} f(c_n)^d c_n^{-d/\alpha} \frac{1}{c_n - c_{n-1}}(c_n - c_{n-1}) \\ &= O(1)\int_2^{\infty} f(t)^d t^{-d/\alpha} t^{\omega'}\, dt = O(1)\int_2^{\infty} f(t)^d t^{-d/\alpha} t^{\omega}\, dt < \infty. \end{aligned}$$

The inequality $(1-\gamma)+(\alpha+1)\gamma = 1+\alpha\gamma > 1$ and the formula $c_n^{-1} = O(n^{\gamma-1})$ imply

$$\mathbf{II} = \sum_{n=2}^{\infty} c_n^{-1} \frac{1}{n^{(\alpha+1)\gamma}} = O(1)\sum_{n=2}^{\infty} \frac{1}{n^{1+\alpha\gamma}} < \infty.$$

Using, by (1.8), $(1/\alpha)(1-\gamma) + 2\gamma = 1/\alpha + \gamma(2 - 1/\alpha) > 1$ we get

$$\mathbf{III} = \sum_{n=2}^{\infty} c_n^{-(1/\alpha)} \frac{1}{n^{2\gamma}} < \infty.$$

Finally, the inequality $(d/\alpha)(1-\gamma) + d\gamma > 1$ yields

$$\mathbf{IV} = \sum_{n=2}^{\infty} c_n^{-d/\alpha} \frac{1}{n^{d\gamma}} < \infty.$$

Clearly $\mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV} < \infty$. Therefore series (1.9) converges. The first Borel-Cantelli lemma implies that only a finite number of the events $B_{A,f}^{c_n,n^{-\gamma}}$ can hold. It means the existence of a constant M such that for $t > c_M$ $|x(t)| \geqslant Af(c_m)$, where $c_m \leqslant t < c_{m+1}$. This implies the statement of Theorem 1.1.1. □

of Theorem 1.1.2. Due to Proposition 1.1.2, $\sum_{n=1}^{\infty} \mathsf{P}\{B_{A,f}^{n,1}\} < \infty$. The first Borel – Cantelli lemma implies that only a finite number of the events $B_{A,f}^{n,1}$ can hold. It means the existence of a constant M such that for $[t] > M$, $|x(t)| \geqslant Af([t])$. This implies the statement of Theorem 1.1.2. □

When Theorems 1.1.1 and 1.1.2 are proved the following results can be obtained by the usual arguments of the scattering theory (see e.g. [AK, AHK1] for details in the case of the Wiener stable process with $\alpha = 2$).

Theorem 1.1.3. *Suppose $d \geqslant 3$, $0 < \alpha < 2$ or $d = 2$, $0 < \alpha < 3/2$. Let the function $K(x)$ be Lipschitz continuous and there exist positive constants C_1, C_2 and a constant*

$$r > \left(\frac{1}{\alpha} - \frac{1}{d} - \frac{1}{d}\max\left\{1 - \frac{1}{\alpha}, 0\right\}\right)^{-1}$$

such that

1. $\|K(x)\| \leqslant C_1 \exp\{-2C_2\|x\|^r\}$ *for all* $x \in \mathbb{R}^d$
2. $\|K(x_1) - K(x_2)\| \leqslant C_1 \exp\{-2C_2\Gamma^r\}\|x_1 - x_2\|$ *for* $\|x_1\|, \|x_2\| > \Gamma$.

Then for any pair $(v_0, x_0) \in \mathbb{R}^{2d}$ and for almost all w_α there exists a unique solution $(\widetilde{v}(t), \widetilde{x}(t))$ of equation (0.4) with $\beta > 0$ such that

$$\lim_{t\to\infty}(\widetilde{v}(t) - v(t)) = 0, \tag{1.10}$$
$$\lim_{t\to\infty}(\widetilde{x}(t) - x(t)) = 0, \tag{1.11}$$

where $(v(t), x(t))$ is given by formulae (1.5), (1.6) and is the solution of (0.3) with the initial condition (v_0, x_0).

The proof of Theorem 1.1.3 is given in section 5.3.

Theorem 1.1.4. *Suppose $d \geqslant 2$. Let the function $K(x)$ be Lipschitz continuous and there exist a positive constant C and a constant*

$$r > 2\left(1 + \frac{1}{\alpha} - \frac{1}{d}\right)^{-1}$$

such that

1. $\|K(x)\| \leqslant C\|x\|^{-r}$ *for all* $x \in \mathbb{R}^d$
2. $\|K(x_1) - K(x_2)\| \leqslant C\Gamma^{-r}\|x_1 - x_2\|$ *for* $\|x_1\|, \|x_2\| > \Gamma$.

Then for any pair $(v_0, x_0) \in \mathbb{R}^{2d}$ *and for almost all* w_α *there exists a unique solution* $(\widetilde{v}(t), \widetilde{x}(t))$ *of equation* (0.4) *with* $\beta = 0$ *such that*

$$\lim_{t\to\infty} (\widetilde{v}(t) - v_0 - w_\alpha(t)) = 0, \tag{1.12}$$

$$\lim_{t\to\infty} (\widetilde{x}(t) - x_0 - v_0 t - \int_0^t w_\alpha(\tau)d\tau) = 0. \tag{1.13}$$

In terms of the scattering theory these results state the existence of the random wave operator $\Omega_{w_\alpha} : (v_0, x_0) \to (\widetilde{v}(0), \widetilde{x}(0))$ for system (0.4). Clearly Ω_{w_α} is an injective measure preserving random map $\mathbb{R}^{2d} \to \mathbb{R}^{2d}$. Note that unlike the case with deterministic Newton equation the Coulomb potential in $\mathbb{R}^3$ is included in the class of functions K, satisfying the assumptions of the theorem 1.1.4. The question whether or not the operator Ω_{w_α} is surjective (in the language of scattering theory the question of the completeness of the random wave operator), i.e. each solution of (0.2) or (0.3) has some 'free motion' limit is an interesting open problem. A partial solution to this question for the Wiener noise $w = w_\alpha$ is given in [AHK2].

1.2 Auxiliary results

In this section $(x(t), v(t))$ will be a solution of system (1.4) such that $x(0) = 0$, $v(0) = 0$. Let $p_X(\cdot, t)$, $p_V(\cdot, t)$, $p_{(V,X)}(\cdot, t)$ be probability densities of processes $x(t)$, $v(t)$, $(v(t), x(t))$ respectively.

Lemma 1.2.1. *The following formulae give the characteristic functions of the transition probability densities of the* $(v(t), x(t))$ *and its projections:*

$$\widetilde{p}_V(q,t) = \exp\Big\{ - \Lambda^\alpha(q) \int_0^t \exp\{-\beta\alpha\tau\} d\tau \Big\}, \tag{1.14}$$

$$\widetilde{p}_X(q,t) = \exp\Big\{ - \Lambda^\alpha(q) \int_0^t \Big(\int_\tau^t \exp\{-\beta(s-\tau)\}\, ds \Big)^\alpha d\tau \Big\} \tag{1.15}$$

and

$$\widetilde{p}_{(V,X)}(q_1, q_2, t) = \exp\Big\{ - \int_0^t \Lambda^\alpha\Big(\exp\{-\beta\tau\}\, q_1 + \int_\tau^t \exp\{-\beta(s-\tau)\} ds\ q_2 \Big)\, d\tau \Big\}, \tag{1.16}$$

where $\Lambda = \Lambda(q)$ *is given by* (1.2).

Corollary 1.2.1. *The density* $p_X(x,t)$ *enjoys the following scaling property*

$$c(t)^d p_X(c(t)x, t) = p_X(x, \sigma), \tag{1.17}$$

where σ *is determined from the equation*

$$1 = \int_0^\sigma \Big(\int_\tau^\sigma \exp\{-\beta(s-\tau)\}\, ds \Big)^\alpha d\tau$$

and

$$c(t) = \Big(\int_0^t \Big(\int_\tau^t \exp\{-\beta(s-\tau)\}\, ds \Big)^\alpha d\tau \Big)^{1/\alpha}. \tag{1.18}$$

In particular the last formula implies that for $\beta > 0$,

$$c(t)^{-1} = O(t^{-1/\alpha}) \tag{1.19}$$

and for $\beta = 0$,

$$c(t)^{-1} = O(t^{-(1+1/\alpha)}). \tag{1.20}$$

proof of Lemma 1.2.1. Formulae (1.14) and (1.15) are direct consequences of formula (1.16). The latter can be obtained by the general technique developed in [ST] . For completeness we shall give a direct proof.

We denote the right-hand side of (1.16) by **I**. Let $0 = t_1 < \ldots < t_{2n} < t_{2n+1} = t$ be a decomposition of the segment $[0,t]$. Take $\delta_k = w_\alpha(t_{k+1}) - w_\alpha(t_k)$, $a_k = \exp\{-\beta(t-t_k)\}$, $b_k = \int_{t_k}^t \exp\{-\beta(\tau - t_k)\}d\tau$, $v_{2n} = \sum_{k=1}^{2n} a_k\delta_k$, $x_{2n} = \sum_{k=1}^{2n} b_k\delta_k$. The random variables $\delta_k, k = 1, \ldots, 2n$ are independent. Therefore the $2d$-dimensional random variables $\Delta_k = (a_{2k-1}\delta_{2k-1} + a_{2k}\delta_{2k}, b_{2k-1}\delta_{2k-1} + b_{2k}\delta_{2k})$, $k = 1, \ldots, n$ are also independent. Hence

$$\widetilde{p}_{(V_{2n}, X_{2n})}(q_1, q_2) = \prod_{k=1}^n \widetilde{p}_{\Delta_k}(q_1, q_2).$$

Since $\widetilde{p}_{A\xi}(x) = \widetilde{p}_\xi(A^T x)$ for any random variable $\xi \in \mathbb{R}^m$ and any matrix $A \in \mathbb{R}^{m \times m}$, it follows that

$$\begin{aligned} \widetilde{p}_{\Delta_k}(q_1, q_2) &= \widetilde{p}_{(\delta_{2k-1}, \delta_{2k})}(a_{2k-1}q_1 + b_{2k-1}q_2, a_{2k}q_1 + b_{2k}q_2) \\ &= \widetilde{p}_{\delta_{2k-1}}(a_{2k-1}q_1 + b_{2k-1}q_2)\widetilde{p}_{\delta_{2k}}(a_{2k}q_1 + b_{2k}q_2). \end{aligned}$$

Consequently

$$\widetilde{p}_{(V_{2n},X_{2n})}(q_1,q_2)=\prod_{k=1}^{2n}\exp\Big\{-(t_{k+1}-t_k)\Lambda^\alpha(a_kq_1+b_kq_2)\Big\}.$$

Therefore

$$\widetilde{p}_{(V,X)}(q_1,q_2,t)=\lim_{n\to\infty}\widetilde{p}_{(V_{2n+1},X_{2n+1})}(q_1,q_2)=\mathbf{I}.$$

□

Lemma 1.2.2. *The density p_X, $d\in\mathbb{N}$ enjoys the following estimate*

$$\begin{aligned} p_X(x,t) &= O(t^{-d/\alpha}) \quad \mathit{for}\ \ \beta>0, && (1.21)\\ p_X(x,t) &= O(t^{-d(1+1/\alpha)}) \quad \mathit{for}\ \ \beta=0 && (1.22)\end{aligned}$$

uniformly for all x.

Proof. Let $\beta>0$. Changing the variable of integration to $q_1=t^{1/\alpha}q$ in the right hand side of the inequality $p_X(x,t)\leqslant(1/2\pi)^d\int_{\mathbb{R}^d}|\widetilde{p}_X(q,t)|dq$ and using formula (1.15), yields

$$\begin{aligned} p_X(x,t) \leqslant{}& (1/2\pi)^d t^{-d/\alpha}\times\\ &\times\int_{\mathbb{R}^d}\exp\Big\{-\Lambda^\alpha(q_1)\,t^{-1}\int_0^t\Big(\int_\tau^t\exp\{-\beta(s-\tau)\}\,ds\Big)^\alpha d\tau\Big\}\,dq_1.\end{aligned}$$

Since

$$t^{-1}\int_0^t\Big(\int_\tau^t\exp\{-\beta(s-\tau)\}\,ds\Big)^\alpha d\tau\geqslant C(\alpha,\beta)$$

for some $C(\alpha,\beta)>0$, we obtain

$$p_X(x,t)\leqslant(1/2\pi)^d t^{-d/\alpha}\int_{\mathbb{R}^d}\exp\{-C(\alpha,\beta)\Lambda^\alpha(q_1)\}\,dq_1$$

and estimate (1.21) follows. The same method can be used to prove estimate (1.22). □

Lemma 1.2.3. *For process $x(\cdot)$ with $\beta \geqslant 0$ the following inequality is true*

$$\mathsf{P}\Big\{\min_{\tau\in[t,t+l]} |x(\tau)| \leqslant Af(t) \,\big|\, v(t)=v, x(t)=x\Big\}$$
$$\leqslant \quad \mathsf{P}\left\{2l \max_{0\leqslant\tau\leqslant l} |w_\alpha(\tau)| \geqslant \min_{\tau\in[0,l]} \left|x + v\int_0^\tau \exp\{-\tau_1\beta\}d\tau_1\right| - Af(t)\right\}.$$

Proof. Applying formula (1.6) with $x_0 = x$, $v_0 = v$ yields

$$\min_{\tau\in[t,t+l]} |x(\tau)| \geqslant \min_{\tau\in[0,l]} \left|x + v\int_0^\tau \exp\{-\beta\tau_1\}\,d\tau_1\right| \tag{1.23}$$
$$+ \min_{\tau\in[0,l]} -\left|\int_0^\tau \int_t^{t+\tau_1} \exp\{-\beta(t+\tau_1-\tau_2)\}\,dw_\alpha(\tau_2)d\tau_1\right|.$$

Using integration by parts we get

$$\left|\int_0^\tau \int_t^{t+\tau_1} \exp\{-\beta(t+\tau_1-\tau_2)\}\,dw_\alpha(\tau_2)d\tau_1\right|$$
$$= \left|\int_0^\tau \Big(\hat{w}_\alpha(\tau_1) - \beta\int_0^{\tau_1} \exp\{-\beta(\tau_1-\tau_2)\}\hat{w}_\alpha(\tau_2)\,d\tau_2\Big)d\tau_1\right|$$
$$\leqslant 2l \max_{\tau\in[0,l]} |\hat{w}_\alpha(\tau)|, \tag{1.24}$$

where $\hat{w}_\alpha(\tau) = w_\alpha(t+\tau) - w_\alpha(t)$. Formulae (1.23) and (1.24) yield

$$\min_{\tau\in[t,t+l]} |x(\tau)| \geqslant \min_{\tau\in[0,l]} \left|x + v\int_0^\tau \exp\{-\beta\tau_1\}\,d\tau_1\right| - 2l \max_{\tau\in[0,l]} |\hat{w}_\alpha(\tau)|.$$

The last inequality and the fact that $\hat{w}_\alpha(\tau)$, $w_\alpha(\tau)$ have the same distributions imply the statement of the lemma. □

In our proof of Propositions 1.1.1 and 1.1.2 we shall use the following well known fact (see e.g. [Ber1]):

Lemma 1.2.4. *For $\alpha \in (0,2)$ there exists $C > 0$ such that for all λ*

$$\mathsf{P}\Big\{\max_{0\leqslant\tau\leqslant l}|w_\alpha(\tau)| > \lambda\Big\} < \frac{Cl}{\lambda^\alpha}. \tag{1.25}$$

Lemma 1.2.5. *Let $f(t)$ be a function from Proposition 1.1.1 for $\beta > 0$ (respectively Proposition 1.1.2 for $\beta = 0$), $c(t)$ defined in Corollary* 1.2.1 *and $0 < l(t) \leqslant 1$ for $\beta > 0$ (respectively $l(t) = 1$ for $\beta = 0$). For brevity we shall omit t in functions $f(t)$, $c(t)$, $l(t)$. Then*

$$\int\limits_{|x|\geqslant Af} p_X(x,t)\,\mathsf{P}\left\{2l\max_{0\leqslant\tau\leqslant l}|w_\alpha(\tau)| > (1/2)|x| - Af\right\}dx \tag{1.26}$$

$$= Q + O(c^{-\alpha}l^{\alpha+1}) + O(c^{-1}l^2),$$

where $Q = O\left(f^d t^{-d/\alpha}\right)$ for $\beta > 0$ (respectively $Q = O\left(f^d t^{-(1+1/\alpha)d}\right)$ for $\beta = 0$).

Proof. We represent integral (1.26) as the sum of two $J_1 + J_2$, whose domain of integration are $\{Af \leqslant |x| \leqslant 2(Af+l)\}$ and $\{|x| \geqslant 2(Af+l)\}$. Formula (1.21) for $\beta > 0$ (respectively (1.22) for $\beta = 0$) implies that

$$J_1 \leqslant \mathsf{P}\left\{Af \leqslant |x(\cdot)| \leqslant 2(Af+l)\right\} \leqslant \mathsf{P}\left\{|x(\cdot)| \leqslant 2(Af+1)\right\} = Q.$$

By Lemma 1.2.4 we see

$$J_2 = O(1)\int\limits_{|x|\geqslant 2(Af+l)} \frac{l^{\alpha+1}p_X(x,t)}{(|x|-2Af)^\alpha}\,dx. \tag{1.27}$$

Changing the variable x to $x_1 = c^{-1}x$ and using the scaling property (1.17) we rewrite expression (1.27) in the form

$$\begin{aligned} O(c^{-\alpha}) & \int\limits_{|x_1|\geqslant 2(Af+l)c^{-1}} \frac{l^{\alpha+1}c^d p_X(cx_1,t)}{(|x_1|-2Afc^{-1})^\alpha}\,dx_1 \\ = \ & O(c^{-\alpha}) \int\limits_{|x_1|\geqslant 2(Af+l)c^{-1}} \frac{l^{\alpha+1}p_X(x_1,\sigma)}{(|x_1|-2Afc^{-1})^\alpha}\,dx_1 \\ = \ & O(c^{-\alpha})\Big[\int\limits_{|x_1|\geqslant 1} + \int\limits_{1\geqslant|x_1|\geqslant 2(Af+l)c^{-1}}\Big]. \end{aligned} \tag{1.28}$$

By (1.21) (respectively (1.22) for $\beta = 0$) we see $fc^{-1} = o(1)$ when $t \to \infty$. Hence the first integral in (1.28) equals $O(c^{-\alpha}l^{\alpha+1})$. The second integral in (1.28) is equal to

$$
\begin{aligned}
O(c^{-\alpha}l^{\alpha+1}) \int\limits_{1 \geqslant |x_1| \geqslant 2(Af+l)c^{-1}} & \frac{\max\limits_{|x_1| \in [0,1]} p_X(x_1, \sigma)\, dx_1}{(|x_1| - 2Afc^{-1})^{\alpha}} \\
&= O(c^{-\alpha}l^{\alpha+1})[(1 - 2Afc^{-1})^{-\alpha+1} - (2c^{-1}l)^{-\alpha+1}] \\
&= O(c^{-\alpha}l^{\alpha+1}) + O(c^{-1}l^2).
\end{aligned}
$$

□

Lemma 1.2.6. *Let $\xi \in \mathbb{R}^{2d}$, $d \geqslant 2$ be a random variable with density $p_\xi(x)$, such that*

$$\widetilde{p}_\xi(q) = \exp\{-\lambda^\alpha(q)\}, \qquad \lambda^\alpha(q) = |q|^\alpha \lambda^\alpha\Big(\frac{q}{|q|}\Big).$$

Denote $h_U(q) = (Uq_1, Uq_2)$, where $U \in \mathbb{R}^{d\times d}$, $q = (q_1, q_2)$, $q_1, q_2 \in \mathbb{R}^d$. Assume that

$$K_1 \leqslant \lambda(q/|q|) \leqslant K_2, \tag{1.29}$$

for some constants $K_1, K_2 > 0$ and

$$\lambda(q(\cdot)) \in C^{2d+3}(\mathbb{R}) \tag{1.30}$$

for every $U \in \mathbb{R}^{d\times d}$ and for every integral curve $q(\tau)$ of the equation $\dot{q}(\tau) = h_U(q(\tau))$. Then

$$p_\xi(R, \psi) = O(R^{-(2d+\alpha)}), \tag{1.31}$$

where $R = |x|$, $\psi = x/|x|$ and $p_\xi(R, \psi) = p_\xi(x(R, \psi))$.

Proof. Step 1. One has

$$p_\xi(R, \psi) = \frac{1}{(\sqrt{2\pi})^{2d}} \int\limits_{S^{2d-1}} \int\limits_0^\infty \exp\{-r^\alpha \lambda^\alpha(\phi)\} \exp\{iRr < \phi, \psi >\} r^{2d-1}\, dr d\phi,$$

where $r = |q|$, $\phi = q/|q|$. Using for ϕ spherical coordinates (ρ, θ), $\rho \in [0, \pi]$, $\theta \in S^{2d-2}$ with main axis directed along ψ and changing the variable ρ to the variable $\kappa = \cos\rho$ give

$$\begin{aligned} p_\xi(R, \psi) &= \frac{1}{(\sqrt{2\pi})^{2d}} \int\limits_{S^{2d-2}} \int\limits_{-1}^{1} (1-\kappa^2)^{\frac{2d-3}{2}} J_0(\kappa, \theta)\, d\kappa d\theta, \\ J_0(\kappa, \theta) &= \int\limits_0^\infty \exp\{-r^\alpha \lambda^\alpha(\kappa, \theta)\} \exp\{irR\kappa\} r^{2d-1}\, dr, \end{aligned}$$

where $\lambda(\kappa, \theta) = \lambda(\phi)$. Changing the variable r to the variable $r \to rR$ we get

$$J_0(\kappa, \theta) = \frac{1}{R^{2d}} \int\limits_0^\infty \exp\{-r^\alpha R^{-\alpha} \lambda^\alpha(\kappa, \theta)\} \exp\{ir\kappa\} r^{2d-1}\, dr.$$

We finally arrive at

$$\begin{aligned} p_\xi(R, \psi) &= \sum_{i=1}^{2} \lim_{\varepsilon \to 0+} \frac{1}{(\sqrt{2\pi} R)^{2d}} \\ &\quad \times \int\limits_{S^{2d-2}} \int\limits_{-1}^{1} \int\limits_0^\infty (1-\kappa^2)^{\frac{2d-3}{2}} Z_i \exp\{ir\kappa\} \exp\{-\varepsilon r\} r^{2d-1}\, dr d\kappa d\theta \\ &= \mathbf{I} + \mathbf{II}, \end{aligned} \tag{1.32}$$

where

$$Z_1 = (\exp\{-r^\alpha R^{-\alpha} \lambda^\alpha(\kappa, \theta)\} - 1), \qquad Z_2 = 1.$$

One can show (see e.g. [K3]) that $\mathbf{II} = 0$. Changing the variable r to the variable $r \to r\lambda(\kappa, \theta)$, yields

$$p_\xi(R, \psi) = \mathbf{I} = \lim_{\varepsilon \to 0+} \frac{1}{(\sqrt{2\pi} R)^{2d}} \int\limits_{S^{2d-2}} \int\limits_{-1}^{1} (1-\kappa^2)^{\frac{2d-3}{2}} \lambda^{-2d}(\kappa, \theta) J(\kappa, \theta, \varepsilon)\, d\kappa d\theta, \tag{1.33}$$

$$J(\kappa,\theta,\varepsilon)=\int\limits_0^\infty(\exp\{-r^\alpha R^{-\alpha}\}-1)\exp\{ir(\kappa+i\varepsilon)\lambda^{-1}(\kappa,\theta)\}r^{2d-1}\,dr. \quad (1.34)$$

Step 2. Let $D=\{q\in S^{2d-1}:\kappa=0\}$. We now show that for each $q\in D$ one can choose $U\in\mathbb{R}^{d\times d}$ such that

$$|(h_U(q),\psi)|>1/2. \quad (1.35)$$

Let $F(\tau,p)$, $\tau\in\mathbb{R}$, $p\in D$ be a solution of the equation on S^{2d-1}

$$\begin{cases}\dfrac{d}{d\tau}F(\tau,p) &= h_U(F(\tau,p))\\ F(0,p) &= p.\end{cases} \quad (1.36)$$

and

$$\mathfrak{D}(\tau,p)=\frac{F(\tau,p)}{|F(\tau,p)|}.$$

Since $h_U(\cdot)$ is a smooth vector field in some neighbourhood of q, there exist open neighbourhoods of q $V(q)\subset S^{2d-1}$, $O(q)\subset D$ and $T=T(q)>0$ such that

$$\mathfrak{D}:(-T,T)\times O(q)\to V(q) \quad (1.37)$$

is a diffeomorphism. Without loss of generality one can assume that for any $\hat q=(\kappa,\theta)\in V(q)$

$$|(h_U(\hat q),\psi)| > 1/2, \quad (1.38)$$

$$|\kappa| < 1/2, \quad (1.39)$$

$$|\kappa|\max\left|\frac{d}{d\tau}|F(\tau,p)|^{-\alpha}\lambda(F(\tau,p))\right| < K_1/4, \quad (1.40)$$

where the maximum is taken over all $(\tau,p)\in(-T,T)\times O(q)$.

Since D is a compact and $D\subset\cup_{q\in D}V(q)$, one can choose a finite subcovering $D\subset V=\cup_{n=1}^m V(q_n)$. Denote $T_n=T(q_n)$. One can choose smooth functions $\chi_n(\kappa,\theta)$, $n=1,\ldots,m$ on S^{2d-1} such that

$$0\leqslant\chi_n(\kappa,\theta)\leqslant 1,\qquad \sum_{n=1}^m\chi_n(\kappa,\theta)\equiv 1 \quad (1.41)$$

for $(\kappa, \theta) \in V$ and supp $\chi_n \subset V(q_n)$. Let

$$K_3 = \inf\{|\kappa| : \text{for some } \theta \quad \hat{q} = (\kappa, \theta) \in S^{2d-1} \setminus V\}.$$

Since $D \subset V$, it follows $K_3 > 0$.

Let $\zeta(\kappa)$ be a smooth function $\mathbb{R} \to [0,1]$ that equals one (respectively zero) for $|\kappa| \leqslant \min\{1/3, K_3/3\}$ (respectively for $|\kappa| \geqslant \min\{1/2, K_3/2\}$). We put

$$f_1(\kappa, \theta) = A\zeta(\kappa) \quad \text{and} \quad f_2(\kappa, \theta) = A(1 - \zeta(\kappa)),$$

where $A = (1-\kappa^2)^{\frac{2d-3}{2}} \lambda^{-2d}(\kappa, \theta)$, and

$$S_i = \operatorname{Re} \lim_{\varepsilon \to 0+} \frac{1}{(\sqrt{2\pi}R)^{2d}} \int\limits_{S^{2d-2}} \int\limits_{-1}^{1} f_i(\kappa, \theta) J(\kappa, \theta, \varepsilon)\, d\kappa d\theta. \tag{1.42}$$

Recall that $J(\kappa, \theta, \varepsilon)$ is given by (1.34). From (1.33) we find

$$p_\xi(R, \psi) = S_1 + S_2.$$

Step 3. Changing the order of the integration gives

$$S_1 = \operatorname{Re} \lim_{\varepsilon \to 0+} \frac{1}{(\sqrt{2\pi}R)^{2d}} \int\limits_0^\infty (\exp\{-r^\alpha R^{-\alpha}\} - 1) r^{2d-1} G(r, \varepsilon)\, dr,$$

where

$$G(r, \varepsilon) = \int\limits_V f_1(\kappa, \theta) \exp\{ir(\kappa + i\varepsilon)\lambda^{-1}(\kappa, \theta)\}\, d\kappa d\theta.$$

We now show that

$$G(r, \varepsilon) = O(r^{-(2d+2)}) \tag{1.43}$$

uniformly for all $0 \leqslant \varepsilon < \varepsilon_0$ for some ε_0. Then (1.43) and the elementary estimate $1 - \exp\{-z\} \leqslant z$, $z \geqslant 0$ imply $S_1 = O(R^{-(2d+\alpha)}) \int_1^\infty r^{-(3-\alpha)} dr = O(R^{-(2d+\alpha)})$.

Using (1.41) we have $G(r, \varepsilon) = \sum\limits_{n=1}^{m} G_n(r, \varepsilon)$, where

$$G_n(r, \varepsilon) = \int\limits_{V(q_n)} f_1(\kappa, \theta) \chi_n(\kappa, \theta) \exp\{ir(\kappa + i\varepsilon)\lambda^{-1}(\kappa, \theta)\}\, d\kappa d\theta. \tag{1.44}$$

Let us make the change of the variables $(\kappa,\theta) \to (\tau,\theta_0) = F^{-1}(\kappa,\theta)$ in (1.44), where $F^{-1} : V(q_n) \to (-T_n, T_n) \times O(q_n)$ is the inverse map of F given by (1.37). Denote the amplitude of the Jacobian of this change of the variables by $\mathcal{J}_n(\tau,\theta_0)$,

$$g_n(\tau) = g_n(\tau,\theta_0) = \mathcal{J}_n(\tau,\theta_0)\,[f_1(\kappa,\theta)\chi_n(\kappa,\theta)]_{\kappa=\kappa(\tau,\theta_0),\theta=\theta(\tau,\theta_0)}$$

and $\lambda(\tau) = \lambda(F(\tau,\theta_0)/|F(\tau,\theta_0)|) = |F(\tau,\theta_0)|^{-\alpha}\lambda(F(\tau,\theta_0))$. Then

$$G_n(r,\varepsilon) = \int\limits_{O(q_n)}\int\limits_{-T_n}^{T_n} g_n(\tau)\exp\{ir\varpi(\tau)\}\,d\tau d\theta_0, \tag{1.45}$$

$$\varpi(\tau) = (\kappa(\tau,\theta_0)+i\varepsilon)\lambda^{-1}(\tau). \tag{1.46}$$

Condition (1.30) implies that $\lambda(\tau) \in C^{2d+3}([-T_n,T_n])$. Therefore $g_n(\tau),\ \varpi(\tau) \in C^{2d+3}([-T_n,T_n])$. By (1.38) we see

$$\Big|\frac{\partial\kappa(\tau,\theta_0)}{\partial\tau}\Big| = \Big|\Big(\frac{\partial F(\tau,\theta_0)}{\partial\tau},\psi\Big)\Big| = |(h_U(F(\tau,\theta_0)),\psi)| > 1/2. \tag{1.47}$$

Using (1.40) we find $\varepsilon_0 > 0$ such that for all $0 \leqslant \varepsilon \leqslant \varepsilon_0$

$$\max\Big|(\kappa(\tau,\theta_0)+i\varepsilon)\frac{\partial\lambda(\tau)}{\partial\tau}\Big| < \frac{K_1}{3},$$

where maximum is taken over all $(\tau,\theta_0) \in (-T_n,T_n)\times O(q_n)$. Consequently

$$\begin{aligned}\Big|\frac{\partial\varpi(\tau)}{\partial\tau}\Big| &= \Big|\frac{\partial\kappa(\tau,\theta_0)}{\partial\tau}\lambda(\tau) - (\kappa+i\varepsilon)\frac{\partial\lambda(\tau)}{\partial\tau}\Big|\lambda^{-2}(\tau)\\ &\geqslant \Big(\frac{1}{2}K_1 - \frac{1}{3}K_1\Big)K_2^{-2} > 0\end{aligned}$$

for $0 \leqslant \varepsilon < \varepsilon_0$. Applying integration by parts formula $2d+2$ times and using the fact that $g_n(\pm T_n) = \ldots = g_n^{(2d+2)}(\pm T_n) = 0$, give

$$\int\limits_{-T_n}^{T_n} g_n(\tau)\exp\{ir\varpi(\tau)\}\,d\tau = (ir)^{-(2d+2)}\int\limits_{-T_n}^{T_n} P_{2d+2}(\tau)\exp\{ir\varpi(\tau)\}\,d\tau, \tag{1.48}$$

where

$$P_0(\tau) = g_n(\tau), \quad P_{m+1}(\tau) = \frac{\partial}{\partial\tau}\Big[P_m(\tau)\Big(\frac{\partial\varpi(\tau)}{\partial\tau}\Big)^{-1}\Big].$$

The estimates above imply $P_{2d+2}(\tau) = O(1)$. Combining (1.45) and (1.48) we get (1.43).

Step 4. We proceed with S_2. Applying integration by parts formula $2d$ times we find from (1.34)

$$J(\kappa,\theta,\varepsilon) = (iB)^{-2d} \int_0^\infty \Big(\frac{d^{2d}}{dr^{2d}} r^{2d-1}(\exp\{-r^\alpha R^{-\alpha}\} - 1)\Big) \exp\{irB\}\, dr, \quad (1.49)$$

where $B = (\kappa+i\varepsilon)\lambda^{-1}(\kappa,\theta)$. The inequality $|\kappa\lambda^{-1}(\kappa,\theta)| \geqslant K_2^{-1}\max\{K_3/3, 1/3\} > 0$ for $\kappa \in \operatorname{supp} f_2$ and

$$\frac{d^{2d}}{dr^{2d}} r^{2d-1}(\exp\{-r^\alpha R^{-\alpha}\} - 1) = \Big(\sum_{m=1}^{2d} a_m r^{\alpha m-1} R^{-\alpha m}\Big) \exp\{-r^\alpha R^{-\alpha}\}$$

for some $a_m = a_m(\alpha, 2d)$ imply that there exists $J(\kappa,\theta) = \lim_{\varepsilon\to+0} J(\kappa,\theta,0)$ for $(\kappa,\theta) \in \operatorname{supp} f_2$ and

$$S_2 = \operatorname{Re} \frac{(-1)^d}{(\sqrt{2\pi R})^{2d}} \int_{S^{2d-2}} \int_{-1}^{1} (1-\kappa^2)^{\frac{2d-3}{2}} \kappa^{-2d} J(\kappa,\theta,0)\, d\kappa d\theta,$$

where

$$J(\kappa,\theta,0) = \int_0^\infty \Big(\sum_{m=1}^{2d} a_m r^{\alpha m-1} R^{-\alpha m}\Big) \exp\{-r^\alpha R^{-\alpha}\} \exp\{ir\kappa\lambda^{-1}(\kappa,\theta)\}\, dr. \quad (1.50)$$

We are going to show that $J(\kappa,\theta,0) = O(R^{-\alpha})$ uniformly for all $(\kappa,\theta) \in \operatorname{supp} f_2$, which evidently completes the proof of Lemma 1.2.6. Without loss of generality one can assume that $\kappa > 0$.

Case 1. Let $\alpha \in (0,1)$. The integral (1.50) along the curve $r = L\exp\{i\tau\pi/2\}$, $L > 0$, $\tau \in [0,1]$

$$\int_0^1 \Big(\sum_{m=1}^{2d} a_m L^{\alpha m-1} \exp\Big\{i(\alpha m-1)\tau\frac{\pi}{2}\Big\} R^{-\alpha m}\Big) \exp\Big\{-L^\alpha \exp\Big\{i\tau\alpha\frac{\pi}{2}\Big\} R^{-\alpha}\Big\}$$
$$\times \exp\Big\{iL\exp\Big\{i\tau\frac{\pi}{2}\Big\}\kappa\lambda^{-1}(\kappa,\theta)\Big\} Li\frac{\pi}{2}\exp\Big\{i\tau\frac{\pi}{2}\Big\}\, d\tau$$

does not exceed in magnitude

$$\Big(\sum_{m=1}^{2d}|a_m|L^{\alpha m-1}R^{-\alpha m}\Big)L\frac{\pi}{2}\exp\Big\{-L^{\alpha}\cos\Big\{\alpha\frac{\pi}{2}\Big\}R^{-\alpha}\Big\}.$$

The latter expression tends to 0 when $L\to\infty$. Then for any $(\kappa,\theta)\in \mathrm{supp} f_2$, $\kappa>0$ one can rotate the contour of integration in (1.50) to the imaginary axis. Changing the variable r to the variable $r\to ir$ yields

$$\int_0^\infty\Big(\sum_{m=1}^{2d}a_m\exp\Big\{i(\alpha m-1)\frac{\pi}{2}\Big\}r^{\alpha m-1}R^{-\alpha m}\Big)$$
$$\times\exp\Big\{-\exp\Big\{i\alpha\frac{\pi}{2}\Big\}r^{\alpha}R^{-\alpha}\Big\}\exp\{-r\kappa\lambda^{-1}(\kappa)\}i\,dr=O(R^{-\alpha}).$$

Case 2. Let $\alpha\in[1,2)$. Similarly, for any $(\kappa,\theta)\in\mathrm{supp} f_2$ one can rotate the contour of integration in (1.50) through the angle $\exp\{i\pi/2\alpha\}$. Changing the variable r to the variable $r\to\exp\{i\pi/2\alpha\}r$ yields

$$\int_0^\infty\Big(\sum_{m=1}^{2d}a_m\exp\Big\{i(\alpha m-1)\,\frac{\pi}{2\alpha}\Big\}r^{\alpha m-1}R^{-\alpha m}\Big)$$
$$\times\exp\{-ir^{\alpha}R^{-\alpha}\}\exp\Big\{i\exp\Big\{i\frac{\pi}{2\alpha}\Big\}r\kappa\lambda^{-1}(\kappa)\Big\}dr=O(R^{-\alpha}).$$

$\square$

Lemma 1.2.7. *Denote by* $\hat{x}(t)=t^{-1/\alpha}x(t)$, $\hat{v}(t)=v(t)$ *for* $\beta>0$ *and* $\hat{x}(t)=t^{-(1+1/\alpha)}x(t)$, $\hat{v}(t)=t^{-1/\alpha}v(t)$ *for* $\beta=0$. *Then* $\xi=(\hat{x}(t),\hat{v}(t))$ *satisfy the conditions* (1.29), (1.30) *of Lemma 1.2.6, where* K_1, K_2 *do not depend on* t.

Proof. Step 1. Formula (1.16) implies that

$$\widetilde{p}_{\xi}(q_1,q_2)=\exp\Big\{-\int_0^1\Lambda^{\alpha}(\gamma_1q_1+\gamma_2q_2)\,ds\Big\},$$

where $\Lambda(\cdot)$ is given by formula (1.2), $\gamma_1=\gamma_1(s)=t^{1/\alpha}\exp\{-\beta st\}$, $\gamma_2=\gamma_2(s)=t\int_s^1\exp\{-\beta(\tau-s)t\}\,d\tau$ for $\beta>0$ and $\gamma_1=1$, $\gamma_2=1-s$ for $\beta=0$.

Applying condition (1.3) to the formula

$$\lambda(q) = \int_0^1 \Lambda^\alpha(\gamma_1 q_1 + \gamma_2 q_2)\, ds = \int_0^1 |\gamma_1 q_1 + \gamma_2 q_2|^\alpha \Lambda^\alpha\Big(\frac{\gamma_1 q_1 + \gamma_2 q_2}{|\gamma_1 q_1 + \gamma_2 q_2|}\Big)\, ds, \tag{1.51}$$

$q = (q_1, q_2)$, we get condition (1.29) with

$$K_1 = C_1 \min_{q \in S^{2d-1}} \int_0^1 |\gamma_1 q_1 + \gamma_2 q_2|^\alpha\, ds, \quad K_2 = C_2 \max_{q \in S^{2d-1}} \int_0^1 |\gamma_1 q_1 + \gamma_2 q_2|^\alpha\, ds,$$

where we used

$$\int_0^1 |\gamma_1(s)|^\alpha\, ds = O(1), \qquad \int_0^1 |\gamma_2(s)|^\alpha\, ds = O(1).$$

Step 2. Let $U \in \mathbb{R}^{d\times d}$ and $q(\tau)$ be an integral curve of the equation $\dot{q}(\tau) = h_U(q(\tau))$. We will show that $\lambda(q(\cdot)) \in C^{2d+3}(\mathbb{R})$.

Given τ_0 we take τ such that

$$\sum_{n=1}^\infty \frac{\|U\|^n}{n!} |\tau - \tau_0|^n < 1. \tag{1.52}$$

Since

$$q_i(\tau) = q_i(\tau_0) + \sum_{n=1}^\infty \frac{q_i^{(n)}(\tau_0)}{n!}(\tau - \tau_0)^n = q_i(\tau_0) + \sum_{n=1}^\infty \frac{U^n(q_i(\tau_0))}{n!}(\tau - \tau_0)^n,$$

$i = 1, 2$, we deduce from (1.51)

$$\begin{aligned} &\lambda(q(\tau)) \\ &= \int_0^1 \Lambda^\alpha\Big(\gamma_1 q_1(\tau_0) + \gamma_2 q_2(\tau_0) + \sum_{n=1}^\infty \frac{U^n(\gamma_1 q_1(\tau_0) + \gamma_2 q_2(\tau_0))}{n!}(\tau - \tau_0)^n\Big)\, ds \\ &= \int_0^1 |\gamma_1 q_1(\tau_0) + \gamma_2 q_2(\tau_0)|^\alpha \Lambda^\alpha(f(s, \tau))\, ds, \end{aligned}$$

where

$$f(s,\tau) = \frac{\gamma_1 q_1(\tau_0) + \gamma_2 q_2(\tau_0)}{|\gamma_1 q_1(\tau_0) + \gamma_2 q_2(\tau_0)|} + \sum_{n=1}^{\infty} \frac{U^n(\gamma_1 q_1(\tau_0) + \gamma_2 q_2(\tau_0))}{|\gamma_1 q_1(\tau_0) + \gamma_2 q_2(\tau_0)|} \frac{(\tau - \tau_0)^n}{n!}.$$

Using (1.52) we find

$$|f(s,\tau)| \geqslant 1 - \sum_{n=1}^{\infty} \frac{\|U\|^n}{n!} |\tau - \tau_0|^n > 0$$

for $s \neq s_0$, where s_0 is a solution (if it exists) of the equation

$$\gamma_1(s_0) q_1(\tau_0) + \gamma_2(s_0) q_2(\tau_0) = 0.$$

This and (1.3) imply that for any $s \neq s_0$ the function $\Lambda^\alpha(f(s,\tau))$ is $d+2$ times differentiable at $\tau = \tau_0$ and so $\lambda(q(\tau))$ is $d+2$ times differentiable at $\tau = \tau_0$. $\square$

Lemma 1.2.8. *Let $d \geqslant 2$. Denote by*

$$\mathbf{I} = \mathsf{P}\left\{ \min_{\tau \in [0,l]} \left| x(t) + v(t) \int_0^\tau \exp\{-\beta s\}\, ds \right| \leqslant \frac{1}{2} |x(t)| \right\}. \tag{1.53}$$

Then $\mathbf{I} = O(t^{-d/\alpha} l^d)$ for $\beta > 0$ and $\mathbf{I} = O(t^{-d} l^d)$ for $\beta = 0$.

Proof. Using the notations of Lemma 1.2.7 we have

$$\mathbf{I} = \mathsf{P}\left\{ \min_{\tau \in [0,l]} |\hat{x}(t) + A(\tau)\hat{v}(t)| \leqslant \frac{1}{2} |\hat{x}(t)| \right\},$$

where $A(\tau) = t^{-1}\tau$ for $\beta = 0$ and $A(\tau) = t^{-1/\alpha} \int_0^\tau \exp\{-\beta s\}\, ds$ for $\beta > 0$. Using the coordinates $\hat{x} = R\cos\vartheta N_1$, $\hat{v} = R\sin\vartheta N_2$, $R \in [0,\infty)$, $\vartheta \in [0,\pi/2]$, $N_1, N_2 \in S^{d-1}$ we obtain

$$\mathbf{I} = \int_{\mathcal{G}} \int_0^\infty (\cos\vartheta \sin\vartheta)^{d-1} p_{\hat{v}(t),\hat{x}(t)}(R,\vartheta,N_1,N_2) R^{2d-1}\, dR d\vartheta dN_1 dN_2, \tag{1.54}$$

where

$$\mathcal{G} = \left\{ (\vartheta, N_1, N_2) : \min_{\tau \in [0,l]} |\cos\vartheta N_1 + A(\tau) \sin\vartheta N_2| \leqslant \frac{1}{2} \cos\vartheta \right\}.$$

Lemma 1.2.7 and formula (1.31) yields

$$\int_0^\infty p_{\hat{v}(t),\hat{x}(t)}(R,\vartheta,N_1,N_2)R^{2d-1}\,dR = O(1)$$

uniformly with respect to ϑ, N_1, N_2. The inequalities

$$\frac{1}{2} \geqslant \min_{\tau\in[0,l]} |N_1 + A(\tau)\tan\vartheta N_2| \geqslant \min_{\tau\in[0,l]}(1 - A(\tau)\tan\vartheta)$$

and $\tau \leqslant l$ imply that $\mathcal{G} \subset \mathcal{G}_1 = \{(\vartheta, N_1, N_2) : 1 - A(l)\tan\vartheta \leqslant 1/2\}$. Applying $|\mathcal{G}_1| = O(A(l))$ and $\cos\vartheta = O(A(l))$ for $(\vartheta,\cdot,\cdot) \in \mathcal{G}_1$, we estimate expression (1.54) by

$$O(1)\int_{\mathcal{G}_1} (\cos\vartheta\sin\vartheta)^{d-1}\,d\vartheta dN_1 dN_2 = O(1)|\mathcal{G}_1|(A(l))^{d-1} = O((A(l))^d).$$

Observing $A(l) \leqslant t^{-1/\alpha}l$ for $\beta > 0$ and $A(l) = t^{-1}l$ for $\beta = 0$, we complete the proof. □

1.3 Estimates for $P(B_{A,f}^{t,l})$

Let $0 < l(t) \leqslant 1$ for $\beta > 0$ (resp. $l(t) = 1$ for $\beta = 0$). It is clear that

$$\mathsf{P}\{B_{A,f}^{t,l}\} = \mathsf{P}\{|x(t)| \leqslant Af(t)\} + \int_{|x|\geqslant Af(t)}\int_{\mathbb{R}^d} \mathcal{J}(v,x)p_{(V,X)}(v,x,t)\,dvdx, \quad (1.55)$$

where

$$\mathcal{J}(v,x) = \mathsf{P}\{\min_{\tau\in[t,t+l]} |x(\tau)| \leqslant Af(t)\,|\,v(t) = v, x(t) = x\}.$$

Due to Lemma 1.2.2 $\mathsf{P}\{|x(t)| \leqslant Af(t)\} = O\big(f(t)t^{-1/\alpha}\big)^d$ for $\beta > 0$ and $\mathsf{P}\{|x(t)| \leqslant Af(t)\} = O\big(f(t)t^{-(1+1/\alpha)}\big)^d$ for $\beta = 0$. We represent the integral in expression (1.55) as the sum $I_1 + I_2$ of two integrals, whose domain of integration in the variable (v,x) are

$$D_1 = \Big\{\min_{\tau\in[0,l]} |x + v\int_t^{t+\tau} \exp\{-\beta s\}\,ds| \leqslant (1/2)|x|\Big\}$$

and

$$D_2 = \left\{ \min_{\tau\in[0,l]} |x + v \int_t^{t+\tau} \exp\{-\beta s\}\, ds| > (1/2)|x| \right\}.$$

Lemma 1.2.8 implies that $I_1 \leqslant \int_{D_1} p_{(V,X)}(v,x,t)\, dvdx = O(t^{-d/\alpha} l^d)$ for $\beta > 0$ and $I_1 = O(t^{-d} l^d)$ for $\beta = 0$. By Lemma 1.2.3 we see

$$\mathcal{J}(v,x) \leqslant \mathsf{P}\left\{ 2l \max_{0\leqslant\tau\leqslant l} |w_\alpha(\tau)| > \frac{1}{2}|x| - Af(t) \right\}$$

for $(v,x) \in D_2$ and so, integrating the estimate for I_2 over v we get

$$I_2 \leqslant \int_{|x|\geqslant Af(t)} p_X(x,t)\, \mathsf{P}\left\{ 2l \max_{0\leqslant\tau\leqslant l} |w_\alpha(\tau)| > \frac{1}{2}|x| - Af(t) \right\} dx. \qquad (1.56)$$

Applying Lemma 1.2.5 and formula (1.19) (respectively (1.20) for $\beta = 0$) to estimate (1.56) yield

$$I_2 = O(f(t)^d t^{-d/\alpha}) + O(t^{-1} l^{\alpha+1}) + O(t^{-1/\alpha} l^2)$$

for $\beta > 0$ and

$$I_2 = O(f(t)^d t^{-(1+1/\alpha)d}) + O(t^{-(1+\alpha)}) + O(t^{-(1+1/\alpha)})$$

for $\beta = 0$.

Piecing together the estimates above gives the proof.

1.4 Existence of the wave operator

The proof of Theorems 1.1.3 and 1.1.4 follows a well known pattern.

Proof of Theorems 1.1.3, 1.1.4. The change of variables

$$\begin{aligned} y &= x - \int_0^t \int_0^\tau \exp\{-\beta(\tau - s)\}\, dw_\alpha(s) d\tau \\ h &= v - \int_0^t \exp\{-\beta(t-\tau)\}\, dw_\alpha(\tau) \end{aligned}$$

transform system (0.4) into the system

$$\begin{cases} \dot{y} &= h \\ \dot{h} &= K\Big(y + \int\limits_0^t \int\limits_0^\tau \exp\{-\beta(\tau - s)\}\, dw_\alpha(s)\, d\tau\Big) - \beta h. \end{cases}$$

Making the change of the second variable $h = \exp\{-\beta t\} z$, we get the equivalent system

$$\begin{cases} \dot{y} &= \exp\{-\beta t\} z \\ \dot{z} &= \exp\{\beta t\} K\Big(y + \int\limits_0^t \int\limits_0^\tau \exp\{-\beta(\tau - s)\}\, dw_\alpha(s)\, d\tau\Big). \end{cases} \tag{1.57}$$

Let $T > 0$. Denote by $C([T,\infty))$ the Banach space of bounded continuous function from $[T,\infty)$ into $\mathbb{R}^d$ with the norm $||u||_\infty = \sup\{|u(x)| : x \in [T,\infty)\}$ and by B_T the unit ball

$$B_T = \{u \in C([T,\infty)) : ||u||_\infty \leqslant 1\}.$$

Case 1. Let $\beta > 0$. It is clear that if some function $u \in B_T$ is a fixed point of the map

$$(\mathcal{F}u)(t) = \int\limits_t^\infty \exp\{-\beta\tau\}\Big\{\int\limits_\tau^\infty \exp\{\beta s\} K\Big(u(s) + y_0$$

$$+ z_0 \int\limits_0^s \exp\{-\beta s_1\} ds_1 + \int\limits_0^s \int\limits_0^{s_1} \exp\{-\beta(s_1 - s_2)\}\, dw_\alpha(s_2) ds_1\Big) ds\Big\} d\tau,$$

then $\widetilde{y} = u + y_0 + z_0 \int_0^t \exp\{-\beta s\}\, ds$ and $\widetilde{z} = z_0 + \exp\{\beta t\}\dot{u}$ are the solutions of (1.57) with the asymptotics (1.10) and (1.11) and so the existence of the solution of (0.4) with asymptotics (1.10), (1.11) is equivalent to the existence of a fixed point for the map $\mathcal{F}$.

Let us choose $p \in (0, 1/\alpha - 1/d - \max\{1 - 1/\alpha, 0\}/d)$. Then $rp > 1$. The existence of such a p is assured by the condition on r. Then the function $f(t) = t^p$ satisfies the condition of Theorem 1.1.1 and consequently, there exists a T_0 such that for all $T \geqslant T_0$

$$\Big|y_0 + z_0 \int\limits_0^T \exp\{-\beta s_1\}\, ds_1 + \int\limits_0^T \int\limits_0^{s_1} \exp\{-\beta(s_1 - s_2)\}\, dw_\alpha(s_2) ds_1\Big| > T^p + 1$$

with probability one and, moreover,

$$\beta T_0^{1-pr} \leqslant C_2.$$

Due to condition 1 of Theorem 1.1.3, for these T the integral

$$\int_T^\infty \exp\{\beta s\} K\Big(u(s) + y_0 + z_0 \int_0^s \exp\{-\beta s_1\}\, ds_1 \\ + \int_0^s \int_0^{s_1} \exp\{-\beta(s_1 - s_2)\}\, dw_\alpha(s_2)\, ds_1\Big)\, ds \qquad (1.58)$$

is well defined for any $u \in B_T$ and does not exceed in magnitude

$$C_1 \int_T^\infty \exp\{\beta s - 2C_2 s^{pr}\}\, ds \leqslant C_1 \int_T^\infty \exp\{-C_2 s^{pr}\}\, ds \\ \leqslant C_1 \int_T^\infty \exp\{-C_2 s\}\, ds = \frac{C_1}{C_2} \exp\{-C_2 T\}.$$

Analogously, due to condition 2 of Theorem 1.1.3, the norm of the difference of two integrals of the form (1.58) corresponding to different functions $u_1, u_2 \in B_T$ is bounded from above by

$$\frac{C_1}{C_2} \exp\{-C_2 T\} \|u_1 - u_2\|.$$

This implies that the map $\mathcal{F}$ is well defined on B_T for such T and, moreover, if we take a T such that

$$\frac{1}{\beta + C_2} \frac{C_1}{C_2} \exp\{-T(\beta + C_2)\} < 1,$$

then $\mathcal{F}$ maps B_T to itself and is a contraction on B_T. The contraction mapping principle implies then the existence of a (unique) fixed point to $\mathcal{F}$, which completes the proof of theorem 1.1.3.

Case 2. We proceed with $\beta = 0$. Obviously, if some function $u \in B_T$ is a fixed point of the map

$$(\mathcal{F}u)(t) = \int_t^\infty \int_\tau^\infty K\Big(u(s) + y_0 + z_0 s + \int_0^s w_\alpha(s_1)\, ds_1\Big)\, ds d\tau, \qquad (1.59)$$

then $\widetilde{y} = u + y_0 + z_0 t$ and $\widetilde{z} = z_0 + \dot{u}$ are the solutions of (1.57) with the asymptotics (1.10) and (1.11) and so the existence of the solution of (0.4) with asymptotics (1.12), (1.13) is equivalent to the existence of a fixed point for the map $\mathcal{F}$. Notice that condition 1 of Theorem 1.1.4 implies the existence of the integral in (1.59).

Let us choose $p \in (0, (1+1/\alpha - 1/d)$. Then $rp > 2$. The existence of such a p is assured by the condition on r. Then the function $f(t) = t^p$ satisfies the condition of Theorem 1.1.1 and consequently, there exists a T_0 such that for all $T \geqslant T_0$

$$\left| y_0 + z_0 T + \int_0^T w_\alpha(s_1) ds_1 \right| > T^p + 1$$

with probability one. Due to condition 1 of Theorem 1.1.3, for these T, the integral

$$\int_\tau^\infty K\Big(u(s) + y_0 + z_0 s + \int_0^s w_\alpha(s_1) ds_1\Big)\, ds \tag{1.60}$$

is well defined for any $u \in B_T$ and does not exceed in magnitude

$$C \int_T^\infty \int_\tau^\infty s^{-pr}\, ds d\tau \leqslant \frac{C}{(pr-1)(pr-2)} T^{2-pr}.$$

Analogously, due to condition 2 of Theorem 1.1.4, the norm of the difference of two integrals of the form (1.60) corresponding to different functions $u_1, u_2 \in B_T$ is bounded from above by

$$\frac{C}{(pr-1)(pr-2)} T^{2-pr} \|u_1 - u_2\|.$$

This implies that the map $\mathcal{F}$ is well defined on B_T for such T and, moreover,if we take a T such that

$$\frac{C}{(pr-1)(pr-2)} T^{2-pr} < 1,$$

then $\mathcal{F}$ maps B_T to itself and is a contraction on B_T. The contraction mapping principle implies then the existence of a (unique) fixed point to $\mathcal{F}$, which gives the proof of theorem 1.1.4. □

Chapter 2

Transience and Non-explosion of Certain Stochastic Newtonian Systems

2.1 Non-explosion

Let $(X(t), P(t)) = (X(t, x_0, p_0), P(t, x_0, p_0))$ be a solution of the system

$$\begin{cases} dx &= p\,dt \\ dp &= -\frac{\partial V}{\partial x}\,dt - \frac{\partial c}{\partial x}\,d\xi_t, \end{cases} \tag{2.1}$$

with initial condition $(x_0, p_0) \in \mathbb{R}^{2d}$ at $t = 0$, where $\xi_t = (\xi_{1,t}, \dots, \xi_{d,t})$ is a Lévy process, $d \geqslant 1$, $c \in C^2(\mathbb{R}^d, \mathbb{R}^d)$, $V \in C^2(\mathbb{R}^d)$, $V \geqslant 0$ and $\partial c/\partial x$ is uniformly bounded. Due to the smoothness of V and c, this solution exists and is unique locally (i.e. for small times).

Theorem 2.1.1. *We denote by $T_m = \inf\{s \geqslant 0 : |X(s)| \vee |P(s)| \geqslant m\}$ and $T_\infty = \sup_m T_m$ the explosion time of system (2.1). Then $\mathbb{P}(T_\infty = \infty) = 1$.*

Proof. Step 1. We write $\tau_m = \inf\{s \geqslant 0 : |P(s)| \geqslant m\}$ and $\tau_\infty = \sup_m \tau_m$. It is clear that $T_m \leqslant \tau_m$ and so $T_\infty \leqslant \tau_\infty$. Assuming that $T_\infty < \tau < \tau_m$ for some $\tau > 0$, $m \in \mathbb{N}$, we deduce from the first equation in (2.1) that

$$\max_{s\in[0,\tau]} |X(s)| \leqslant |x_0| + \tau \max_{s\in[0,\tau]} |P(s)| \leqslant |x_0| + \tau m.$$

On the other hand $T_\infty < \tau$ implies $\max\limits_{s\in[0,\tau]} |X(s)| = \infty$. The contradiction proves $\tau_\infty = T_\infty$ and we are done if we can show that $\mathbb{P}(\tau_\infty = \infty) = 1$.

Step 2. We put $H = H(x,p) = (p^2/2) + V(x)$. An application of Itô's formula to $H(t) = H(X(t), P(t))$ yields

$$\begin{aligned} dH(t) &= P(t-)\,dP(t) + \frac{1}{2}\operatorname{tr}\left[\frac{\partial c(X(t-))}{\partial x}\,d\mathcal{Z}_t\left(\frac{\partial c(X(t-))}{\partial x}\right)^T\right] \\ &\quad + \frac{\partial V(X(t))}{\partial x}P(t)\,dt + \Sigma, \end{aligned} \tag{2.2}$$

where $\mathcal{Z}_t \in \mathbb{R}^{d\times d}$, $(\mathcal{Z}_t)_{ij} = [\xi_i, \xi_j]_t$ and

$$\begin{aligned} \Sigma = \frac{1}{2}\sum_{0\leqslant\tau\leqslant t}\Big(P^2(\tau) - P^2(\tau-) &- 2P(\tau-)(P(\tau) - P(\tau-)) \\ &- (P(\tau) - P(\tau-))^2\Big) = 0. \end{aligned}$$

Notice first equation in (2.1) implies that $X(t)$ is a continuous function. Replacing dP in (2.2) through the expression in formula (2.1) we arrive at

$$\begin{aligned} &dH(t) \\ &= -P(t-)\frac{\partial c(X(t))}{\partial x}\,d\xi_t + \frac{1}{2}\operatorname{tr}\left[\frac{\partial c(X(t-))}{\partial x}\,d\mathcal{Z}_t\left(\frac{\partial c(X(t-))}{\partial x}\right)^T\right]. \end{aligned} \tag{2.3}$$

Setting a stopping time $\sigma = s \wedge \tau_m \wedge \mathcal{Q}_R$, where $\mathcal{Q}_R = \inf\{t : |\xi_t| \geqslant R\}$, $s > 0$, $m \in \mathbb{N}$ we calculate from (2.3) that

$$H(\sigma-) = H(0) - \mathbf{I} + \mathbf{II}, \tag{2.4}$$

where

$$\begin{aligned} \mathbf{I} &= \int_0^{\sigma-} P(t-)\frac{\partial c(X(t))}{\partial x}\,d\xi_t, \\ \mathbf{II} &= \frac{1}{2}\int_0^{\sigma-}\operatorname{tr}\left[\frac{\partial c(X(t-))}{\partial x}\,d\mathcal{Z}_t\left(\frac{\partial c(X(t-))}{\partial x}\right)^T\right]. \end{aligned}$$

Step 3. Denote by $\psi(D)$ a generator of ξ_t and by $\mathfrak{D}(\psi(D))$ its domain. We want to estimate $|\mathbb{E}\mathbf{I}|$. For this purpose let us take a function $\phi \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$

such that $\phi(x) = x$ if $|x| \leqslant 1$, $\operatorname{supp}\phi \subset \{x : |x| \leqslant 2\}$ and define $\phi_R(x) = R\phi\left(\frac{x}{R}\right)$. Clearly,

$$\phi_R(\xi_t) = \xi_t \qquad \text{for any} \quad t < \mathcal{Q}_R$$

and since $\phi_R \in C_b^\infty(\mathbb{R}^d) \subset \mathfrak{D}(\psi(D))$ is in the domain of the generator of ξ_t we find that

$$M_t^{\phi_R} = \phi_R(\xi_t) - \int_0^t [\psi(D)\phi_R](\xi_t)dt \tag{2.5}$$

is a martingale (w.r.t. the natural filtration of $\{\xi_t\}_{t\geqslant 0}$). Using decomposition (2.5) we have

$$\mathbb{E}\mathbf{I} = \mathbb{E}\int_0^{\sigma-} P(t-)\frac{c(X(t))}{\partial x}\, dM_t^{\phi_R} + \mathbb{E}\int_0^{\sigma-} P(t-)\frac{c(X(t))}{\partial x}[\psi(D)\phi_R](\xi_t)\, dt = \mathbf{I}' + \mathbf{I}''.$$

Applying optimal stopping time to the martingale $\int_0^\sigma P(t-)(\partial c(X(t))/\partial x)\, dM_t^{\phi_R}$ we deduce from

$$\mathbf{I}' = \mathbb{E}\int_0^\sigma P(t-)\,\frac{\partial c(X(t))}{\partial x}\, dM_t^{\phi_R} - \mathbb{E}\, P(\sigma-)\frac{\partial c(X(t))}{\partial x}\,\Delta M_\sigma^{\phi_R}$$

that

$$\mathbf{I}' \leqslant m\left\|\frac{\partial c}{\partial x}\right\|_\infty \mathbb{E}\left|\Delta M_\sigma^{\phi_R}\right| \leqslant 2mR\left\|\frac{\partial c}{\partial x}\right\|_\infty \|\phi\|_\infty, \tag{2.6}$$

where

$$\left\|\frac{\partial c}{\partial x}\right\|_\infty = \max_{i,j=1,\ldots,d}\ \sup_{x\in\mathbb{R}^d}\left|\frac{\partial c_i(x)}{\partial x_j}\right|$$

and we used

$$\left|\Delta M_\sigma^{\phi_R}\right| = |\phi_R(\xi_\sigma) - \phi_R(\xi_{\sigma-})| \leqslant 2R\|\phi\|_\infty.$$

To estimate $\mathbf{I}''$ we first estimate

$$\mathbf{III} = |[\psi(D)\phi_R](x)| = (2\pi)^{-\frac{d}{2}}\left|\int_{\mathbb{R}^d} \exp\{ix\zeta\}\psi(\zeta)\hat{\phi}_R(\zeta)\, d\zeta\right|,$$

where $\hat{\phi}_R$ is a Fourier transform of ϕ_R. Since $\hat{\phi}_R(\zeta) = R^{d+1}\hat{\phi}(R\zeta)$, it follows

$$\mathbf{III} \leqslant (2\pi)^{-\frac{d}{2}} R \int\limits_{\mathbb{R}^d} R^d \left|\psi(\zeta)\hat{\phi}(R\zeta)\right| \, d\zeta = (2\pi)^{-\frac{d}{2}} R \int\limits_{\mathbb{R}^d} \left|\psi\left(\frac{\eta}{R}\right)\hat{\phi}(\eta)\right| \, d\eta.$$

Lévy-Khinchine formula implies that $|\psi(v)| \leqslant C_1(1+|v|^2)$ for some constant $C_1 = C_1(\psi) > 0$ and so we find

$$|[\psi(D)\phi_R](x)| \leqslant C_1 R \int\limits_{\mathbb{R}^d} \left(1 + \left|\frac{\eta}{R}\right|^2\right) \left|\hat{\phi}(\eta)\right| \, d\eta.$$

Since $\hat{\phi} \in \mathcal{S}(\mathbb{R}^d)$, we obtain

$$|[\psi(D)\phi_R](x)| \leqslant C_1 R \int\limits_{\mathbb{R}^d} \left(1 + |\eta|^2\right) \left|\hat{\phi}(\eta)\right| \, d\eta = RC_2 < \infty$$

for some constant $C_2 = C_2(\psi, \phi)$ uniformly for all $x \in \mathbb{R}^d$. Hence

$$\mathbf{I}'' \leqslant \left\|\frac{\partial c}{\partial x}\right\|_\infty m C_2 R\, s. \tag{2.7}$$

Combining (2.6), (2.7), gives

$$|\mathbb{E}\mathbf{I}| \leqslant C_3 mR + C_4 mRs, \tag{2.8}$$

where $C_3 = 2\|\partial c/\partial x\|_\infty \|\phi\|_\infty$, $C_4 = \|\partial c/\partial x\|_\infty C_2$.

Step 4. We proceed with $|\mathbb{E}\mathbf{II}|$. Formula

$$\|AB\|_\infty \leqslant d\|A\|_\infty \|B\|_\infty, \qquad A, B \in \mathbb{R}^{d\times d},$$

where $\|A\|_\infty = \max\limits_{i,j=1,\dots,d} (A)_{ij}$, implies that

$$\operatorname{tr}\left[\frac{\partial c(X(t-))}{\partial x} \, d\mathcal{Z}_t \left(\frac{\partial c(X(t-))}{\partial x}\right)^T\right] \leqslant d^3 \left\|\frac{\partial c}{\partial x}\right\|_\infty^2 d[\xi, \xi]_t.$$

Using

$$\mathbb{E}\left([\xi, \xi]_{s\wedge\sigma_R -}\right) \leqslant s \int\limits_{|y|\leqslant 2R} |y|^2 \, \nu(dy),$$

we get

$$\mathbb{E}\,\mathrm{II} \leqslant C_5\, s \int\limits_{|y|\leqslant 2R} |y|^2\, \nu(dy) \tag{2.9}$$

for some $C_5 = (d^3/2)\, \|\partial c/\partial x\|^2$.

Step 5. Combining (2.4), (2.8), (2.9) we obtain

$$\mathbb{E}\,(H(\sigma-)) \leqslant H(0) + C_3 mR + C_4 mR\, s + C_5\, s \int\limits_{|y|\leqslant 2R} |y|^2 \nu(dy). \tag{2.10}$$

On the other hand,

$$\begin{aligned} \mathbb{E}\,(H(\sigma-)) &= \frac{1}{2}\,\mathbb{E}\,P^2(\sigma-) + \mathbb{E}\,V(X(\sigma-)) \geqslant \frac{1}{2}\,\mathbb{E}\,P^2(\sigma-) \\ &\geqslant \frac{1}{2}\,\mathbb{E}\left(P^2(s\wedge\tau_m\wedge\mathcal{Q}_R-)\mathbf{1}_{\{\tau_m<s\wedge\mathcal{Q}_R\}}\right) \geqslant \frac{m^2}{2}\mathbb{P}\left(\tau_m < s\wedge\mathcal{Q}_R\right). \end{aligned} \tag{2.11}$$

Piecing together (2.10) and (2.11) finally gives

$$\mathbb{P}(\tau_m < s\wedge\mathcal{Q}_R) \leqslant \frac{2H(0)}{m^2} + \frac{2C_3 R}{m} + \frac{2C_4 R\, s}{m} + \frac{2C_5\, s}{m^2} \int\limits_{|y|\leqslant 2R} |y|^2 \nu(dy).$$

Let first $m\to\infty$ and then $R\to\infty$ shows

$$\mathbb{P}(\tau_\infty < s) = 0$$

for any fixed s, so $\mathbb{P}(\tau_\infty = \infty) = 1$, and the claim follows. □

2.2 Transience

Our proof of the transience for system

$$\left\{ \begin{array}{rcl} dx &=& p\,dt \\ dp &=& -\frac{\partial V}{\partial x}\,dt - \frac{\partial c}{\partial x}\,dw_{\alpha,t}\,, \end{array} \right. \tag{2.12}$$

where $w_{\alpha,t}$ is α-stable process, will be based on the following statement which is a natural extension to general Markov processes of a criterion which is well known for diffusion processes (see e.g. [Fr], [Kha]).

Lemma 2.2.1. *Let $\{\eta_t\}_{t\geqslant 0}$ be a predictable, $\mathbb{R}^n$-valued, càdlàg strong Markov process generated by $(A, \mathfrak{D}(A))$. Let $D \subset \mathbb{R}^n$ be a relatively compact Borel set and assume that there exists some $u \in \mathfrak{D}(A)$ with the following properties:*

(i) $u \in C_b(\mathbb{R}^n)$

(ii) $\inf_D u > a > 0$ *in* D

(iii) $u \geqslant 0$ *in* D^c

(iv) $u(y_0) < a$ *for some* $y_0 \notin \bar{D}$

(v) $Au \in C(\mathbb{R}^n)$ *and satisfies* $Au \leqslant 0$ *in* D^c

Then $\{\eta_t\}_{t\geqslant 0}$ *is transient.*

Proof. Since $u \in \mathfrak{D}(A)$ we find that

$$M_t = u(\eta_t) - \int_0^t Au(\eta_s)\,ds$$

is a martingale. We set

$$\tau_D = \inf\{t > 0 : \eta_t \in D\}.$$

An application of optional stopping time shows that for any fixed $T > 0$

$$\mathbb{E}^{y_0} M_{\tau_D \wedge T} = \mathbb{E}^{y_0} M_0 = \mathbb{E}^{y_0} u(y_0) = u(y_0) < a.$$

On the other hand

$$\begin{aligned}
\mathbb{E}^{y_0} M_{\tau_D \wedge T} &= \mathbb{E}^{y_0}\left(u(\eta_{\tau_D \wedge T}) - \int_0^{\tau_D \wedge T} Au(\eta_s)ds\right)\\
&\geqslant \mathbb{E}^{y_0} u(\eta_{\tau_D \wedge T}) \geqslant \mathbb{E}^{y_0}\left(u(\eta_{\tau_D \wedge T})\mathbf{1}_{\tau_D < \infty}\right).
\end{aligned}$$

Since $u \in C_b$ we can use Lebesgue Theorem and let $T \to \infty$. This gives

$$\begin{aligned}
a &\geqslant \lim_{T\to\infty} \mathbb{E}^{y_0}\left(u(\eta_{\tau_D \wedge T})\mathbf{1}_{\tau_D < \infty}\right) = \mathbb{E}^{y_0}\left(u(\eta_{\tau_D})\mathbf{1}_{\tau_D < \infty}\right)\\
&\geqslant (\inf_D u)\, \mathbb{P}^{y_0}(\tau_D < \infty) > a\mathbb{P}^{y_0}(\tau_D < \infty).
\end{aligned}$$

Therefore $\mathbb{P}^{y_0}(\tau_D < \infty) < 1$, that is (see e.g [AKR]) $\{\eta_t\}_{t\geqslant 0}$ is transient. $\square$

Lemma 2.2.2. *Let $\{\xi_t\}_{t\geqslant 0}$, be a Lévy process with $Q=0$, that is its Lévy measure has no Brownian part. The generator of the process $(X(t),P(t)) = (X(t,x_0,p_0),P(t,x_0,p_0))$ solving* (2.1) *is given by the formula*

$$Au(x,p) = \frac{\partial u(x,p)}{\partial x}p - \frac{\partial u(x,p)}{\partial p}\frac{\partial V(x)}{\partial x} - \frac{\partial u(x,p)}{\partial p}\frac{\partial c(x)}{\partial x}\mathbb{E}\xi_1 \qquad (2.13)$$
$$+ \int\limits_{|\zeta|\neq 0}\left(u\left(x,p+\frac{\partial c(x)}{\partial x}\zeta\right) - u(x,p) - \frac{\partial u(x,p)}{\partial p}\frac{\partial c(x)}{\partial x}\zeta\right)\nu(d\zeta).$$

Proof. Let $u(x_0,p_0)\in C^2(\mathbb{R}^d)$. Since $[\xi,\xi]^c=0$, an application of Itô's formula shows

$$u(X(t),P(t)) = \int\limits_0^t \frac{\partial u}{\partial x}P\,d\tau - \int\limits_0^t \frac{\partial u}{\partial p}\frac{\partial V}{\partial x}\,d\tau - \int\limits_0^t \frac{\partial u}{\partial p}\frac{\partial c}{\partial x}\,d\xi_\tau + \mathbf{I},$$

where

$$\mathbf{I} = \sum_{0\leqslant\tau\leqslant t}\Big(u(X(\tau),P(\tau)) - u(X(\tau-),P(\tau-)) + \frac{\partial u(X(\tau-),P(\tau-))}{\partial p}\frac{\partial c}{\partial x}\Delta\xi_\tau\Big).$$

One readily sees,

$$\frac{d}{dt}\mathbb{E}u(X(t),P(t))\Big|_{t=0} \qquad (2.14)$$
$$= \frac{\partial u(x_0,p_0)}{\partial x}p_0 - \frac{\partial u(x_0,p_0)}{\partial p}\frac{\partial V(x_0)}{\partial x} - \frac{\partial u(x_0,p_0)}{\partial p}\frac{\partial c(x_0)}{\partial x}\mathbb{E}\xi_1 + \frac{d}{dt}(\mathbb{E}\mathbf{I})\Big|_{t=0}.$$

The compensation formula (see [Ber1], page 7) gives

$$\mathbb{E}\mathbf{I} = \mathbb{E}\int\limits_0^t\int\limits_{\mathbb{R}^d}\Bigg[u\left(X(\tau-),P(\tau-)+\frac{\partial c(X(\tau-))}{\partial x}y\right) - u(X(\tau-),P(\tau-)) + \frac{\partial u(X(\tau-),P(\tau-))}{\partial p}\frac{\partial c(X(\tau))}{\partial x}y\Bigg]\nu(dy)\,d\tau$$

and so

$$\frac{d}{dt}(\mathbb{E}\mathbf{I})\bigg|_{t=0} \tag{2.15}$$
$$= \int\limits_{\mathbb{R}^d} \left[u\left(x_0, p_0 + \frac{\partial c(x_0)}{\partial x} y\right) - u(x_0, p_0) + \frac{\partial u(x_0, p_0)}{\partial p} \frac{\partial c(x_0)}{\partial x} y \right] \nu(dy).$$

Combining (2.14) and (2.15) we complete the proof. □

Theorem 2.2.1. *Let $d \geqslant 3$, $V \in C^2(\mathbb{R}^d)$, $c \in C^2(\mathbb{R}^d, \mathbb{R}^d)$. Then the process solving* (2.12) *is transient.*

Proof. We are going to apply Lemma 2.2.1. Take the function

$$u(x,p) = (H(x,p) - V_0)^{-\gamma} = \left(\frac{p^2}{2} + V(x) - V_0\right)^{-\gamma}, \quad \gamma > 1,$$

where $V_0 = \inf V - 1$, and

$$D = \{(x,p) : |x| + |p| \leqslant 1\} \subset \mathbb{R}^{2d}, \qquad a = (1/2) \min_{(x,p)\in D} u(x,p).$$

Conditions (i)-(iv) of Lemma 2.2.1 obviously hold. For chosen $u = u(x,p)$ we get

$$\frac{\partial u}{\partial x} p - \frac{\partial u}{\partial p}\frac{\partial V}{\partial x} = 0.$$

The Lévy measure $\nu(d\zeta)$ for α-stable process is given by the formula $\nu(d\zeta) = |\zeta|^{-d-\alpha}\, d\zeta$. Using this and the symmetry of ν, we deduce from (2.13) that

$$[Au](x,p) = \int\limits_{|\zeta|\neq 0} \left(u(x, p + \frac{\partial c}{\partial x}\zeta) - u(x,p) \right) \frac{1}{|\zeta|^{d+\alpha}}\, d\zeta.$$

An application of Corollary $A.1$ with $B = (\partial c/\partial x)$ with $b = 2(V(x) - V_0)$ gives $Au \leqslant 0$ for any $x, p \in \mathbb{R}^d$ and so condition (v) of Lemma 2.2.1 holds. Applying Lemma 2.2.1 we complete the proof. □

Chapter 3

Estimates for Multiple Stochastic Integrals

Here we derive some estimates for multiple stochastic integrals which will be needed later on. We use the following notation. For any $A \in \mathbb{R}^{M\times N}$ we write

$$||A||_\infty = \max_{\substack{i=1,\dots,M\\ j=1,\dots,N}} |(A)_{i,j}|.$$

We will always consider k-fold stochastic integrals driven by (general) real-valued semimartingales $\{\eta_{j,t}\}_{t\geqslant 0}$, $j = 1,\dots,d$ with càdlàg paths or by the deterministic process $\eta_{0,t} = t$. We assume that all semimartingales are on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are adapted to the same filtration $(\mathcal{F})_{t\geqslant 0}$. The filtration is assumed to satisfy the usual conditions, i.e., it is right-continuous and augmented. Since the $d\eta_{j,t}$ $j = 0,\dots,d$ may appear in any order we want to keep track when we deal with $d\tau = d\eta_{0,\tau}$ and $d\eta_{j,\tau}$, $j = 1,\dots,d$. To do so we introduce a sequence $\ell_n \in \mathbb{N}$ in the following way: if

$$d\eta_{j_1,\tau_1} d\eta_{j_2,\tau_2} \dots d\eta_{j_k,\tau_k} \tag{3.1}$$

is the integrator in our k-fold integral, then

$$\begin{aligned}
\ell_1 &= \min\{s : j_s \neq 0\}\\
\ell_2 &= \min\{s > \ell_1 : j_s = 0\} - 1\\
&\;\;\vdots\\
\ell_{2n-1} &= \min\{s > \ell_{2n-2} : j_s \neq 0\}\\
\ell_{2n} &= \min\{s > \ell_{2n-1} : j_s = 0\} - 1
\end{aligned}$$

i.e., we observe runs of general semimartingale integrators between ℓ_{2n-1} and ℓ_{2n} (inclusive) and of $d\tau$'s otherwise. Note that $\ell_{2n-1} \leqslant \ell_{2n}$ and $\ell_{2n}+1 < \ell_{2n+1}$. We set $\mathcal{M}_i = \{1,\dots,d\}$ if $\ell_{2n-1} \leqslant i \leqslant \ell_{2n}$ for some n and $\mathcal{M}_i = \{0\}$ otherwise. Finally, set

$$\mathfrak{m} = \mathfrak{m}(k) = \#\{s : j_s \neq 0\}$$

i.e. $\mathfrak{m}$ is the number of non-trivial integrals in (3.1).

Let $W_{j,\tau} = W_{j,\tau}(\omega)$, $I_{0,\tau} = I_{0,\tau}(\omega)$ be $\mathbb{R}^{M\times M}$-valued continuous processes, such that for $0 \leqslant t_0 \leqslant \tau$

$$\sup_{t_0 \leqslant s \leqslant \tau} \|W_{j,s}\|_\infty \leqslant 1, \quad j = 0,\dots,d, \tag{3.2}$$

and v_τ be some real-valued adapted increasing process such that for any $a, b \in \mathbb{R}_+$

$$\|W_{j,b} - W_{j,a}\|_\infty \leqslant v_b - v_a, \quad j = 1,\dots,d. \tag{3.3}$$

We also assume that

$$2 \sup_{\tau_1 \in [t_0,\tau]} |\eta_{i,\tau_1} - \eta_{i,t_0}| \leqslant 1, \qquad [\eta_i, \eta_i]^c = 0, \quad i = 1,\dots,d. \tag{3.4}$$

Notice that the assumptions (3.2), (3.4) can always be achieved by suitable (pre-)stopping arguments.

For $0 \leqslant t_0 \leqslant \tau$ we set

$$I_{k,\tau} = \sum_{(j_1,\dots,j_k)\in\mathcal{M}} \int_{t_0}^{\tau}\int_{t_0}^{\tau_k -} \cdots \int_{t_0}^{\tau_2 -} W_{j_k,\tau_k} \cdots W_{j_1,\tau_1} I_{0,\tau_1}\, d\eta_{j_1,\tau_1} \cdots d\eta_{j_k,\tau_k},$$

where $\mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_k$, and

$$D_\tau \tag{3.5}$$

$$= dM \left[v_\tau - v_{t_0} + 4 \sum_{j=1}^{d} \left(\sup_{\tau_1 \in [t_0,\tau]} |\eta_{j,\tau_1} - \eta_{j,t_0}| + \left([\eta_j\,,\eta_j]_\tau - [\eta_j\,,\eta_j]_{t_0}\right)^{\frac{1}{2}} \right) \right].$$

We will use the abbreviation *BV-process* for a process with (almost surely) paths of bounded variation on compact time-intervals.

Proposition 3.0.1. *Let $W_{j,\tau}$, $\eta_{i,\tau}$ be as above and assume that*

$$W_{j,\tau}W_{i,\tau} = W_{i,\tau}W_{j,\tau} \qquad j,i = 1,\dots,d.$$

Then

$$||I_{k,\tau}||_\infty \leqslant b_k\, D_\tau^{\mathrm{m}}\{M(\tau - t_0)\}^{k-\mathrm{m}}||I^*_{0,\tau}||_\infty, \tag{3.6}$$

where $||I^*_{0,\tau}||_\infty = \sup\limits_{t_0 \leqslant s \leqslant \tau} ||I_{0,s}||_\infty$ *and*

$$b_k = \frac{(2^7)^k}{(\ln\{\ln\{k+2\}\})^{\frac{k}{16}}}, \tag{3.7}$$

provided that at least one of the following two conditions hold:

$$I_{0,\tau} = E_M \quad \text{or} \quad \ell_1 > 1.$$

For the proof of Proposition 3.0.1 we need some technical lemmas. Here and later we assume tacitly $t_0 = 0$.

Lemma 3.0.3. *Let $\{U_\tau\}_{\tau \geqslant 0}$, $\{\Phi_\tau\}_{\tau \geqslant 0}$ be predictable $\mathbb{R}^{M\times M}$-valued processes, ν_τ, κ_τ be real-valued semimartingales and*

$$Q_\tau = \int\limits_0^\tau \int\limits_0^{\tau_2 -} U_{\tau_2}\Phi_{\tau_1 -}\, d\nu_{\tau_1} d\kappa_{\tau_2}.$$

If U_τ is a continuous BV random process, then

$$\begin{aligned} Q_\tau \;&=\; -\int\limits_0^\tau U_{\tau_2}\Phi_{\tau_2 -}\, d[\nu,\kappa]_{\tau_2} - \int\limits_0^\tau dU_{\tau_2} \times \int\limits_0^{\tau_2 -} (\kappa_{\tau_2} - \kappa_\tau)\Phi_{\tau_1 -}\, d\nu_{\tau_1} \\ &\quad - \int\limits_0^\tau (\kappa_{\tau_2 -} - \kappa_\tau) U_{\tau_2}\Phi_{\tau_2 -}\, d\nu_{\tau_2}. \end{aligned} \tag{3.8}$$

Remarks.

1. Note that dU_{τ_2} and $\int\limits_0^{\tau_2 -} (\kappa_{\tau_2} - \kappa_\tau)\, \Phi_{\tau_1 -}\, d\nu_{\tau_1}$ are matrix-valued non-commutative objects.

2. All stochastic integrals, where the integrand is a vector (or a matrix) and the integrator is an $\mathbb{R}$-valued semimartingale will be understood coordinate-wise. In a similar way, brackets of vectors and $\mathbb{R}$-semimartingales or matrices of $\mathbb{R}$-semimartingales are understood coordinatewise. The bracket of two matrices A, B is defined as a matrix

$$[A,B]_{ik} = \sum_{j=1}^{M} [a_{ij}, b_{jk}], \quad i,k = 1,\ldots,M,$$

which is compatible with the rules of stochastic calculus and matrix algebra.

Proof. We use the following integration by parts formula for $\mathbb{R}$-valued semimartingales:

$$\int_0^{\tau} Y_{\tau_2 -}\, dZ_{\tau_2} = Y_{\tau} Z_{\tau} - Y_0 Z_0 - [Y,Z]_{\tau} - \int_0^{\tau} Z_{\tau_2 -}\, dY_{\tau_2}.$$

With the coordinate conventions detailed in the above remark we may choose

$$Y_{\tau_2} = U_{\tau 2} \int_0^{\tau 2} \Phi_{\tau_1 -}\, d\nu_{\tau_1},$$

and $Z_{\tau_2} = \kappa_{\tau_2} - \kappa_{\tau}$. Clearly, $Y_0 = Z_{\tau} = 0$ and therefore

$$\begin{aligned} Q_{\tau} &= -\left[U_{\bullet} \int_0^{\bullet} \Phi_{\tau_1 -}\, d\nu_{\tau_1}, \kappa_{\bullet} \right]_{\tau} - \int_0^{\tau} (\kappa_{\tau 2 -} - \kappa_{\tau})\, d\left(U_{\tau_2} \int_0^{\tau 2} \Phi_{\tau_1 -}\, d\nu_{\tau_1} \right) \\ &= \mathbf{I} + \mathbf{II}. \end{aligned}$$

An application of Itô's formula shows

$$\begin{aligned} & d\left(U_{\tau 2} \int_0^{\tau 2} \Phi_{\tau_1 -}\, d\nu_{\tau_1} \right) \\ &= dU_{\tau_2} \times \int_0^{\tau 2 -} \Phi_{\tau_1 -}\, d\nu_{\tau_1} + U_{\tau_2} \Phi_{\tau_2 -}\, d\nu_{\tau_2} + d\left[U_{\bullet}, \int_0^{\bullet} \Phi_{\tau_1 -}\, d\nu_{\tau_1} \right]_{\tau 2}. \end{aligned} \tag{3.9}$$

Since $U_\bullet$ is a continuous BV-process, the bracket in (3.9) vanishes. Thus

$$\mathbf{II} = -\int_0^\tau dU_{\tau_2} \times \int_0^{\tau_2-} (\kappa_{\tau_2} - \kappa_\tau)\Phi_{\tau_1-}\, d\nu_{\tau_1} - \int_0^\tau (\kappa_{\tau_2-} - \kappa_\tau) U_{\tau_2}\Phi_{\tau_2-}\, d\nu_{\tau_2}. \quad (3.10)$$

From (3.9) we find

$$\begin{aligned} -\mathbf{I} &= \left[\left(U_\bullet \int_0^\bullet \Phi_{\tau_1-}\, d\nu_{\tau_1}\right), \kappa_\bullet\right]_\tau = \left[\left(\int_0^\bullet U_{\tau_2}\Phi_{\tau_2-}\, d\nu_{\tau_2}\right), \kappa_\bullet\right]_\tau \\ &+ \left[\int_0^\bullet dU_{\tau_1} \times \int_0^{\tau_1} \Phi_{\tau_2-}\, d\nu_{\tau_2}, \kappa_\bullet\right]_\tau . \end{aligned}$$

Since $U_\bullet$ is a continuous BV-process, so is the stochastic integral driven by dU, and the last bracket above vanishes. So,

$$\mathbf{I} = -\int_0^\tau U_{\tau_2}\Phi_{\tau_2-}\, d[\nu, \kappa]_{\tau_2}.$$

Combining this and (3.10) completes the proof. $\square$

For the multi-index $J = (j_1, \ldots, j_d) \in \mathbb{N}_0^d$ and $i \in \{1, \ldots, d\}$, $k \in \mathbb{N}$ we denote by

$$Q_{J,i,\tau} = \int_0^\tau \int_0^{\tau_2-} U_{J,\tau_2} W_{i,\tau_1} I_{k-1,\tau_1-}\, d\eta_{i,\tau_1}\, d\lambda^J_{\tau_2,\tau},$$

where

$$\begin{aligned} U_{J,\tau_2} &= \frac{1}{J!}(W_{1,\tau_2})^{j_1} \ldots (W_{d,\tau_2})^{j_d}, \quad J! = j_1! \ldots j_d!, \\ \lambda^J_{\tau_2,\tau} &= (\eta_{1,\tau_2} - \eta_{1,\tau})^{j_1} \ldots (\eta_{d,\tau_2} - \eta_{d,\tau})^{j_d}. \end{aligned}$$

Lemma 3.0.4. *For any $m \in \mathbb{N}$ the stochastic integrals $Q_{J,i,\tau}$ satisfy*

$$\begin{aligned} \left\| \sum_{i=1}^d \sum_{J \in \mathcal{B}_m} Q_{J,i,\tau} \right\|_\infty &\leqslant \frac{D_\tau^m}{m!} \int_0^\tau \|I_{k-1,\tau_1-}\|_\infty\, dD_{\tau_1} \quad (3.11) \\ &+ \frac{D_\tau^{m-1}}{(m-1)!} \int_0^\tau \|I_{k,\tau_2-}\|_\infty\, dD_{\tau_2} + \left\| \sum_{L \in \mathcal{B}_{m+1}} \int_0^\tau U_{L,\tau_2} I_{k-1,\tau_2-}\, d\lambda^L_{\tau_2,\tau} \right\|_\infty , \end{aligned}$$

where

$$\mathcal{B}_m = \{J = (j_1, \ldots, j_d) : j_1 + \ldots + j_d = m, \quad j_1, \ldots, j_d \geqslant 0\}$$

and with D_τ *as in* (3.5).

Proof. We write for the right-hand side of (3.11) **I**+**II**+**III**. An application of Lemma 3.0.3 with

$$U_{\tau_2} = U_{J,\tau_2}, \qquad \Phi_{\tau_1} = W_{i,\tau_1} I_{k-1,\tau_1}, \qquad \kappa_{\tau_2} = \lambda^J_{\tau_2,\tau}, \qquad \nu_{\tau_1} = \eta_{i,\tau_1}$$

gives

$$Q_{J,i,\tau} = A_{J,i,\tau} + B_{J,i,\tau} + C_{J,i,\tau} \tag{3.12}$$

with

$$A_{J,i,\tau} = -\int_0^\tau U_{J,\tau_1} W_{i,\tau_1} I_{k-1,\tau_1 -} \, d[\lambda^J_{\bullet,\tau}, \eta_{i,\bullet}]_{\tau_1}, \tag{3.13}$$

$$B_{J,i,\tau} = -\int_0^\tau dU_{J,\tau_2} \times \int_0^{\tau_2} \lambda^J_{\tau_2,\tau} W_{i,\tau_1} I_{k-1,\tau_1 -} \, d\eta_{i,\tau_1}, \tag{3.14}$$

$$C_{J,i,\tau} = -\int_0^\tau \lambda^J_{\tau_2,\tau} U_{J,\tau_2} W_{i,\tau_2} I_{k-1,\tau_2} d\eta_{i,\tau_2} \tag{3.15}$$

from the right-hand side of (3.8).

From (3.5) we get

$$|\eta_{l,\bullet} - \eta_{l,\tau}| \leqslant (2dM)^{-1} D_\tau, \qquad |\Delta\eta_l| \leqslant (2dM)^{-1} D_\tau, \quad l = 1, \ldots, d.$$

Using this, the elementary identity

$$\Delta(a_1 \cdot a_2 \cdot \ldots \cdot a_m) = \sum_{r=1}^m a_0 \ldots a_{r-1} (\Delta a_r)(a_{r+1} - \Delta a_{r+1}) \ldots (a_{m+1} - \Delta a_{m+1}),$$

where $a_0 = a_{m+1} = 1$, $\Delta a_0 = \Delta a_{m+1} = 0$, and $|J| = j_1 + \ldots + j_d = m$, we find

$$|\Delta\lambda^J_{\bullet,\tau}| \leqslant \sum_{r=1}^m (2dM)^{-m+1} D_\tau^{m-1} \max_{l=1,\ldots,d} |\Delta\eta_l| \leqslant (dM)^{-m+1} D_\tau^{m-1} \max_{l=1,\ldots,d} |\Delta\eta_l|,$$

and so

$$|\Delta\eta_i \Delta\lambda^J_{\bullet,\tau}| \leqslant (dM)^{-m+1} D_\tau^{m-1} \sum_{l=1}^{d} |\Delta\eta_l|^2.$$

Since $\eta_{i,\bullet}$ and $\lambda^J_{\bullet,\tau}$ are pure jump semimartingales, the above estimate implies

$$\begin{aligned} \|A_{J,i,\tau}\|_\infty &= \left\| \int_0^\tau U_{J,\tau_1} W_{i,\tau_1} I_{k-1,\tau_1-}\, d[\lambda^J_{\bullet,\tau}, \eta_{i,\bullet}]_{\tau_1} \right\|_\infty \\ &\leqslant \sum_{l=1}^{d} (dM)^{-m+1} D_\tau^{m-1} \int_0^\tau \|U_{J,\tau_1} W_{i,\tau_1} I_{k-1,\tau_1-}\|_\infty \, d[\eta_l, \eta_l]_{\tau_1}. \end{aligned}$$

From condition (3.2) and the formula

$$\|YZ\|_\infty \leqslant M \|Y\|_\infty \|Z\|_\infty, \qquad Y, Z \in \mathbb{R}^{M\times M} \tag{3.16}$$

we obtain $\|U_{J,\tau_2}\| \leqslant (J!)^{-1} M^{m-1}$, hence

$$\|A_{J,i,\tau}\|_\infty \leqslant \frac{1}{2} \frac{d^{-m-1}}{J!} D_\tau^m \int_0^\tau \|I_{k-1,\tau_1-}\|_\infty \, dD_{\tau_1},$$

where we used the fact that for $\tau > s_1 > s_2 \geqslant 0$

$$\begin{aligned} [\eta_l, \eta_l]_{s_1} - [\eta_l, \eta_l]_{s_2} &\leqslant 2\left([\eta_l, \eta_l]_{s_1}\right)^{\frac{1}{2}} \left(\left([\eta_l, \eta_l]_{s_1}\right)^{\frac{1}{2}} - \left([\eta_l, \eta_l]_{s_2}\right)^{\frac{1}{2}} \right) \\ &\leqslant 2(4dM)^{-2} D_\tau \left(D_{s_1} - D_{s_2}\right) \end{aligned} \tag{3.17}$$

holds. From the multinomial identity

$$\sum_{J\in\mathcal{B}_m} \frac{1}{J!} = \frac{d^m}{m!} \tag{3.18}$$

we finally obtain

$$\left\| \sum_{i=1}^{d} \sum_{J\in\mathcal{B}_m} A_{J,i,\tau} \right\|_\infty \leqslant \frac{1}{2m!} D_\tau^m \int_0^\tau \|I_{k-1,\tau_1-}\|_\infty \, dD_{\tau_1} = \frac{1}{2} \mathbf{I}. \tag{3.19}$$

To estimate $B_{J,i,\tau}$, we observe that for the continuous BV-processes $W_{j,\tau}$

$$dU_{J,\tau_2} = dU_{(j_1,\dots,j_d),\tau_2} = \frac{1}{J!}\sum_{r=1}^{d}(W_{1,\tau_2})^{j_1}\dots\left(d(W_{j_r,\tau_2})^{j_r}\right)\dots(W_{d,\tau_2})^{j_d},$$

and so, by (3.3), for any integrand $f(\tau)\in\mathbb{R}^{M\times M}$

$$\left\|\int_0^\tau dU_{J,\tau_1}\times f(\tau_1)\right\|_\infty \leqslant \frac{m}{J!}M^m\int_0^\tau \|f(\tau_1)\|_\infty\, dv_{\tau_1}.$$

Since (3.14) we have

$$\begin{aligned}\left\|\sum_{i=1}^{d}B_{J,i,\tau}\right\|_\infty &= \left\|\sum_{i=1}^{d}\int_0^\tau dU_{J,\tau_2}\times(\lambda^J_{\tau_2,\tau}I_{k,\tau_2-})\right\|_\infty\\ &\leqslant \frac{m}{J!}M^m\sup_{\tau_2\in[0,\tau]}\left|\lambda^J_{\tau_2,\tau}\right|\int_0^\tau \|I_{k,\tau_2}\|_\infty\, dv_{\tau_2}.\end{aligned}$$

Using (3.4) we estimate $|\lambda^J_{\tau_2,\tau}|$ by

$$2\max_{l=1,\dots,d}\{\eta^*_{l,\tau}\}^m \leqslant 2\max_{l=1,\dots,d}\{\eta^*_{l,\tau}\}^{m-1}\leqslant (dM)^{-m+1}D_\tau^{m-1},$$

where $\eta^*_{l,\tau}=\sup\limits_{0\leqslant s\leqslant\tau}|\eta_{l,s}|$. This and the multinomial identity (3.18) imply

$$\sum_{J\in\mathcal{B}_m}\frac{m}{J!}M^m\sup_{\tau_2\in[0,\tau]}\left|\lambda^J_{\tau_2,\tau}\right|\leqslant\frac{dM}{(m-1)!}D_\tau^{m-1}$$

and so

$$\left\|\sum_{i=1}^{d}\sum_{J\in\mathcal{B}_m}B_{J,i,\tau}\right\|_\infty\leqslant\frac{1}{(m-1)!}D_\tau^{m-1}\int_0^\tau\|I_{k,\tau_2-}\|_\infty\, dD_{\tau_2}=\mathbf{II}. \tag{3.20}$$

We proceed with $C_{J,i,\tau}$. Since the $\eta_{j,\tau}$ are pure-jump semimartingales, a *formal* application of Itô's formula to the function

$$f_L(x_1,\dots,x_d)=x_1^{l_1}\dots x_d^{l_d},\qquad L=(l_1,\dots,l_d)\in\mathcal{B}_{m+1} \tag{3.21}$$

and the process $\eta_{\tau_2} - \eta_\tau = ((\eta_{1,\tau_2} - \eta_{1,\tau}), \ldots, (\eta_{d,\tau_2} - \eta_{d,\tau}))$ yields

$$
\begin{aligned}
\widetilde{C}_{L,\tau} &:= \int_0^\tau U_{L,\tau_2} I_{k-1,\tau_2-}\, d\lambda^L_{\tau_2,\tau} \\
&= \sum_{i=1}^d \int_0^\tau U_{L,\tau_2} I_{k-1,\tau_2-} \frac{\partial f_L(\eta_{\tau_2} - \eta_\tau)}{\partial x_i}\, d\eta_{i,\tau_2} + \sum_{0 \leqslant \tau_2 \leqslant \tau} U_{L,\tau_2} I_{k-1,\tau_2-} \delta_{\tau_2} \\
&= \widetilde{C}'_{L,\tau} + \widetilde{C}''_{L,\tau}, \qquad (3.22)
\end{aligned}
$$

where

$$\delta_{\tau_2} = \Delta[f_L(\eta_\bullet - \eta_\tau)]_{\tau_2} - \sum_{i=1}^d \frac{\partial f_L(\eta_{\tau_2-} - \eta_\tau)}{\partial x_i} \Delta\eta_{i,\tau_2}.$$

Notice that the last calculation is formal. It can, however, be (though quite laboriously) justified by considering the stochastic differential

$$d_{\tau_2} \prod_{j=1}^d (\eta_{j,\tau_2} - \eta_{j,\tau})^{l_j}$$

after multiplying out the product. Then the usual Itô rules apply to the semimartingales η_{j,τ_2} and their various mixed products. The process $\eta_{i,\tau}$ and their products are, w.r.t. τ_2, constants.

From the first line in (3.22) we conclude that

$$\mathbf{III} = \left\| \sum_{L \in \mathcal{B}_{m+1}} \widetilde{C}_{L,\tau} \right\|_\infty .$$

Since

$$U_J W_i = U_{(j_1,\ldots,j_d)} W_i = (j_i + 1) U_{(j_1,\ldots,j_i+1,\ldots,j_d)}$$

and

$$
\begin{aligned}
\sum_{L \in \mathcal{B}_{m+1}} l_i U_{L,\tau_2} \lambda^{l_1,\ldots,l_i-1,\ldots,l_d}_{\tau_2,\tau}\, d\eta_{i,\tau_2} &= \sum_{J \in \mathcal{B}_m} (j_i + 1) U_{(j_1,\ldots,j_i+1,\ldots,j_d)} \lambda^J_{\tau_2,\tau}\, d\eta_{i,\tau_2} \\
&= \sum_{J \in \mathcal{B}_m} W_{i,\tau} U_{J,\tau} \lambda^J_{\tau_2,\tau}\, d\eta_{i,\tau_2},
\end{aligned}
$$

it follows that for the integrals $C_{J,i,\tau}$ from (3.15),

$$\sum_{i=1}^{d}\sum_{J\in\mathcal{B}_m} C_{J,i,\tau} = \sum_{i=1}^{d}\sum_{J\in\mathcal{B}_m}\int_0^\tau \lambda^J_{\tau_2,\tau} U_{J,\tau_2} W_{i,\tau_2} I_{k-1,\tau_2} d\eta_{i,\tau_2} \tag{3.23}$$

$$= \sum_{i=1}^{d}\sum_{L\in\mathcal{B}_{m+1}}\int_0^\tau l_i U_{L,\tau_2}\lambda^{l_1,\dots,l_i-1,\dots,l_d}_{\tau_2,\tau} I_{k-1,\tau_2}\, d\eta_{i,\tau_2} = \sum_{L\in\mathcal{B}_{m+1}} \widetilde{C}'_{L,\tau}.$$

Combining formulae (3.22)-(3.23) we have found

$$\left\|\sum_{i=1}^{d}\sum_{J\in\mathcal{B}_m} C_{J,i,\tau}\right\|_\infty \leqslant \mathbf{III} + \left\|\sum_{L\in\mathcal{B}_{m+1}} \widetilde{C}''_{L,\tau}\right\|_\infty . \tag{3.24}$$

We deduce from (3.12) and (3.19), (3.20), (3.24) that

$$\left\|\sum_{i=1}^{d}\sum_{J\in\mathcal{B}_m} Q_{J,i,\tau}\right\|_\infty \leqslant \frac{1}{2}\mathbf{I} + \mathbf{II} + \mathbf{III} + \left\|\sum_{L\in\mathcal{B}_{m+1}} \widetilde{C}''_{L,\tau}\right\|_\infty . \tag{3.25}$$

It remains to estimate the contribution of the jump terms $\widetilde{C}''_{L,\tau}$.

Definition (3.21) implies that

$$\sum_{i,j=1}^{d}\left|\frac{\partial^2 f_L(x)}{\partial x_i \partial x_j}\right| \leqslant (m+1)^2 \max_{n=1,\dots,d} |x_n|^{m-1}.$$

By Taylor's formula and the definition of D_τ, $\tau_2 \leqslant \tau$,

$$\begin{aligned} |\delta_{\tau_2}| &\leqslant \frac{1}{2}\sup_{0\leqslant\theta\leqslant 1}\sum_{i,j=1}^{d}\left|\frac{\partial^2 f_L(\eta_{\tau_2-} - \eta_\tau + \theta\Delta\eta_{\tau_2})}{\partial x_i\partial x_j}\right| |\Delta\eta_{i,\tau_2}\Delta\eta_{j,\tau_2}| \\ &\leqslant \frac{1}{2}(m+1)^2(2dM)^{-m+1}D_\tau^{m-1}\max_{l=1,\dots,d}(\Delta\eta_{l,\tau_2})^2, \end{aligned}$$

where we used the inequality

$$\max_{l=1,\dots,d}|\eta_{l,\tau_2-} - \eta_{l,\tau} + \theta\Delta\eta_{l,\tau_2}|^{m-1} \leqslant (2dM)^{-m+1}D_\tau^{m-1}, \quad \tau_2 \leqslant \tau.$$

Consequently, using $(m+1)2^{-m-1} \leqslant 1$, (3.17) and the fact that $\eta_{n,\bullet}$ is a pure jump semimartingale,

$$\begin{aligned}
\left\|\widetilde{C}''_{L,\tau}\right\|_\infty &\leqslant \frac{1}{2}\sum_{n=1}^{d}(m+1)(dM)^{-m+1}D_\tau^{m-1}\int_0^\tau \left\|U_{L,\tau_1}I_{k-1,\tau_1-}\right\|_\infty d[\eta_n,\eta_n]_{\tau_1} \\
&\leqslant \frac{1}{2}\frac{m+1}{L!}d^{-m-1}D_\tau^m\int_0^\tau \left\|I_{k-1,\tau_1-}\right\|_\infty dD_\tau,
\end{aligned}$$

by (3.18) we obtain

$$\sum_{L\in\mathcal{B}_{m+1}}\frac{m+1}{L!}d^{-m-1} = \frac{1}{m!},$$

and so

$$\left\|\sum_{L\in\mathcal{B}_{m+1}}\widetilde{C}''_{L,\tau}\right\|_\infty \leqslant \frac{1}{2m!}D_\tau^m\int_0^\tau \left\|I_{k-1,\tau_1-}\right\|_\infty dD_\tau = \frac{1}{2}\mathbf{I}.$$

Substituting this into (3.25), completes the proof. $\square$

Corollary 3.0.1. *If $\ell_{2n} < k' < \ell_{2n+1}$ for some $n \in \mathbb{N}$ or $1 \leqslant k' < \ell_1$ then*

$$\left\|\sum_{J\in\mathcal{B}_m}\int_0^\tau\int_0^{\tau_2-}U_{J,\tau_2}W_{0,\tau_1}I_{k'-1,\tau_1}\,d\tau_1 d\lambda^J_{\tau_2,\tau}\right\|_\infty \tag{3.26}$$

$$\leqslant \frac{M}{m!}D_\tau^m\int_0^\tau \|I_{k'-1,\tau_2}\|_\infty\,d\tau_2 + \frac{1}{(m-1)!}D_\tau^{m-1}\int_0^\tau \|I_{k',\tau_2-}\|_\infty\,dD_{\tau_2}$$

Proof. An application of Lemma 3.0.3 with

$$U_{\tau_2} = U_{J,\tau_2},\quad \Phi_{\tau_1} = W_{0,\tau_1}I_{k'-1,\tau_1},\quad \nu_\tau = \eta_{0,\tau} = \tau \quad\text{and}\quad \kappa_{\tau_2} = \lambda^J_{\tau_2,\tau}$$

shows that we can estimate the left-hand side of (3.26) by $\mathbf{I} + \mathbf{II}$, where

$$\begin{aligned}
\mathbf{I} &\leqslant \left\|\sum_{J\in\mathcal{B}_m}\int_0^\tau \lambda^J_{\tau_2,\tau}U_{J,\tau_2}W_{0,\tau_2}I_{k'-1,\tau_2}\,d\tau_2\right\|_\infty, \\
\mathbf{II} &\leqslant \left\|\sum_{J\in\mathcal{B}_m}\int_0^\tau dU_{J,\tau_2}\times\int_0^{\tau_2-}\lambda^J_{\tau_2,\tau}W_{0,\tau_1}I_{k'-1,\tau_1}\,d\tau_1\right\|_\infty.
\end{aligned}$$

Similar calculations to the ones performed in the proof of Lemma 3.0.4 give

$$\begin{aligned}\mathbf{I} &\leqslant M^{m+1}\sum_{J\in\mathcal{B}_m}\frac{1}{J!}\sup_{\tau_1\in[0,\tau]}|\lambda^J_{\tau_1,\tau}|\int_0^\tau \|I_{k'-1,\tau_2}\|_\infty\,d\tau_2,\\ \mathbf{II} &\leqslant M^m\sum_{J\in\mathcal{B}_m}\frac{m}{J!}\sup_{\tau_1\in[0,\tau]}|\lambda^J_{\tau_1,\tau}|\int_0^\tau \|I_{k',\tau_1-}\|_\infty\,dv_{\tau_2},\end{aligned}$$

and

$$M^m\sum_{J\in\mathcal{B}_m}\frac{1}{J!}\sup_{\tau_2\in[0,\tau]}|\lambda^J_{\tau_2,\tau}|\leqslant\frac{D^m_\tau}{m!},\quad M^{m-1}\sum_{J\in\mathcal{B}_m}\frac{m}{J!}\sup_{\tau_2\in[0,\tau]}|\lambda^J_{\tau_2,\tau}|\leqslant\frac{D^{m-1}_\tau}{(m-1)!}.$$

Thus **I** (resp. **II**) is bounded by the first (resp. second) term in formula (3.26). □

Proof of Proposition 3.0.1. Without loss of generality we assume that $t_0 = 0$. We define p_l, $l \in \mathbb{N}_0$, recursively by $p_0 = 1$ and for

$$l\in(\gamma_{n-1},\gamma_n] := \left(\sum_{i=1}^{n-1}(\ell_{2i}-\ell_{2i-1}+1),\sum_{i=1}^{n}(\ell_{2i}-\ell_{2i-1}+1)\right],\quad \gamma_0=0$$

by

$$p_l = p_{\gamma_{n-1}}q_{l-\gamma_{n-1}},\tag{3.27}$$

where q_k are defined by formula (B.9).

Recall that $\mathfrak{m}=\mathfrak{m}(k)$, D_τ are defined at the beginning of this section. Throughout this proof we suppress the argument in $\mathfrak{m}(\cdot)$ if the argument is k, i.e. $\mathfrak{m}=\mathfrak{m}(k)$. We split the proof into two steps.

Step 1. We show by induction that

$$\|I_{k,\tau}\|_\infty\leqslant p_{\mathfrak{m}}D^{\mathfrak{m}}_\tau\frac{(M\tau)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!}\|I^*_{0,\tau}\|_\infty,\qquad k\in\mathbb{N}.\tag{3.28}$$

Clearly, (3.28) holds for $k = 0$. Assume that we have (3.28) for $0,\dots,k-1$.

Case 1. $\ell_{2n} < k < \ell_{2n+1}$ for some $n \in \mathbb{N}$ or $1 \leqslant k < \ell_1$. In this case $\mathfrak{m}(k) = \mathfrak{m}(k-1)$ and $\eta_{j_k,\tau} = \tau$ in the definition of $I_{k,\tau}$. Therefore,

$$\begin{aligned}\|I_{k,\tau}\| &\leqslant M\int_0^\tau\|I_{k-1,s}\|\,ds\leqslant M\int_0^\tau p_{\mathfrak{m}}D^{\mathfrak{m}}_s\frac{(Ms)^{k-1-\mathfrak{m}}}{(k-1-\mathfrak{m})!}\,ds\,\|I^*_{0,\tau}\|_\infty\\ &\leqslant p_{\mathfrak{m}}D^{\mathfrak{m}}_\tau\frac{(M\tau)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!}\|I^*_{0,\tau}\|_\infty.\end{aligned}$$

Case 2. $\ell_{2n-1} \leqslant k \leqslant \ell_{2n}$ for some $n \in \mathbb{N}$. For $m' \in \mathbb{N}$ and $r' = \ell_{2n-1}, \ldots, k$ we denote by

$$z^{m'}_{r',\tau} = \sum_{J \in \mathcal{B}_{m'}} \int_0^\tau U_{J,s} I_{r'-1,s-}\, d\lambda^J_{s,\tau}.$$

Note that $z^1_{r',\tau} = I_{r',\tau}$. Applying Lemma 3.0.4 with

$$Q_{J,i,\tau} = \int_0^\tau \int_0^{\tau_2-} U_{J,\tau_2} W_{r',\tau_1} I_{r'-1,\tau_1-}\, d\eta_{i,\tau_1}\, d\lambda^J_{\tau_2,\tau}$$

we get

$$\|z^{m'}_{r'+1,\tau}\|_\infty \leqslant \frac{D^{m'}_\tau}{m'!} \int_0^\tau \|I_{r'-1,s-}\|_\infty\, dD_s + \frac{D^{m'-1}_\tau}{(m'-1)!} \int_0^\tau \|I_{r',s-}\|_\infty\, dD_s + \|z^{m'+1}_{r',\tau}\|_\infty,$$

where we used

$$z^{m'}_{r'+1,\tau} = \sum_{i=1}^d \sum_{J \in \mathcal{B}_{m'}} Q_{J,i,\tau} \qquad \text{and} \qquad z^{m'+1}_{r',\tau} = \sum_{L \in \mathcal{B}_{m'+1}} \int_0^\tau U_{L,\tau_2} I_{r'-1,\tau_2-}\, d\lambda^L_{\tau_2,\tau}.$$

If $k > \ell_{2n-1}$ then after the change of indices $m' = m+1$, $r'+1 = k-m$, $m = 0, 1, \ldots, k - \ell_{2n-1} - 1$, we obtain

$$\|z^{m+1}_{k-m,\tau}\|_\infty - \|z^{m+2}_{k-(m+1),\tau}\|_\infty \leqslant \beta_{m+1,\tau} + \beta_{m,\tau}, \tag{3.29}$$

where

$$\beta_{m,\tau} = \frac{D^m_\tau}{m!} \int_0^\tau \|I_{k-m-1,s}\|_\infty\, dD_s. \tag{3.30}$$

Summing (3.29) over $m = 0, \ldots, (k - \ell_{2n-1} - 1)$ we get

$$\|z^1_{k,\tau}\|_\infty - \|z^{k-\ell_{2n-1}+1}_{\ell_{2n-1},\tau}\|_\infty \leqslant \beta_{0,\tau} + \beta_{k-\ell_{2n-1},\tau} + 2 \sum_{m=1}^{k-\ell_{2n-1}-1} \beta_{m,\tau}. \tag{3.31}$$

By Corollary 3.0.1, applied to $\ell_{2n-2} < k' < \ell_{2n-1}$, $k' = \ell_{2n-1} - 1$ we find

$$\|z^{k-\ell_{2n-1}+1}_{\ell_{2n-1},\tau}\|_\infty \leqslant \beta_{k-\ell_{2n-1},\tau} + \frac{D^{k-\ell_{2n-1}+1}_\tau}{(k-\ell_{2n-1}+1)!} M \int_0^\tau \|I_{\ell_{2n-1}-2,\tau_2}\|_\infty\, d\tau_2. \tag{3.32}$$

Here and later on in case 2 $\quad \ell_{2n-1} \leqslant k \leqslant \ell_{2n}$. Combining (3.31), (3.32) we arrive at

$$\|z^1_{k,\tau}\|_\infty \leqslant \frac{D_\tau^{k-\ell_{2n-1}+1}}{(k-\ell_{2n-1}+1)!} M \int_0^\tau \|I_{\ell_{2n-1}-2,\tau_2}\|_\infty \, d\tau_2 + 2 \sum_{m=0}^{k-\ell_{2n-1}} \beta_{m,\tau}. \tag{3.33}$$

Since

$$\mathfrak{m}(k-m-1) = \mathfrak{m}(k) - m - 1 \qquad m = 0, \ldots, k - \ell_{2n-1}, \tag{3.34}$$

we can use the induction hypothesis (3.28) and deduce from (3.30) for $m = 0, 1, \ldots, k - \ell_{2n-1}$

$$\begin{aligned} \beta_{m,\tau} &\leqslant \frac{D_\tau^m}{m!} p_{\mathfrak{m}-m-1} \int_0^\tau D_s^{\mathfrak{m}-m-1} \frac{(Ms)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!} \, dD_s \, \|I^*_{0,\tau}\|_\infty \\ &\leqslant \frac{p_{\mathfrak{m}-m-1}}{(\mathfrak{m}-m)\, m!} D_\tau^{\mathfrak{m}} \frac{(M\tau)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!} \|I^*_{0,\tau}\|_\infty . \end{aligned} \tag{3.35}$$

The identity $\mathfrak{m}(\ell_{2n-1}-2) = \mathfrak{m}(\ell_{2n-1}-1) = \mathfrak{m}(k) - (k - \ell_{2n-1}+1)$ implies

$$\begin{aligned} &M \int_0^\tau \|I_{\ell_{2n-1}-2,\tau_2}\|_\infty \, d\tau_2 \\ &\leqslant p_{\mathfrak{m}-(k-\ell_{2n-1}+1)} D_\tau^{\mathfrak{m}-(k-\ell_{2n-1}+1)} M \int_0^\tau \frac{(Ms)^{k-\mathfrak{m}-1}}{(k-\mathfrak{m}-1)!} \, ds \, \|I^*_{0,\tau}\|_\infty . \end{aligned} \tag{3.36}$$

Together (3.33), (3.35) and (3.36) now show

$$\begin{aligned} \|z^1_{k,\tau}\|_\infty &\leqslant p_{\mathfrak{m}-(k-\ell_{2n-1}+1)} \frac{D_\tau^{\mathfrak{m}}}{(k-\ell_{2n-1}+1)!} \frac{(M\tau)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!} \|I^*_{0,\tau}\|_\infty \\ &+ \sum_{m=0}^{k-\ell_{2n-1}} \frac{2}{(\mathfrak{m}-m)\, m!} p_{\mathfrak{m}-m-1} D_\tau^{\mathfrak{m}} \frac{(M\tau)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!} \|I^*_{0,\tau}\|_\infty . \end{aligned} \tag{3.37}$$

Since $\mathfrak{m}-(k-\ell_{2n-1}+1) = \gamma_{n-1}$ it follows from (3.27), (3.34) with $l = \mathfrak{m}-m-1$ that

$$p_{\mathfrak{m}-m-1} = p_{\gamma_{n-1}} q_{(k-\ell_{2n-1}+1)-m-1}, \qquad m = 0, \ldots, k - \ell_{2n-1}.$$

From the definition (B.9) we conclude

$$\begin{aligned}
&\frac{1}{(k-\ell_{2n-1}+1)!}\,p_{\mathfrak{m}-(k-\ell_{2n-1}+1)} + \sum_{m=0}^{k-\ell_{2n-1}} \frac{2}{(\mathfrak{m}-m)\,m!}\,p_{\mathfrak{m}-m-1} \\
&\leqslant\ p_{\gamma_{n-1}} \sum_{m=0}^{k-\ell_{2n-1}} \frac{3}{((k-\ell_{2n-1}+1)-m)\,m!}\,q_{(k-\ell_{2n-1}+1)-m-1} \\
&=\ p_{\gamma_{n-1}} q_{k-\ell_{2n-1}+1} = p_{\mathfrak{m}}, \qquad (3.38)
\end{aligned}$$

where we used $\mathfrak{m} \geqslant k-\ell_{2n-1}+1$. Combining (3.37) and (3.38) we arrive at (3.28).

Step 2. We are going to prove that

$$\frac{p_{\mathfrak{m}}}{(k-\mathfrak{m})!} \leqslant \frac{(2^7)^k}{(\ln\ln(k+1))^{\frac{k}{16}}}. \qquad (3.39)$$

If $1 \leqslant k < \ell_1$ then $\mathfrak{m}(k) = 0$, $p_{\mathfrak{m}} = p_0 = 1$, and estimate (3.39) is clear. From definition (3.27) we deduce

$$p_{\mathfrak{m}} = q_{k-\ell_{2n-1}+1} \prod_{j=1}^{n-1} q_{\ell_{2j}-\ell_{2j-1}+1} \qquad \text{for} \quad \ell_{2n-1} \leqslant k \leqslant \ell_{2n}$$

and

$$p_{\mathfrak{m}} = \prod_{j=1}^{n} q_{\ell_{2j}-\ell_{2j-1}+1} \qquad \text{for} \quad \ell_{2n} < k < \ell_{2n+1}.$$

From (B.10) we know

$$\frac{p_{\mathfrak{m}}}{(k-\mathfrak{m})!} \leqslant (2^6)^k Z^{-1}, \qquad (3.40)$$

where

$$Z = (k-\mathfrak{m})! \prod_{j=1}^{n} (\ln\{\alpha_j+1\})^{\frac{\alpha_j}{2}}.$$

Here $\alpha_j = \ell_{2j} - \ell_{2j-1} + 1$, $j = 1, \dots, n-1$, and $\alpha_n = k - \ell_{2n-1} + 1$ for $\ell_{2n-1} \leqslant k \leqslant \ell_{2n}$, $\alpha_n = \ell_{2n} - \ell_{2n-1} + 1$ for $\ell_{2n} < k < \ell_{2n+1}$. Clearly $k - \mathfrak{m} \geqslant n-1$. Using the estimate from Lemma $B.4$ with $m = k - \mathfrak{m}$ gives

$$Z \geqslant \frac{1}{2^k} (\ln\ln(k+1))^{\frac{k}{16}} \qquad (3.41)$$

and (3.40), (3.41) show (3.39). The assertion now follows from (3.28) and (3.39). □

Chapter 4

Stochastic Hamilton-Jacobi Equations

4.1 Boundary value problems for stochastic Hamilton systems (theorem on a diffeomorphism)

We consider the following Hamiltonian system

$$\begin{cases} dx &= p\,dt \\ dp &= \frac{\partial V(x)}{\partial x}\,dt - \frac{\partial c(x)}{\partial x}d\xi_t, \end{cases} \tag{4.1}$$

with initial condition $(x_0, p_0) \in \mathbb{R}^{2d}$ at $t = t_0$. We write $(X, P) = (X(t, t_0, x_0, p_0), P(t, t_0, x_0, p_0)) \in \mathbb{R}^{2d}$ for its solution. The coefficients $(\partial V/\partial x) \in \mathbb{R}^d$, $(\partial c/\partial x) \in \mathbb{R}^{d\times d}$ are derivatives of functions $V : \mathbb{R}^d \to \mathbb{R}^1$ and $c = (c_1, \ldots, c_d) : \mathbb{R}^d \to \mathbb{R}^d$ which admit (at least) continuous partial derivatives up to order 3 such that

$$\left|\frac{\partial^{|L|}V(x)}{\partial x^L}\right|, \left|\frac{\partial^{|L|}c(x)}{\partial x^L}\right| \leqslant K \quad |L| = 2, 3 \tag{4.2}$$

and

$$\frac{\partial c(x)}{\partial x} = 0, \qquad \forall \quad |x| > K \tag{4.3}$$

for some constant $K > 0$. The driving noise $\xi_t = (\xi_{1,t}, \ldots, \xi_{d,t})$ is a d-dimensional Lévy process such that

$$[\xi_j, \xi_j]^c = 0 \qquad j = 1, \ldots, d, \tag{4.4}$$

i.e. it contains no Brownian component. The main result of this section is the following

Theorem 4.1.1. *Under assumptions* (4.2)-(4.4), *there exists a stopping time* T *such that* $\mathbb{P}(T > 0) = 1$ *and for* $0 \leqslant t_0 < t < T(\omega)$, $x_0 \in \mathbb{R}^d$,

(i) the system (4.1) *has a solution (X,P),*

$$\begin{aligned} \frac{\partial X}{\partial x_0} &= E_d + O(t - t_0), \quad \frac{\partial P}{\partial p_0} = E_d + O(t - t_0), &(4.5)\\ \frac{\partial X}{\partial p_0} &= (t - t_0)E_d + O((t - t_0)^2), &(4.6)\end{aligned}$$

where $O(\cdot)$ *is uniform with respect to* x_0, p_0,

(ii) the map

$$\mathfrak{D} : \mathbb{R}^d \mapsto \mathbb{R}^d, \quad p_0 \to X(t, t_0, x_0, p_0),$$

is a diffeomorphism.

Remark. We can rewrite the system (4.1) in the following form

$$\begin{pmatrix} X(t) \\ P(t) \end{pmatrix} = \int_{t_0}^{t} \mathcal{V}(X(s), P(s))\, ds - \int_{t_0}^{t} \gamma(X(s), P(s)) \cdot d\zeta_s$$

with coefficients

$$\mathcal{V}(x,p) = \begin{pmatrix} p \\ \partial V(x)/\partial x \end{pmatrix} \in \mathbb{R}^{2d}, \qquad \gamma(x,p) = \begin{pmatrix} 0 & 0 \\ 0 & \partial c(x)/\partial x \end{pmatrix} \in \mathbb{R}^{2d \times 2d}.$$

and the (degenerate) Lévy noise

$$\zeta_t = \begin{pmatrix} 0 \\ \xi_t \end{pmatrix} \in \mathbb{R}^{2d}.$$

Notice that $\mathcal{V}$ and γ are globally Lipschitz continuous; Theorem 7 of [Pro], p. 197/8 guarantees existence and uniqueness of a solution $\left\{ \begin{pmatrix} X(t) \\ P(t) \end{pmatrix} \right\}_{t \geqslant 0}$.

Moreover, if the coefficients have globally Lipschitz continuous partial derivatives up to order $N+2$, than we may differentiate $\left\{\begin{pmatrix} X(t) \\ P(t) \end{pmatrix}\right\}_{t\geqslant 0}$ w.r.t. the initial conditions up to order N, cf. [Pro], p.254, Theorem 40.

For the proof of Theorem 4.1.1 we need the following auxiliary result.

Lemma 4.1.1. *There exists a constant* $K_1 = K_1(K,d)$ *such that for* $t_0 \leqslant a \leqslant b < \mathcal{R} \wedge K_1^{-1}$ *($\mathcal{R}$ being the stopping time from Lemma* 0.0.1 *and* K *being the constant from* (4.2)*)*

$$\int_a^b |P(\tau)|\,d\tau \leqslant 3|X(a)-X(b)| + K_1(b-a),$$

where

$$X(\tau) = X(\tau, t_0, x_0, p_0), \qquad P(\tau) = P(\tau, t_0, x_0, p_0). \tag{4.7}$$

Proof. Step 1. From system (4.1) we find

$$P(\tau) - P(a) = \int_a^\tau \frac{\partial V(X(s))}{\partial x}\,ds - \int_a^\tau \frac{\partial c(X(s))}{\partial x}\,d\xi_s.$$

Since $X(t)$, $\partial V(X(t))/\partial x$ and $\partial c(X(t))/\partial x$ are continuous BV-processes, we find by integration by parts

$$\begin{aligned} P(\tau) - P(a) &= \frac{\partial V(X(a))}{\partial x}(\tau - a) - \frac{\partial c(X(a))}{\partial x}(\xi_\tau - \xi_a) \\ &- \int_a^\tau (s-\tau)\frac{\partial^2 V(X(s))}{\partial x^2}P(s)\,ds + \int_a^\tau \frac{\partial^2 c(X(s))(\xi_s - \xi_\tau)}{\partial x^2}P(s)\,ds. \end{aligned} \tag{4.8}$$

Here $\partial^2 c(X(s))(\xi_s - \xi_\tau)/\partial x^2 = \sum_{i=1}^d \partial^2 c_i(X(s))(\xi_{i,s} - \xi_{i,\tau})/\partial x^2 \in \mathbb{R}^{d\times d}$. We know from Lemma 0.0.1 that $2\sup_{0\leqslant s\leqslant \tau}|\xi_{i,s}| \leqslant 1$ for $i = 1,\dots,d$ $\tau < \mathcal{R}$ and so

$$|P(\tau)| \leqslant |P(a)| + C_1 + C_1\int_a^\tau |P(s)|\,ds, \tag{4.9}$$

where

$$C_1 = \max_{i=1,\dots,d} \sup_{x\in\mathbb{R}^d} \left(d^2 \left\| \frac{\partial^2 V}{\partial x^2} \right\|_\infty + d^2 \left\| \frac{\partial^2 c_i}{\partial x^2} \right\|_\infty \right) \vee \left(d \left\| \frac{\partial V}{\partial x} \right\|_\infty + d \left\| \frac{\partial c}{\partial x} \right\|_\infty \right).$$

Integrating (4.9) we have for $b < \mathcal{R}$

$$\int_a^b |P(\tau)|\, d\tau \leqslant |P(a)|(b-a) + C_1(b-a) + C_1 \int_a^b \int_a^\tau |P(s)|\, ds d\tau. \tag{4.10}$$

Since for $b - a < (3C_1)^{-1}$

$$\begin{aligned} C_1 \int_a^b \int_a^\tau |P(s)|\, ds d\tau &= C_1(b-a) \int_a^b |P(s)|\, ds - C_1 \int_a^b (\tau - a)|P(\tau)|\, d\tau \\ &\leqslant \frac{1}{3} \int_a^b |P(s)|\, ds, \end{aligned} \tag{4.11}$$

we deduce from (4.10) that

$$\int_a^b |P(\tau)|\, d\tau \leqslant \frac{3}{2} (|P(a)| + C_1)(b-a). \tag{4.12}$$

Step 2. Similarly, we find from (4.8)

$$|P(\tau) - P(a)| \leqslant C_1 + C_1 \int_a^\tau |P(s)|\, ds. \tag{4.13}$$

We integrate (4.13) to get

$$\begin{aligned} \int_a^b |P(\tau) - P(a)|\, d\tau &\leqslant C_1(b-a) + C_1 \int_a^b \int_a^\tau |P(s)|\, ds d\tau \\ &\leqslant C_1(b-a) + \frac{1}{3} \int_a^b |P(s)|\, ds, \end{aligned}$$

where we used (4.11) and so, by (4.12)

$$\int_a^b |P(\tau) - P(a)|\, d\tau \leqslant \frac{1}{2}(|P(a)| + 3C_1)(b-a).$$

Thus

$$\begin{aligned}\left|\int_a^b P(\tau)\, d\tau\right| &\geqslant |P(a)|(b-a) - \int_a^b |P(\tau) - P(a)|\, d\tau \\ &\geqslant \frac{1}{2}(|P(a)| - 3C_1)(b-a). \end{aligned} \tag{4.14}$$

Combining (4.12) and (4.14) we arrive at

$$\int_a^b |P(\tau)|\, d\tau \leqslant 6C_1(b-a) + 3\left|\int_a^b P(\tau)\, d\tau\right|.$$

The assertion follows with $K_1 = 6C_1$. □

Corollary 4.1.1. *Let $f : \mathbb{R}^d \to \mathbb{R}$, $f \in C^1$ and*

$$f^{(1)}(x) = 0 \quad \text{if} \quad |x| > K$$

(K being the the constant from (4.2)). Then for $0 \leqslant t_0 \leqslant t < \mathcal{R} \wedge K_2^{-1}$

$$Var_{[t_0,t]}\, f(X(\cdot)) \leqslant K_2, \tag{4.15}$$

where $K_2 = K_2(K, d, f)$ is some constant.

Proof. Let $\mathcal{B} = \{\tau \in [t_0, t] : |X(\tau)| \leqslant K\}$. If $\mathcal{B} = \emptyset$ then the left-hand side of (4.15) vanishes and the assertion of the corollary is clear. Otherwise we set $a = \inf\{\tau : \tau \in \mathcal{B}\}$, $b = \sup\{\tau : \tau \in \mathcal{B}\}$. Since $\operatorname{supp} f^{(1)} \subset \{x : |x| \leqslant K\}$,

$$\begin{aligned}\operatorname{Var}_{[t_0,t]} f(X(\cdot)) &\leqslant d \sup_{x\in\mathbb{R}^d} |f^{(1)}(x)| \max_{i=1,\dots,d} \operatorname{Var}_{\mathcal{B}} X_i(\cdot) \\ &\leqslant d \sup_{x\in\mathbb{R}^d} |f^{(1)}(x)| \int_a^b |P(\tau)|\, d\tau \end{aligned}$$

and (4.15) follows from Lemma 4.1.1. □

We introduce a new stopping time

$$T = \mathcal{R} \wedge \widetilde{K}_2^{-1} \wedge 1,$$

where $\widetilde{K}_2 = \max\limits_{i,j=1,\dots,d} K_2(K, d, \partial c_i/\partial x_j)$ and K_2 is defined in Corollary 4.1.1.

Proof of Theorem 4.1.1. *Step1.* Since (X, P) can be differentiated with respect to the initial data (x_0, p_0), we find that the matrix-valued process

$$G = \frac{\partial(X,P)}{\partial(x_0,p_0)} = \begin{pmatrix} \partial X/\partial x_0 & \partial X/\partial p_0 \\ \partial P/\partial x_0 & \partial P/\partial p_0 \end{pmatrix}$$

satisfies the formally differentiated system (4.1) (cf. also [Pro], proof of Theorem 39, p.250):

$$dG = W_{0,t} G\, dt + \sum_{j=1}^{d} W_{j,t} G\, d\xi_{j,t}, \qquad G\big|_{t=t_0} = G_0 = \begin{pmatrix} E_d & 0 \\ 0 & E_d \end{pmatrix}, \tag{4.16}$$

where

$$W_{0,t} = \begin{pmatrix} 0 & E_d \\ \partial^2 V(X(t))/\partial x^2 & 0 \end{pmatrix}, \qquad W_{j,t} = \begin{pmatrix} 0 & 0 \\ -\partial^2 c_j(X(t))/\partial x^2 & 0 \end{pmatrix}. \tag{4.17}$$

A solution of the system (4.16) can be given by the following (formal) series expansion,

$$G = \sum_{k=0}^{\infty} G_k \tag{4.18}$$

with $G_0 = G\big|_{t=t_0}$ and

$$G_1 = \sum_{j=0}^{d} \int_{t_0}^{t} W_{j,\tau} G_0\, d\eta_{j,\tau}, \quad G_k = \sum_{j=0}^{d} \int_{t_0}^{t} W_{j,\tau} G_{k-1}\, d\eta_{j,\tau} \quad (k \in \mathbb{N})$$

where $\eta_\tau = (\eta_{0,\tau}, \eta_{1,\tau}, \dots, \eta_{d,\tau}) = (\tau, \xi_{1,\tau}, \dots, \xi_{d,\tau})$ is a $(d+1)$-dimensional semimartingale. Indeed, it is immediate that

$$\sum_{j=0}^{d} \int_{t_0}^{t} W_{j,\tau}(G_0 + G_1 + \ldots + G_k)\, d\eta_{j,\tau} = G_1 + \ldots + G_{k+1},$$

so (4.18) will give a solution of (4.16) whenever it converges uniformly (on compact intervals) in t.

Since the terms of the series (4.18) are k-fold integrals, we get

$$G = E_{2d} + \sum_{k=1}^{\infty} \sum_{j_1,\dots,j_k=0}^{d} \widetilde{I}_{j_1,\dots,j_k,t} = E_{2d} + \widetilde{I}_{0,t} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{4.19}$$

where $A_{ij} \in \mathbb{R}^{d\times d}$, are suitable (series of) block-matrices and

$$\widetilde{I}_{j_1,\dots,j_k,t} = \int_{t_0}^{t} \int_{t_0}^{\tau_k-} \cdots \int_{t_0}^{\tau_2-} W_{j_k,\tau_k} \dots W_{j_1,\tau_1}\, d\eta_{j_1,\tau_1} \dots d\eta_{j_k,\tau_k}. \tag{4.20}$$

Because of the particular form of the $W_{j,\tau}$'s in (4.17), we know more about the structure of A_{ij} in (4.19). Let

$$\mathcal{J}_1 = \{(j_1,\dots,j_k) : k \in \mathbb{N}, \quad \text{none of } \ j_1,\dots,j_k \text{ equals to } \ 0\}$$

(i.e. all integrators in (4.20) are Lévy processes) and

$$\mathcal{J}_2 = \{(j_1,\dots,j_k) : \ k \geqslant 2 \text{ at most one } j_1,\dots,j_k \text{ equals to } \ 0\}$$

(i.e. at most one $d\tau$ integration happens). If $(j_1,\dots,j_k) \in \mathcal{J}_1$, then the iterated integrals have the form

$$\widetilde{I}_{j_1,\dots,j_k,t} = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}, \quad r \in \mathbb{R}^{d\times d}$$

and if $(j_1,\dots,j_k) \in \mathcal{J}_2$, they are of the form

$$\widetilde{I}_{j_1,\dots,j_k,t} = \begin{pmatrix} r_{11} & 0 \\ r_{21} & r_{22} \end{pmatrix}, \quad r_{11}, r_{21}, r_{22} \in \mathbb{R}^{d\times d}.$$

Thus

$$\|A_{11}\|_\infty, \|A_{22}\|_\infty \leqslant \left\| \sum_{(j_1,\dots,j_k)\in\mathcal{J}\setminus\mathcal{J}_1} \widetilde{I}_{j_1,\dots,j_k,t} \right\|_\infty, \tag{4.21}$$

$$\|A_{12}\|_\infty \leqslant \left\| \sum_{(j_1,\dots,j_k)\in\mathcal{J}\setminus\mathcal{J}_2} \widetilde{I}_{j_1,\dots,j_k,t} \right\|_\infty, \tag{4.22}$$

where $\mathcal{J} = \cup_{k=1}^{\infty}\{0,\ldots,d\}^k$.

Step 2. Let us now verify the conditions needed in Proposition 3.0.1. Lemma 0.0.1 and condition (4.4) imply that $\eta_t = (t,\xi_t)$ satisfies (3.4) for $0 \leqslant t_0 \leqslant t < \mathcal{R}$.

Condition (4.2) implies

$$\|W_{j,t}\|_\infty \leqslant K, \quad j = 0,\ldots,d$$

and by (4.17) we find

$$W_{j,\tau}W_{i,\tau} = 0 = W_{i,\tau}W_{j,\tau} \quad i,j = 1,\ldots,d.$$

Definition (3.5) (with M=2d) and formula (4.15) with $f = (\partial c_i/\partial x_j)$, $i,j = 1,\ldots,d$ give

$$D_t \leqslant 2d^2(K_2 + \vartheta_t + \vartheta_{t_0}),$$

where ϑ_t, ϑ_{t_0} are given by formula (0.13), and Lemma 0.0.1 shows

$$D_t \leqslant 2d^2(K_2+2) = O(1), \qquad 0 \leqslant t_0 \leqslant t < T. \tag{4.23}$$

Step 3. An application of Proposition 3.0.1 to matrices $K^{-1}W_{j,\tau} \in \mathbb{R}^{2d\times 2d}$, $j = 1,\ldots,d$ shows

$$\left\| \sum_{(j_1,\ldots,j_d)\in\mathcal{M}} \widetilde{I}_{j_1,\ldots,j_k,t} \right\|_\infty \leqslant K^k b_k D_t^{\mathfrak{m}}\{2d(t-t_0)\}^{k-\mathfrak{m}},$$

where b_k are given by (3.7),

$$\mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_k \quad \text{and} \quad \mathcal{M}_i = \{1,\ldots,d\} \quad \text{or} \quad \mathcal{M}_i = \{0\}. \tag{4.24}$$

One readily sees

$$\mathcal{M}\cap\mathcal{J}_1 = \emptyset \Longrightarrow \mathfrak{m} \leqslant k-1 \quad \text{and} \quad \mathcal{M}\cap\mathcal{J}_2 = \emptyset \Longrightarrow \mathfrak{m} \leqslant k-2. \tag{4.25}$$

Consequently we find from (4.21)

$$\|A_{11}\|_\infty, \|A_{22}\|_\infty \leqslant \sum_{k=2}^{\infty} 2^k K^k b_k (D_t + 2d(t-t_0))^{k-1}(2d)(t-t_0), \tag{4.26}$$

where we used, by (4.25),

$$D_t^{\mathfrak{m}}\{2d(t-t_0)\}^{k-\mathfrak{m}} \leqslant (D_t + 2d(t-t_0))^{k-1}(2d)(t-t_0)$$

and the fact that the set $\{0, \dots, d\}$ is the union of subsets of type (4.24); the number of such subsets equal to 2^k. Clearly the series $\Phi_1(x) = \sum\limits_{k=2}^{\infty} 2^k K^k b_k x^{k-1}$ converges for any $x \in \mathbb{R}$ and, by (4.23), $\Phi_1(D_t + 2d(t - t_0)) = O(1)$. Then we deduce from (4.26) that

$$\|A_{11}\|_\infty, \|A_{22}\|_\infty = O(t - t_0), \quad 0 \leqslant t_0 \leqslant t < T. \tag{4.27}$$

Similarly, we have

$$\begin{aligned} \|A_{12}\|_\infty &\leqslant \sum_{k=2}^{\infty} 2^k K^k b_k (D_t + 2d(t - t_0))^{k-2} (2d)^2 (t - t_0)^2 \\ &\leqslant (2d)^2 \Phi_2(D_t + 2d(t - t_0))(t - t_0)^2 = O((t - t_0)^2), \end{aligned} \tag{4.28}$$

where $\Phi_2(x) = \sum\limits_{k=2}^{\infty} 2^k K^k b_k x^{k-2}$.

Substituting estimates (4.27), (4.28) into (4.19) we arrive at (4.5), (4.6).

Step 4. From (4.6), we conclude (using the implicit function theorem) that the map $\mathfrak{D} : p_0 \to X(t, p_0) = X(t, t_0, x, x_0)$ is a local diffeomorphism. Let us prove that it is injective. Since

$$X(t, p_2) - X(t, p_1) = \int_0^1 \frac{\partial X}{\partial p_0} (p_1 + \tau(p_2 - p_1)) (p_2 - p_1) \, d\tau,$$

we have

$$\begin{aligned} |X(t, p_2) - X(t, p_1)|^2 = \int_0^1 \int_0^1 (p_2 - p_1)^T \left(\frac{\partial X}{\partial p_0} \right)^T (p_1 + s(p_2 - p_1)) \times \\ \times \left(\frac{\partial X}{\partial p_0} \right) (p_1 + \tau(p_2 - p_1)) (p_2 - p_1) \, d\tau ds \geqslant C \|p_2 - p_1\|^2, \end{aligned} \tag{4.29}$$

for some constant $C = C(t, t_0) > 0$. The last inequality is due to (4.6). This shows that $\mathfrak{D}$ is injective and so $\mathfrak{D} : \mathbb{R}^d \to \mathfrak{D}(\mathbb{R}^d) \subset \mathbb{R}^d$ is a global diffeomorphism. It follows from estimate (4.29) that $X(t, p_0) \to \infty$ as $p_0 \to \infty$. Since $\mathfrak{D}$ is open and closed, $\mathfrak{D}(\mathbb{R}^d) \subset \mathbb{R}^d$ is open and closed. As $\mathfrak{D}(\mathbb{R}^d) \neq \emptyset$ we have $\mathfrak{D}(\mathbb{R}^d) = \mathbb{R}^d$. This finishes the proof of Theorem 4.1.1. $\square$

We assume that for any multi-index $I \in \mathbb{N}_0^d$, $2 \leqslant |I| \leqslant q+2$ the partial derivatives are bounded

$$\left|\frac{\partial^{|I|}V}{\partial x^I}\right| \leqslant K, \qquad \left|\frac{\partial^{|I|}c_i}{\partial x^I}\right| \leqslant K, \quad i=1,\dots,d \tag{4.30}$$

and continuous.

Lemma 4.1.2. *If the coefficients V, c_i satisfy the above mentioned assumptions, we have for any $0 \leqslant t_0 \leqslant t < T$*

$$\frac{\partial^{|I|}X(t,t_0,x_0,p_0)}{\partial p_0^I} = O((t-t_0)^{|I|+1}), \tag{4.31}$$

$$\frac{\partial^{|I|}P(t,t_0,x_0,p_0)}{\partial p_0^I} = O((t-t_0)^{|I|}), \tag{4.32}$$

where $2 \leqslant |I| \leqslant q$ and $O(\cdot)$ is uniform with respect to x_0 and p_0.

For the proof we need the following lemma.

Lemma 4.1.3. *Let $\widetilde{Y}_\tau, Y_{1,\tau}, \dots, Y_{n,\tau}$ be $\mathbb{R}^{M\times M}$-valued locally integrable processes, and ν be a real-valued semimartingale. Then*

$$\left\| \int_{t_0}^{s} \widetilde{Y}_\tau \left(\int_{t_0}^{\tau} Y_{1,\tau_1}\, d\tau_1\right) \dots \left(\int_{t_0}^{\tau} Y_{n,\tau_1}\, d\tau_1\right) d\nu_\tau \right\|_\infty$$

$$\leqslant n(\nu-\nu_s)_s^* M^n \left(\|\widetilde{Y}_s^*\|_\infty + Var_{[t_0,s]}\widetilde{Y}\right) \int_{t_0}^{s} \|Y_{1,\tau_1}\|_\infty\, d\tau_1 \dots \int_{t_0}^{s} \|Y_{n,\tau_1}\|_\infty\, d\tau_1,$$

where $(\nu-\nu_s)_s^* = \sup_{\tau\in[t_0,s]} |\nu_\tau - \nu_s|$, $\widetilde{Y}_s^* = \sup_{\tau\in[t_0,s]} \|Y_\tau\|_\infty$, provided that $\widetilde{Y}_\tau$ is a continuous BV-process.

Proof. We set $U_{i,\tau} = \int_{t_0}^{\tau} Y_{i,\tau_1}\, d\tau_1$, $i=1,\dots,n$ and $\mathbf{I} = \int_{t_0}^{s} \widetilde{Y}_\tau U_{1,\tau}\dots U_{n,\tau}\, d\nu_\tau$. Since $\widetilde{Y}, U_1, \dots, U_n$ are continuous BV-processes, integration by parts gives

$$\mathbf{I} = -\int_{t_0}^{s} (\nu_\tau - \nu_s) d\widetilde{Y}_\tau \times U_{1,\tau}\dots U_{n,\tau} - \sum_{i=1}^{n} \int_{t_0}^{s} (\nu_\tau-\nu_s)\widetilde{Y}_\tau U_{1,\tau}\dots Y_{i,\tau}\dots U_{n,\tau}\, d\tau.$$

The assertion of the Lemma follows now from (3.16). $\square$

Proof of Lemma 4.1.2. For notational convenience we set $c_0(X) = -V(X)$.

Step 1.

Let us choose and fix a sequence $(j_1, j_2, \ldots) \in \{1, \ldots, d\}^{\mathbb{N}}$. Write

$$\mathcal{A}_0(\tau) = \frac{\partial X(\tau)}{\partial p_0}, \quad \mathcal{B}_0(\tau) = \frac{\partial P(\tau)}{\partial p_0}, \quad \mathcal{A}_m(\tau) = \frac{\partial^{|J|}\mathcal{A}_0(\tau)}{\partial p_0^J}, \quad \mathcal{B}_m(\tau) = \frac{\partial^{|J|}\mathcal{B}_0(\tau)}{\partial p_0^J},$$

$\mathcal{A}_0(\tau), \mathcal{B}_0(\tau), \mathcal{A}_m(\tau), \mathcal{B}_m(\tau) \in \mathbb{R}^{d\times d}$, where $J = (j_1, \ldots, j_m)$ and $p_0 = (p_{0,1}, \ldots, p_{0,d}) \in \mathbb{R}^d$.

From equation (4.16) we get

$$d_\tau \begin{pmatrix} \mathcal{A}_0(\tau) \\ \mathcal{B}_0(\tau) \end{pmatrix} = \sum_{i=0}^{d} W_{i,\tau} \begin{pmatrix} \mathcal{A}_0(\tau) \\ \mathcal{B}_0(\tau) \end{pmatrix} d\eta_{i,\tau}. \tag{4.33}$$

Differentiating (4.33) with respect to $p_{0,j_1}, \ldots, p_{0,j_m}$ we get the following system of SDE

$$\begin{aligned} d_\tau \begin{pmatrix} \mathcal{A}_m(\tau) \\ \mathcal{B}_m(\tau) \end{pmatrix} &= \sum_{i=0}^{d} W_{i,\tau} \begin{pmatrix} \mathcal{A}_m(\tau) \\ \mathcal{B}_m(\tau) \end{pmatrix} d\eta_{i,\tau} - \sum_{i=0}^{d} \widetilde{A}_{m,i,\tau}\, d\eta_{i,\tau} \quad (4.34) \\ \mathcal{A}_m(t_0) &= \mathcal{B}_m(t_0) = 0, \end{aligned}$$

where $\widetilde{A}_{m,i,\tau} \in \mathbb{R}^{2d\times d}$ is given by the recurrence relation

$$\widetilde{A}_{0,i,\tau} = 0, \quad \widetilde{A}_{m,i,\tau} = -\frac{\partial W_{i,\tau}}{\partial p_{0,j_m}} \begin{pmatrix} \mathcal{A}_{m-1}(\tau) \\ \mathcal{B}_{m-1}(\tau) \end{pmatrix} + \frac{\partial \widetilde{A}_{m-1,i,\tau}}{\partial p_{0,j_m}} \quad m > 0.$$

The interchange of stochastic and ordinary differentials (with respect to the initial conditions) is possible since the coefficients of the system (4.33) smooth enough cf. Protter [Pro] p.245 Theorem 40. Using (4.17) gives

$$\widetilde{A}_{m,i,\tau} = \begin{pmatrix} 0 \\ A_{m,i,\tau} \end{pmatrix}, \qquad i = 0, \ldots, d,\ A_{m,i,\tau} \in \mathbb{R}^{d\times d}, \tag{4.35}$$

where

$$A_{0,i,\tau} = 0, \quad A_{m,i,\tau} = \frac{\partial c_i^{(2)}(X(\tau))}{\partial p_{0,j_m}} \mathcal{A}_{m-1}(\tau) + \frac{\partial A_{m-1,i,\tau}}{\partial p_{0,j_m}} \quad m > 0. \tag{4.36}$$

Here $c_i^{(2)}(x) = (\partial^2 c_i(x)/\partial x^2) \in \mathbb{R}^{d\times d}$. From (4.36) we find by induction

$$A_{m,i,\tau} = \sum_{k=2}^{m+1} \frac{\partial^{m+1-k}}{\partial p_{0,j_m} \cdots \partial p_{0,j_k}} \left(\frac{\partial c_i^{(2)}(X(\tau))}{\partial p_{0,j_{k-1}}} \frac{\partial^{k-2}\mathcal{A}_0(\tau)}{\partial p_{0,j_{k-2}} \cdots \partial p_{0,j_1}} \right). \tag{4.37}$$

A solution of (4.34) is given by the following (formal) series expansion

$$G = \sum_{k=1}^{\infty} G_k \tag{4.38}$$

with

$$G_1 = -\sum_{i=0}^{d} \int_{t_0}^{t} \widetilde{A}_{m,i,\tau_0}\, d\eta_{i,\tau_0}, \qquad G_k = \sum_{i=0}^{d} \int_{t_0}^{t} W_{i,\tau} G_{k-1}\, d\eta_{i,\tau}, \;\; k \geqslant 2.$$

This can be seen as in the proof of Theorem 4.1.1 and (4.38) gives the solution of (4.34) whenever it converges uniformly (on compact intervals) in t. Thus

$$\begin{pmatrix} \mathcal{A}_m(t) \\ \mathcal{B}_m(t) \end{pmatrix} = -\sum_{i=0}^{d} \int_{t_0}^{t} \widetilde{A}_{m,i,\tau_0}\, d\eta_{i,\tau_0} - \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_k=0}^{d} \sum_{i=0}^{d} \widetilde{I}^{i,m}_{i_1,\dots,i_k,t},$$

where

$$\widetilde{I}^{m,i}_{i_1,\dots,i_k,t} = \int_{t_0}^{t} \int_{t_0}^{\tau_k} \cdots \int_{t_0}^{\tau_1} W_{i_k,\tau_k} \cdots W_{i_1,\tau_1} \widetilde{A}_{m,i,\tau_0}\, d\eta_{i,\tau_0} d\eta_{i_1,\tau_1} \cdots d\eta_{i_k,\tau_k}.$$

Since $W_{i_1,\tau}\widetilde{A}_{m,i,\tau} = 0, \quad i_1 = 1,\dots,d$, it follows that

$$\widetilde{I}^{m,i}_{i_1,\dots,i_k,t} = 0 \qquad \text{for} \quad i_1 > 0.$$

Observe that formula (3.6) still holds for $I_{0,t} \in \mathbb{R}^{M\times N}, \quad \forall N \in \mathbb{N}$. For fixed $i = 1,\dots,d$ an application of Proposition 3.0.1 to matrices $K^{-1}W_{j,\tau} \in \mathbb{R}^{2d\times 2d}$, $j = 1,\dots,d$ with $I_{0,t} = \int_{t_0}^{t} \widetilde{A}_{m,i,\tau_0}\, d\eta_{i,\tau_0} \in \mathbb{R}^{2d\times d}$ shows

$$\left\| \sum_{(i_1,\dots,i_d)\in\mathcal{M}} \widetilde{I}^{m,i}_{i_1,\dots,i_k,t} \right\|_\infty \leqslant K^k b_k D_t^{\mathrm{m}} \{2d(t-t_0)\}^{k-\mathrm{m}} \|I^*_{0,t}\|_\infty,$$

where b_k are given by (3.7), $\mathcal{M} = \mathcal{M}_1 \times \dots \times \mathcal{M}_k$, $\mathcal{M}_1 = \{0\}$ and $\mathcal{M}_i = \{1,\dots,d\}$ or $\mathcal{M}_i = \{0\}$ for $i > 1$. Thus

$$\|\mathcal{A}_m(t)\|_\infty \vee \|\mathcal{B}_m(t)\|_\infty \leqslant \sup_{t_0\leqslant s\leqslant t} \left\| \sum_{i=0}^{d} \int_{t_0}^{s} \widetilde{A}_{m,i,\tau_0}\, d\eta_{i,\tau_0} \right\|_\infty \hat{\Phi}(D_t + t),$$

where $\hat{\Phi}(x) = 1 + \sum_{k=1}^{\infty} 2^k K^k b_k x^k$, D_τ and b_k are given by (3.5) and (3.7) respectively. Using, see (4.23), $\hat{\Phi}(D_t + t) \leqslant \hat{\Phi}(2d^2(K_2+2)+1) = O(1)$ we arrive at

$$\left\| \frac{\partial^{|J|}\mathcal{B}_0(t)}{\partial p_0^J} \right\|_\infty = \|\mathcal{B}_m(t)\|_\infty = O(1) \sup_{t_0 \leqslant s \leqslant t} \left\| \sum_{i=0}^{d} \int_{t_0}^{s} \widetilde{A}_{m,i,\tau_0}\, d\eta_{i,\tau_0} \right\|_\infty . \tag{4.39}$$

Step 2. By induction in m we now show

$$\left\| \int_{t_0}^{s} A_{m,i,\tau}\, d\eta_{i,\tau} \right\|_\infty = O((t-t_0)^{m+1}), \tag{4.40}$$

where $O(\cdot)$ is taken uniformly with respect to $s < t$, $i = 0, \ldots, d$, $J \in \mathbb{N}_0^d$, $x_0, p_0 \in \mathbb{R}^d$. Then combining (4.35), (4.39), (4.40) we obtain (4.32) and, by (4.1), get (4.31).

Let us first check (4.40) for $m = 1$. Estimate (4.40) for $i = 0$ directly follows from (4.6). From (4.37) we find

$$A_{1,i,\tau} = \sum_{j=1}^{d} \frac{\partial c_i^{(2)}(X(\tau))}{\partial x_j} \frac{\partial X_j(\tau)}{\partial p_{0,j_1}} \frac{\partial X(\tau)}{\partial p_0} = \sum_{j=1}^{d} Q_{i,j,j_1}(\tau).$$

For fixed $i, j, j_1 = 1, \ldots, d$ an application of Lemma 4.1.3 with $n = 2$, $M = d$, $\nu_\tau = \eta_{i,\tau}$,

$$\widetilde{Y}_\tau = \frac{\partial c_i^{(2)}(X(\tau))}{\partial x_j}, \qquad Y_{1,\tau} = \frac{\partial P_j(\tau)}{\partial p_{0,j_1}} E_d, \qquad Y_{2,\tau} = \frac{\partial P(\tau)}{\partial p_0}$$

shows

$$\left\| \int_{t_0}^{s} Q_{i,j,j_1}(\tau)\, d\eta_{i,\tau} \right\|_\infty = O(1)(\eta_i - \eta_{i,s})_s^* \left(\int_{t_0}^{s} \left\| \frac{\partial P(\tau)}{\partial p_0} \right\|_\infty d\tau \right)^2,$$

where we used that, by Corollary 4.1.1, $\mathrm{Var}_{[t_0,s]}\widetilde{Y} = O(1)$ and, by (4.30), $\|\widetilde{Y}_s^*\|_\infty = \sup_{\tau \in [t_0,s]} \|\widetilde{Y}_\tau\|_\infty = O(1)$. Recall that $\eta_{i,s}$ is a Lévy process. Therefore by Lemma 0.0.1 we have $(\eta_i - \eta_{i,s})_s^* = O(1)$ for $0 \leqslant t_0 \leqslant s < \mathcal{R}$. Using (4.5) we see that $\|\partial P/\partial p_0\|_\infty = O(1)$ and so $\| \int_{t_0}^{s} Q_{i,j,j_1}\, d\eta_{i,\tau}\|_\infty = O((t-t_0)^2)$. Hence

$$\left\| \int\limits_{t_0}^{s} A_{1,i,\tau}\, d\eta_{i,\tau} \right\|_\infty \leqslant \sum_{j=1}^{d} \left\| \int\limits_{t_0}^{s} Q_{i,j,j_1}(\tau)\, d\eta_{i,\tau} \right\|_\infty = O((t-t_0)^2),$$

for $i = 1, \ldots, d$.

We now assume that (4.40) holds for $n = 1, \ldots, m-1$. Estimate (4.40) for $i = 0$ immediately follows from (4.6) and the induction assumption. From (4.37) we find that $A_{m,i,\tau}$ is a sum of the terms of the form

$$\begin{aligned} & Q_{i,i_1,\ldots,i_r,\rho,\ldots,\mu,L}(\tau) \\ &= \frac{\partial^{|\rho|} c_i^{(2)}(X(\tau))}{\partial x^\rho} \frac{\partial^{|\mu|} X_{i_1}(\tau)}{\partial p_0^\mu} \cdots \frac{\partial^{|\lambda|} X_{i_r}(\tau)}{\partial p_0^\lambda} \frac{\partial^{|L|} \mathcal{A}_0(\tau)}{\partial p_0^L}, \end{aligned} \tag{4.41}$$

where

$$|\rho| = r \leqslant m, \qquad |\mu| + \ldots + |\lambda| + |L| = m. \tag{4.42}$$

We put $\widetilde{Y}_\tau = (\partial^{|\rho|} c_i^{(2)}(X(\tau))/\partial x^\rho)$. By (4.30) we see $\|\widetilde{Y}_s^*\|_\infty = O(1)$. Using again Corollary 4.1.1 we obtain

$$\sup_{p,q=1,\ldots,d} \mathrm{Var}_{[t_0,t]}(\widetilde{Y})_{pq} = O(1)$$

and so for $\nu_\tau = \eta_{i,\tau}$, $n = r+1$, $M = d$ we have

$$n(\eta_i - \eta_{i,s})_s^* M^n \left(\|\widetilde{Y}_s^*\|_\infty + \mathrm{Var}_{[t_0,s]} \widetilde{Y} \right) = O(1),$$

where we used Lemma 0.0.1 in the form $(\eta_i - \eta_{i,s})_s^* \leqslant \vartheta_s < 1$ for $s < \mathcal{R}$. From equation (4.1) we find

$$\frac{\partial^{|\mu|} X(\tau)}{\partial p_0^\mu} = \int\limits_{t_0}^{\tau} \frac{\partial^{|\mu|} P(\tau_1)}{\partial p_0^\mu}\, d\tau_1$$

and so for fixed $i, i_1, i_r = 1, \ldots, d$ and $\rho, \mu, \ldots, \lambda, L \in \mathbb{N}_0^d$ an application of Lemma 4.1.3 with $n = r+1$

$$\begin{aligned} \widetilde{Y}_\tau &= \frac{\partial^{|\rho|} c_i^{(2)}(X(\tau))}{\partial x^\rho}, \\ Y_{1,\tau} &= \frac{\partial^{|\mu|} P_{i_1}(\tau)}{\partial p_0^\mu} E_d, \ldots, Y_{r,\tau} = \frac{\partial^{|\lambda|} P_{i_r}(\tau)}{\partial p_0^\lambda} E_d, Y_{r+1,\tau} = \frac{\partial^{|L|} \mathcal{B}_0(\tau)}{\partial x^L} \end{aligned}$$

shows

$$\left\| \int_{t_0}^{s} Q_{i,i_1,\ldots,i_r,\rho,\ldots,\mu,L}(\tau)\, d\eta_{i,\tau} \right\|_\infty \tag{4.43}$$
$$= O(1) \int_{t_0}^{s} \left\| \frac{\partial^{|\mu|} P(\tau)}{\partial p_0^\mu} \right\|_\infty d\tau \ldots \int_{t_0}^{s} \left\| \frac{\partial^{|\lambda|} P(\tau)}{\partial p_0^\lambda} \right\|_\infty d\tau \int_{t_0}^{s} \left\| \frac{\partial^{|L|} \mathcal{B}_0(\tau)}{\partial x^L} \right\|_\infty d\tau.$$

By induction assumption and by (4.6) we have

$$\left\| \frac{\partial^{|\mu|} P(\tau)}{\partial p_0^\mu} \right\|_\infty = O((t-t_0)^{\phi(|\mu|)}) \qquad \left\| \frac{\partial^{|L|} \mathcal{B}_0(\tau)}{\partial x^L} \right\|_\infty = O((t-t_0)^{|L|}),$$

where $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(1) = 0$, $\phi(n) = n$ for all $n > 1$. Hence

$$\left\| \int_{t_0}^{s} Q_{i,i_1,\ldots,i_r,\rho,\ldots,\mu,L}(\tau)\, d\eta_{i,\tau} \right\|_\infty = O((t-t_0)^\gamma),$$

where

$$\gamma = \phi(|\mu|) + 1 + \ldots + \phi(|\lambda|) + 1 + |L| + 1.$$

Using that $1 + \phi(n) \geqslant n \ \forall n \in \mathbb{N}$ and (4.42), give

$$\gamma \geqslant |\mu| + \ldots + |\lambda| + |L| + 1 = m + 1.$$

Thus

$$\left\| \int_{t_0}^{s} A_{m,i,\tau}\, d\eta_{i,\tau} \right\|_\infty = O(1) \left\| \int_{t_0}^{s} Q_{i,i_1,\ldots,i_r,\rho,\ldots,\mu,L}(\tau)\, d\eta_{i,\tau} \right\|_\infty = O((t-t_0)^{m+1})$$

for $i = 1, \ldots, d$. □

4.2 The method of stochastic characteristics

As before we denote by $(X, P) = (X(t, t_0, x_0, p_0), P(t, t_0, x_0, p_0))$ the solution of the Hamilton system

$$\begin{cases} dx &= \frac{\partial H}{\partial p}\, dt \\ dp &= -\frac{\partial H}{\partial x}\, dt - \frac{\partial c}{\partial x}\, d\xi_t, \end{cases} \tag{4.44}$$

with initial condition $(x_0, p_0) \in \mathbb{R}^{2d}$ at $t = t_0$, where $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $c : \mathbb{R}^d \to \mathbb{R}^d$. We shall say that H and c satisfy property (D1) if

$$\begin{array}{l}\text{There exists a stopping time } T > 0 \text{ a.s. such that}\\ \text{for any } 0 \leqslant t_0 < t < T,\ \forall x_0 \in \mathbb{R}^d \text{ the map}\\ \mathfrak{D}_1 : \mathbb{R}^d \to \mathbb{R}^d, \quad p_0 \to X(t, t_0, x_0, p_0)\\ \text{is a diffeomorphism.}\end{array} \tag{D1}$$

Next we shall say that H, c and $S_0 : \mathbb{R}^d \to \mathbb{R}$ satisfy property (D2) if

$$\begin{array}{l}\text{There exists a stopping time } T > 0 \text{ a.s. such that}\\ \text{for any } 0 \leqslant t_0 < t < T \text{ the map}\\ \mathfrak{D}_2 : \mathbb{R}^d \to \mathbb{R}^d, \quad x_0 \to X(t, t_0, x_0, \nabla S_0(x_0))\\ \text{is a diffeomorphism.}\end{array} \tag{D2}$$

Remark. *In the literature on Burgers turbulence, the map $\mathfrak{D}_2$ is called Lagrangian function, and its inverse $\mathfrak{D}_2^{-1}$ is called the inverse Lagrangian function [Ber2].*

In the following statement we summarise the main results of the previous section.

Theorem 4.2.1. *Let $S_0 : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable function such that*

$$\frac{\partial^2 S_0(x)}{\partial x^2} \geqslant \Lambda \quad \textit{for some} \quad \Lambda \in \mathbb{R}^{d \times d}, \forall x \in \mathbb{R}^d, \tag{4.45}$$

$H(x,p) = (1/2)p^2 + V(x)$ and the conditions of the Theorem 4.1.1 *hold. Then (D1), (D2) are satisfied.*

Proof. Theorem 4.1.1 immediately implies (D1).

Using formulae (4.5), (4.6) we deduce from

$$\begin{aligned}\frac{\partial X(t, t_0, x_0, \nabla S_0(x_0))}{\partial x_0} &= \left.\frac{\partial X(t, t_0, x_0, p_0)}{\partial x_0}\right|_{p_0 = \nabla S_0(x_0)}\\ &\quad + \frac{\partial X(t, t_0, x_0, \nabla S_0(x_0))}{\partial p_0} \frac{\partial^2 S_0(x_0)}{\partial x_0^2}\end{aligned}$$

that

$$\frac{\partial X(t, t_0, x_0, \nabla S_0(x_0))}{\partial x_0} = E_d + O(t - t_0) + [(t - t_0) + O(t - t_0)] \frac{\partial^2 S_0(x_0)}{\partial x_0^2}$$

and so there exist a constant $C_1 > 0$ such that

$$\frac{\partial X(t,t_0,x_0,p_0)}{\partial x_0} \geqslant \frac{1}{2} E_d \quad \text{for} \quad 0 \leqslant t_0 \leqslant t < T \wedge C_1.$$

Therefore the map $\mathfrak{D}_2 : x_0 \to X(t,t_0,x_0,\nabla S_0(x_0))$ is a local diffeomorphism. Along the same lines as in the proof of Theorem 4.1.1 we conclude that $\mathfrak{D}_2$ is a global diffeomorphism. □

Let $p_0 = p_0(t,t_0,x,x_0)$ be such that

$$X(t,t_0,x_0,p_0(t,t_0,x,x_0)) = x, \qquad t > t_0, \qquad x \in \mathbb{R}^n. \tag{4.46}$$

Moreover, we set

$$p(t,t_0,x,x_0) = P(t,t_0,x_0,p_0(t,t_0,x,x_0)). \tag{4.47}$$

For short we write

$$x(\tau) = X(\tau,t_0,x_0,p_0(t,t_0,x,x_0)), \quad p(\tau) = P(\tau,t_0,x_0,p_0(t,t_0,x,x_0)). \tag{4.48}$$

Recall that $X(\tau)$, $P(\tau)$ are defined by (4.7). We will use this notations throughout this paper. To each pair (X,P) of solutions there corresponds the *action function* defined by the formula

$$\sigma(t,t_0,x_0,p_0) = \int_{t_0}^{t} \left[P(\tau)\frac{\partial X(\tau)}{\partial \tau} - H(X(\tau),P(\tau)) \right] d\tau - \int_{t_0}^{t} c(X(\tau))\, d\xi_\tau. \tag{4.49}$$

If (D1) holds, then one can define locally (for $0 \leqslant t_0 < t < T$) the *two-point function*

$$S(t,t_0,x,x_0) = \sigma(t,t_0,x_0,p_0(t,t_0,x,x_0)); \tag{4.50}$$

finally we set $X(\tau,x_0,p_0) = X(\tau,0,x_0,p_0)$, $P(\tau,x_0,p_0) = P(\tau,0,x_0,p_0)$, $p_0(\tau,0,x_0,p_0) = p_0(\tau,x_0,p_0)$, $\sigma(t,x_0,p_0) = \sigma(t,0,x_0,p_0)$, $S(t,x,x_0) = S(t,0,x,x_0)$ if $t_0 = 0$.

The following results (and their proofs) are stochastic versions of the well known method of characteristics for solving the Hamilton-Jacobi equation (see e.g. [K3]).

Theorem 4.2.2. *Let $H(x,p)$ and $c(x)$ satisfy (D1). The function $(t,x) \to S(t,t_0,x,x_0)$, as a function of the variables (t,x), satisfies the Hamilton-Jacobi equation*

$$dS + H\left(x, \frac{\partial S}{\partial x}\right) dt + c(x)\, d\xi_t = 0 \tag{4.51}$$

in the domain $(t_0, T) \times \mathbb{R}^d$ for stopping time T with $\mathbb{P}(T > 0) = 1$. Moreover, we have

$$\frac{\partial S}{\partial x}(t,t_0,x,x_0) = p(t,t_0,x,x_0), \quad \frac{\partial S}{\partial x_0}(t,t_0,x,x_0) = -p_0(t,t_0,x,x_0). \tag{4.52}$$

Proof. Without loss of generality we assume that $t_0 = 0$.

Step 1. We start with the proof of the first relation in (4.52). This equality can be rewritten as

$$\frac{\partial S}{\partial x}(t, X(t,x_0,p_0), x_0) = P(t,x_0,p_0)$$

which is, by (4.50),

$$\frac{\partial \sigma}{\partial p_0}(t,x_0,p_0)\frac{\partial p_0}{\partial x}(t, X(t,x_0,p_0), x_0) = P(t,x_0,p_0). \tag{4.53}$$

Due to (4.46),

$$\left(\frac{\partial p_0}{\partial x}(t, X(t,x_0,p_0), x_0)\right)^{-1} = \frac{\partial X}{\partial p_0}(t,x_0,p_0). \tag{4.54}$$

It follows that the first equation in (4.52), using (4.53), has the form

$$\frac{\partial \sigma}{\partial p_0}(t,x_0,p_0) = P(t,x_0,p_0)\,\frac{\partial X}{\partial p_0}(t,x_0,p_0). \tag{4.55}$$

Since $X(t)$, $\partial X(t)/\partial p_0$ are continuous and of bounded variation, it follows from Itô's formula that

$$dP\frac{\partial X}{\partial p_0} = P\, d\frac{\partial X}{\partial p_0} + \frac{\partial X}{\partial p_0}\, dP. \tag{4.56}$$

The left-hand side of (4.55) can be expressed using (4.49). Together with (4.56) we calculate that its Itô differential gives:

$$\frac{\partial}{\partial p_0}\left(P\frac{\partial X}{\partial t} - H\right) dt - \frac{\partial c(X)}{\partial p_0}\, d\xi_t = P\, d\frac{\partial X}{\partial p_0} + \frac{\partial X}{\partial p_0}\, dP.$$

Notice that we need the fact that

$$\frac{\partial}{\partial p_0} \int_0^t \frac{\partial c(X(\tau))}{\partial x} d\xi_\tau = \sum_{i=1}^d \int_0^t \frac{\partial^2 c_i(X(\tau))}{\partial p_0 \partial x} d\xi_{i,\tau},$$

which is justified by a special case of Theorem 36.9 [M], p. 258. Since by (4.1) $dP = (\partial H/\partial x)\, dt + (\partial c/\partial x)\, d\xi_t$ we find

$$\begin{aligned} &\frac{\partial P}{\partial p_0}\frac{\partial X}{\partial t} dt + P\frac{\partial^2 X}{\partial p_0 \partial t} dt - \frac{\partial H}{\partial x}\frac{\partial X}{\partial p_0} dt - \frac{\partial H}{\partial p}\frac{\partial P}{\partial p_0} dt - \frac{\partial X}{\partial p_0}\frac{\partial c}{\partial x} d\xi_t \\ = \;& P\frac{\partial^2 X}{\partial p_0 \partial t} dt - \frac{\partial X}{\partial p_0}\left(\frac{\partial H}{\partial x} dt + \frac{\partial c}{\partial x} d\xi_t\right). \end{aligned} \tag{4.57}$$

As $\partial X/\partial t = P$ we find that (4.57) holds for all $t < T(\omega)$, and the first part of (4.52) is established.

Step 2. Using (4.50) we get

$$\frac{\partial S}{\partial x_0} = \frac{\partial \sigma}{\partial x_0} + \frac{\partial \sigma}{\partial p_0}\frac{\partial p_0(t, x, x_0)}{\partial x_0}$$

and so, by (4.46), we rewrite the second formula in (4.52) as

$$\frac{\partial \sigma}{\partial x_0} - \frac{\partial \sigma}{\partial p_0}\left(\frac{\partial X}{\partial p_0}\right)^{-1}\frac{\partial X}{\partial x_0} = -p_0. \tag{4.58}$$

The relation

$$\begin{aligned} &\frac{\partial}{\partial \alpha}\left(\left[P(\tau)\frac{\partial X(\tau)}{\partial \tau} - H(X(\tau), P(\tau))\right] d\tau - c(X(\tau))\, d\xi_\tau\right) \\ = \;& d_\tau\left(P(\tau)\frac{\partial X(\tau)}{\partial \alpha}\right), \end{aligned}$$

where $\alpha = x_0$ or $\alpha = p_0$, and definition (4.49) imply

$$\frac{\partial \sigma}{\partial x_0} = \left(P(\tau)\frac{\partial X(\tau)}{\partial x_0}\right)\Bigg|_0^t, \qquad \frac{\partial \sigma}{\partial p_0} = \left(P(\tau)\frac{\partial X(\tau)}{\partial p_0}\right)\Bigg|_0^t. \tag{4.59}$$

Using (4.59) and the fact that

$$\left.\frac{\partial X(\tau)}{\partial x_0}\right|_{\tau=0} = 1, \qquad \left.\frac{\partial X(\tau)}{\partial p_0}\right|_{\tau=0} = 0$$

give (4.58).

Step 3. To prove (4.51), let us first rewrite it as

$$d\sigma(t,x_0,p_0)+\frac{\partial\sigma}{\partial p_0}\,dp_0+H(x,p(t,x_0,x))dt+c(x)\,d\xi_t=0.$$

Because of (4.49) we find

$$\begin{aligned}&P(t,x_0,p_0)\frac{\partial X}{\partial t}(t,x_0,p_0)\,dt-H(X(t,x_0,p_0),P(t,x_0,p_0))\,dt\\ -\quad&c(X(t,x_0,p_0))\,d\xi_t+\frac{\partial\sigma}{\partial p_0}\,dp_0+H(x,p(t,x,x_0))\,dt+c(x)\,d\xi_t=0.\end{aligned}$$

By construction, $X(\tau,x_0,p_0)=x$, $P(t,x_0,p_0)=p$ and expressing $\partial\sigma/\partial p_0$ by (4.55) gives

$$P(t,x_0,p_0)\frac{\partial X}{\partial t}(t,x_0,p_0)\,dt+P(t,x_0,p_0)\frac{\partial X}{\partial p_0}(t,x_0,p_0)\,dp_0=0. \tag{4.60}$$

Differentiating (4.46) with respect to t we get

$$dX(t,x_0,p_0(t,x_0,x))=\frac{\partial X}{\partial t}(t,x_0,p_0)\,dt+\frac{\partial X}{\partial p_0}(t,x_0,p_0)\,dp_0=0.$$

Thus (4.60) is always satisfied and (4.51) follows. □

Corollary 4.2.1. *Under the assumption of the Theorem 4.2.2 we have for* $0\leqslant t_0<t<T$

$$\frac{\partial^2S(t,t_0,x,x_0)}{\partial x^2}=\frac{1}{t-t_0}(E_d+O(t-t_0)), \tag{4.61}$$

$$\frac{\partial^2S(t,t_0,x,x_0)}{\partial x_0^2}=\frac{1}{t-t_0}(E_d+O(t-t_0)), \tag{4.62}$$

$$\frac{\partial^2S(t,t_0,x,x_0)}{\partial x\partial x_0}=-\frac{1}{t-t_0}(E_d+O(t-t_0)), \tag{4.63}$$

where $O(\cdot)$ *is uniform with respect to* x_0, x.

Proof. Assume again that $t_0=0$. From (4.52) and (4.54) we deduce the equality

$$\frac{\partial^2S(t,x,x_0)}{\partial x^2}=\frac{\partial P}{\partial p_0}(t,x_0,p_0)\left(\frac{\partial X}{\partial p_0}(t,x_0,p_0)\right)^{-1}.$$

Now (4.5), (4.6) imply the first formula in Corollary 4.2.1. The same argument can be used to prove the remaining formulae. □

Theorem 4.2.3. *We assume that $H(x,p)$, $c(x)$ and $S_0(x)$ satisfy conditions (D1), (D2). Then for $0 \leqslant t_0 < t < T(\omega)$ the formula*

$$S(t,t_0,x) = S_0(x_0) + \int_{t_0}^{t} (\widetilde{p}(\tau)\, d\widetilde{x}(\tau) - H(\widetilde{x}(\tau),\widetilde{p}(\tau))\, d\tau - c(\widetilde{x}(\tau))\, d\xi_\tau) \tag{4.64}$$

(where the integral is taken along the trajectory $\widetilde{x}(\tau) = X(\tau,t_0,x_0,\nabla S_0(x))$, $\widetilde{p}(\tau) = P(\tau,t_0,x_0,\nabla S_0(x_0))$ such that $\widetilde{x}(t) = x$ and $x_0 = x_0(t,t_0,x)$ is the inverse map of $\mathfrak{D}_2$) gives a unique classical solution of the Cauchy problem for the equation

$$dS + H\left(x, \frac{\partial S}{\partial x}\right) dt + c(x)\, d\xi_t = 0 \tag{4.65}$$

with initial function $S_0(x)$. One can rewrite formula (4.64) in the equivalent form

$$S(t,t_0,x) = (S_0(x_0) + S(t,t_0,x,x_0))\,\big|_{x_0 = x_0(t,t_0,x)}. \tag{4.66}$$

Proof. Definition of the two-point function (4.50) implies the equivalence of (4.64) and (4.66). From system (4.44) follows that $X(t,t_0,x_0,p_0)$ continuous in t and, using the implicit function theorem, from (D2) we obtain that $x_0 = x_0(t,t_0,x)$ is continuous in t and $[x_0,x_0] = 0$, Itô's differentials for this equation give

$$\begin{aligned} & d_t S(t,t_0,x) \\ &= \nabla S_0(x_0) d_t x_0(t,x) + \frac{\partial S(t,t_0,x,x_0)}{\partial t} dt + \frac{\partial S(t,t_0,x,x_0)}{\partial x_0} d_t x_0(t,t_0,x) \\ &= \frac{\partial S(t,t_0,x,x_0)}{\partial t} dt. \end{aligned}$$

In the last equality we used $\nabla S_0(x_0) = p_0$ in conjunction with (4.52). From Theorem 4.2.2 we know that

$$\frac{\partial S(t,t_0,x,x_0)}{\partial t} dt = d_t S(t,t_0,x,x_0) = -H\left(x, \frac{\partial S(t,t_0,x,x_0)}{\partial x}\right) dt - c(x)\, d\xi_t,$$

and the theorem follows. □

One can find it more convenient to have an alternative representation of the solution for the Cauchy problem in Theorem 4.2.3.

Corollary 4.2.2. *Let the assumptions of Theorem* 4.2.3 *be fulfilled. Then for* $0 \leqslant t_0 < t < T(\omega)$ *there exists a unique classical (i.e. smooth) solution of the Cauchy problem from Theorem* 4.2.3 *with initial function* $S_0(x)$. *This solution is given by the formula*

$$S(t, t_0, x) = \min_{x_0} \left(S_0(x_0) + S(t, t_0, x, x_0) \right). \tag{4.67}$$

Proof. From the definition of $\mathfrak{D}_2$ it follows that $x_0(t, x)$ is a critical point of the function $S_0(x_0) + S(t, t_0, x, x_0)$. Moreover, due to (4.45) and (4.62) this critical point is also the (unique) minimum point. □

Chapter 5

Small time and Semiclassical Asymptotics for Stochastic Heat Equation Driven by a Lévy Noise

5.1 Preliminaries

We write $(X, P) = (X(\tau, t_0, x_0, p_0), P(\tau, t_0, x_0, p_0)) \in \mathbb{R}^{2d}$ for solution of system

$$\begin{cases} dx &= p\,dt \\ dp &= \left(\frac{\partial V}{\partial x} + h\frac{\partial a}{\partial x}\right)\,dt - \frac{\partial c}{\partial x}\,d\xi_t. \end{cases} \tag{5.1}$$

with initial condition $(x_0, p_0) \in \mathbb{R}^{2d}$ at $t = t_0$. We assume that the coefficients V, a, c admit continuous partial derivatives up to order q, $q \geqslant 3$, that is

$$\left|\frac{\partial^{|L|}V(x)}{\partial x^L}\right|, \left|\frac{\partial^{|L|}a(x)}{\partial x^L}\right|, \left|\frac{\partial^{|L|}c(x)}{\partial x^L}\right| = O(1), \quad 1 \leqslant |L| \leqslant q, \tag{5.2}$$

$c(x)$ satisfies (4.3) and the driving noise $\xi_t = (\xi_{1,t}, \dots, \xi_{d,t})$ is a d-dimensional Lévy process such that condition (4.4) holds. Additionally we assume that

$$\int\limits_{|y|\geqslant 1} |y|^2\,\nu(dy) < \infty, \tag{5.3}$$

where ν is a Lévy measure of ξ_t.

Lemma 5.1.1. *For any* $x_0, x \in \mathbb{R}^d$, $0 \leqslant t_0 < t < T$ *we have*

$$\begin{aligned} p(t,t_0,x,x_0) &= p_0(t,t_0,x,x_0) + O(h), & (5.4)\\ p(t,t_0,x_0,x_0) &= O(h), & (5.5)\\ x(\tau) &= x_f(\tau) + O(\tau - t_0), & (5.6) \end{aligned}$$

where

$$x_f(\tau) = x_0 + \frac{\tau - t_0}{t - t_0}(x - x_0), \tag{5.7}$$

and

$$S(t,t_0,x_0,x_0) = [V(x_0) + ha(x_0)](t - t_0) - hc(x_0)\Delta\xi_{t_0} + o(1) \tag{5.8}$$

as $t \to t_0$.

Proof. Step 1. From system (5.1) we get

$$\begin{aligned} &p(t,t_0,x,x_0)\\ &= p_0(t,t_0,x,x_0) + \int_{t_0}^{t} \frac{\partial[V(x(\tau)) + ha(x(\tau))]}{\partial x}\, d\tau - h\int_{t_0}^{t} \frac{\partial c(x(\tau))}{\partial x}\, d\xi_\tau\\ &= \mathbf{I} + \mathbf{II} + \mathbf{III}. \end{aligned}$$

Since $\sup_{x\in\mathbb{R}^d} |\partial V(x)/\partial x|$, $\sup_{x\in\mathbb{R}^d} |\partial a(x)/\partial x| = O(1)$, it follows that $\mathbf{II} = O(t - t_0)$. Integrating by parts we have

$$\begin{aligned} \mathbf{III} &= h\int_{t_0}^{t} \frac{\partial c(x(\tau))}{\partial x}\, d\xi_\tau\\ &= h\frac{\partial c(x_0)}{\partial x}(\xi_t - \xi_{t_0}) - h\int_{t_0}^{t} \frac{\partial^2 c(x(\tau))(\xi_\tau - \xi_t)}{\partial x^2} p(\tau)\, d\tau, \end{aligned}$$

where we used by formula (4.15) with $f(x) = c_i(x)$, $i = 1, \ldots, d$ that

$$[c(x(\cdot)), c(x(\cdot))] = 0.$$

Since (4.3),

$$\int_{t_0}^{t} \frac{\partial^2 c(x(\tau))(\xi_\tau - \xi_t)}{\partial x^2} p(\tau)\, d\tau = \int_{\mathcal{B}} \frac{\partial^2 c(x(\tau))(\xi_\tau - \xi_t)}{\partial x^2} p(\tau)\, d\tau,$$

where $\mathcal{B} = \{\tau \in [t_0, t] : |x(\tau)| \leqslant K\}$. By Lemma 0.0.1, $\sup_{\tau\in[0,t]} |\xi_\tau| < 1$ for $t < \mathcal{R}$ and so, by (5.2),

$$\left|\frac{\partial c(x_0)}{\partial x}(\xi_t - \xi_{t_0})\right|, \left|\frac{\partial^2 c(x(\tau))(\xi_\tau - \xi_t)}{\partial x^2}\right| = O(1).$$

It follows

$$\mathbf{III} = O(h) + O(h) \int_{\mathcal{B}} |p(\tau)|\, d\tau.$$

By Lemma 4.1.1 we see $\int_{\mathcal{B}} |p(\tau)|\, d\tau = O(1)$. Hence $\mathbf{III} = O(h)$. Piecing together estimates for $\mathbf{II}, \mathbf{III}$ we establish (5.4).
Step 2. Applying (5.4) with $x = x_0$ we have

$$P(\tau, t_0, x_0, p_0(t, t_0, x_0, x_0)) = p_0(t, t_0, x_0, x_0) + O(h). \tag{5.9}$$

Integrating the last estimate on the segment $[t_0, t]$ and using

$$\int_{t_0}^{t} P(\tau, t_0, x_0, p_0(t, t_0, x_0, x_0))\, d\tau = x_0 - x_0 = 0$$

we get (5.5).
Step 3. From (5.4) we deduce that

$$x(\tau) - x_0 = \int_{t_0}^{\tau} (p_0 + O(1))\, ds = p_0(\tau - t_0) + O(\tau - t_0).$$

Similarly $x - x_0 = p_0(t - t_0) + O(t - t_0)$ and so

$$x(\tau) - x_f(\tau) = (x(\tau) - x_0) - \frac{\tau - t_0}{t - t_0}(x - x_0) = O(\tau - t_0).$$

Step 4. We now proceed with (5.8). Let $x(\tau)$ be defined by (4.48) with $x = x_0$. The same kind of argument as in step 1 shows that

$$\int_{t_0}^{t} c(x(\tau))\, d\xi_\tau = c(x_0)\Delta\xi_{t_0} + o(1)$$

for $0 \leqslant t_0 < t < \mathcal{R}$, $t \to t_0$. The fact that $x(t) = x_0$ and (5.5) imply

$$\int_{t_0}^{t} V(x(\tau))\,d\tau = V(x_0)t - V(x_0)t_0 - \int_{t_0}^{t} \tau \frac{\partial V}{\partial x} p(\tau) d\tau = V(x_0)(t-t_0) + O(t-t_0).$$

Using the estimates above we deduce from (4.50)

$$S(t, t_0, x_0, x_0) = \frac{1}{2}\int_{t_0}^{t} p^2(\tau)\,d\tau + [V(x_0) + ha(x_0)](t - t_0) - hc(x_0)\Delta\xi_{t_0} + o(1).$$

Using again (5.5), gives the proof. □

Corollary 5.1.1. *For $0 \leqslant t_0 < t < T$ one has*

$$S(t, t_0, x, x_0) = [V(x_0) + ha(x_0)](t - t_0) - hc(x_0)\Delta\xi_{t_0} + o(1) + O(h|x - x_0|) + \frac{(x - x_0)^2}{2(t - t_0)}(1 + O(t - t_0))$$

as $t \to t_0$.

Proof. Expanding $S(t, t_0, x, x_0)$ into Taylor's series with respect to x and applying Corollary 4.2.1, formula (4.52), yield

$$S(t, t_0, x, x_0) = S(t, t_0, x_0, x_0) + p(t, t_0, x_0, x_0)(x - x_0) + \frac{(x - x_0)^2}{2(t - t_0)}(1 + O(t - t_0)).$$

Using (5.5), (5.8) we complete the proof. □

5.2 Formal asymptotics for the Green function of stochastic heat equations

We shall construct WKB-type asymptotics for the stochastic differential equation

$$h\,d\psi = -\left(-\frac{h^2}{2}\mathrm{tr}\frac{\partial^2}{\partial x^2} + V + ha\right)\psi_-\,dt + hc\psi_-\,d\xi_t, \tag{5.10}$$

where $a, V : \mathbb{R}^d \to \mathbb{R}$, $c = (c_1, \dots, c_d) : \mathbb{R}^d \to \mathbb{R}^d$, $\psi_- = \psi(t-, t_0, x, x_0)$. Assume that the functions V, a, c satisfy conditions (4.3), (5.2) and the Lévy process $\{\xi_t\}_{t\geqslant 0}$ satisfy (5.3), (4.4). Additionally we suppose that $V(x)$, $a(x)$ are bounded below and

$$c(x)y \geqslant 0 \qquad \forall x \in \mathbb{R}^d,\ \forall y \in \operatorname{supp} \nu \subset \mathbb{R}^d \tag{5.11}$$

with ν being the Lévy measure of ξ_t. One can read (5.11) as

$$c(x)\Delta\xi_t \geqslant 0 \quad \forall x \in \mathbb{R}^d,\ \forall t \in \mathbb{R}_+. \tag{5.12}$$

Let $S(t, t_0, x, x_0)$ be the two point function (defined in preliminaries) for the Hamilton-Jacobi equation

$$dS + \frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 dt - V\,dt - ha\,dt + hc\,d\xi_t = 0. \tag{5.13}$$

In order to find $S(t, t_0, x, x_0)$ we consider the corresponding Hamilton system

$$\begin{cases} dx &= p\,dt \\ dp &= \frac{\partial V(x)}{\partial x}\,dt + h\frac{\partial a(x)}{\partial x}\,dt - h\frac{\partial c(x)}{\partial x}\,d\xi_t. \end{cases} \tag{5.14}$$

As before we denote by (X, P) the solution of system (5.14) with initial condition (x_0, p_0) at $t = t_0$. We set

$$I(t, t_0, x_0, p_0) = \det \frac{\partial X(t, t_0, x_0, p_0)}{\partial p_0}, \tag{5.15}$$

$$J(t, t_0, x, x_0) = I(t, t_0, x_0, p_0(t, t_0, x, x_0)). \tag{5.16}$$

We will use both notations interchangeably according to the set of variables we want to consider. Formula (4.6) and definition (5.15) immediately imply

Corollary 5.2.1. *For $0 \leqslant t_0 < t < T$ one has*

$$J^{-\frac{1}{2}}(t, t_0, x, x_0) = \frac{1}{(\sqrt{t - t_0})^d}(1 + O(t - t_0)) \tag{5.17}$$

for all $x, x_0 \in \mathbb{R}^d$.

For short we write $I(\tau) = I(\tau, t_0, x_0, p_0)$. Recall that $X(\tau)$ is given by (4.7).

Lemma 5.2.1. *The function $J^{-1/2} = J^{-1/2}(t, t_0, x, x_0)$ satisfies the equation*

$$\frac{\partial J^{-1/2}}{\partial t} + \frac{\partial J^{-1/2}}{\partial x}\frac{\partial S}{\partial x} + \frac{1}{2}J^{-1/2}\,\mathrm{tr}\left(\frac{\partial^2 S}{\partial x^2}\right) = 0. \tag{5.18}$$

Proof. The identity

$$\det M = \exp\{\mathrm{tr}\ln M\}, \tag{5.19}$$

where M is a positive definite matrix, and Itô's formula imply

$$dI(t) = d\det\frac{\partial X(t)}{\partial p_0} = \det\frac{\partial X(t)}{\partial p_0}\,\mathrm{tr}\left(\left(\frac{\partial X(t)}{\partial p_0}\right)^{-1} d\frac{\partial X(t)}{\partial p_0}\right),$$

where we used the fact that $(\partial X(t)/\partial p_0)$ is continuous and of bounded variation. Applying formula (4.52) we find

$$\begin{aligned} dI(t) &= I(t)\,\mathrm{tr}\left(\left(\frac{\partial X(t)}{\partial p_0}\right)^{-1}\frac{\partial}{\partial p_0}\frac{\partial S(t,t_0,X(t),x_0)}{\partial x}\right)dt \\ &= I(t)\,\mathrm{tr}\left(\frac{\partial^2 S(t,t_0,X(t),x_0)}{\partial x^2}\right)dt \end{aligned}$$

or, using that $I(t) = J(t, t_0, X(t), x_0)$

$$\frac{dJ(t,t_0,X(t),x_0)}{dt} = J(t,t_0,X(t),x_0)\,\mathrm{tr}\left(\frac{\partial^2 S}{\partial x^2}(t,t_0,X(t),x_0)\right).$$

Hence

$$\frac{dJ^{-1/2}(t,t_0,X(t),x_0)}{dt} = -\frac{1}{2}J^{-1/2}(t,t_0,X(t),x_0)\,\mathrm{tr}\left(\frac{\partial^2 S}{\partial x^2}(t,t_0,X(t),x_0)\right). \tag{5.20}$$

Combining (5.20) and relation

$$\frac{dJ^{-1/2}(t,t_0,X(t),x_0)}{dt} = \frac{\partial J^{-1/2}(t,t_0,X(t),x_0)}{\partial t} + \frac{\partial J^{-1/2}(t,t_0,X(t),x_0)}{\partial x}\frac{dX(t)}{dt}$$

and the equation

$$dX(t) = P(t)\,dt = \frac{\partial S}{\partial x}(t,t_0,X(t),x_0)\,dt,$$

gives

$$-\frac{1}{2}J^{-1/2}(t,t_0,X(t),x_0)\,\mathrm{tr}\left(\frac{\partial^2 S}{\partial x^2}(t,t_0,X(t),x_0)\right)$$
$$=\frac{\partial J^{-1/2}(t,t_0,X(t),x_0)}{\partial t}+\frac{\partial J^{-1/2}(t,t_0,X(t),x_0)}{\partial x}\frac{\partial S}{\partial x}(t,t_0,X(t),x_0).$$

Making change of the variables $p_0 = p_0(t,t_0,x,x_0)$ we arrive at (5.18). □

We put

$$\rho(t,x) = \prod_{t_0<\tau\leqslant t} \exp\{-c(x)\Delta\xi_\tau\}(1+c(x)\Delta\xi_\tau) \tag{5.21}$$

and $\rho(t,x) = 1$ if there are no jumps of ξ_t on $(t_0,t]$.

Lemma 5.2.2. *For any $t_0 \leqslant t < \mathcal{R}$ we have*

$$\rho(t,x) = 1 + o(1) \qquad as \quad t \to t_0.$$

Proof. The inequality

$$\exp\{-y\}(1+y) < 1 \qquad \text{for} \quad y > 0 \tag{5.22}$$

implies that

$$\exp\{-c(x)\Delta\xi_\tau\}(1+c(x)\Delta\xi_\tau) < 1.$$

On the other hand from

$$\exp\{-y\}(1+y) > \exp\{-y^2\} \qquad \text{for} \quad y > 0 \tag{5.23}$$

we get

$$\exp\{-c(x)\Delta\xi_\tau\}(1+c(x)\Delta\xi_\tau) > \exp\{-|c(x)\Delta\xi_\tau|^2\}.$$

Using that by Lemma 0.0.1 $\sum_{0\leqslant\tau\leqslant t} |\Delta\xi_\tau|^2 < 1$ for $t < \mathcal{R}$ and the fact that $|c(x)| = O(1)$, we establish the lemma. □

Lemma 5.2.3. *For any $0 \leqslant t_0 < \tau < T$ we get*

$$\Delta\left(\rho(\tau)\exp\left\{-\frac{S(\tau)}{h}\right\}\right) + \frac{1}{h}\rho(\tau-)\exp\left\{-\frac{S(\tau-)}{h}\right\}\Delta S(\tau) = 0, \tag{5.24}$$

where for short we write $S(\tau) = S(\tau,t_0,x,x_0)$, $\rho(\tau) = \rho(\tau,t_0,x)$.

Proof. We rewrite the left-hand side of (5.24) as

$$\begin{aligned}
&\rho(\tau)\exp\left\{-\frac{S(\tau)}{h}\right\}-\rho(\tau-)\exp\left\{-\frac{S(\tau-)}{h}\right\}\\
&+\frac{1}{h}\rho(\tau-)\exp\left\{-\frac{S(\tau-)}{h}\right\}\Delta S(\tau)\\
=\;&\exp\left\{-\frac{S(\tau)}{h}\right\}\left[\rho(\tau)-\rho(\tau-)\exp\left\{\frac{\Delta S(\tau)}{h}\right\}(1-\frac{1}{h}\Delta S(\tau))\right]. \qquad (5.25)
\end{aligned}$$

Since, by (5.13), $\Delta S(\tau)=-hc(x)\Delta\xi_\tau$, it follows that

$$\exp\left\{\frac{\Delta S(\tau)}{h}\right\}\left(1-\frac{1}{h}\Delta S(\tau)\right)=\exp\{-c(x)\Delta\xi_\tau\}\left(1+c(x)\Delta\xi_\tau\right)$$

and so the square brackets in (5.25) vanishes. □

Let

$$\phi(t,t_0,x,x_0)=\frac{\exp\{-c(x_0)\Delta\xi_{t_0}\}}{(\sqrt{2\pi h})^d}\rho(t,x)\lambda(t,x,x_0)J^{-\frac{1}{2}}(t,t_0,x,x_0), \qquad (5.26)$$

where

$$\lambda(t,x,x_0)=\exp\left\{-\int_{t_0}^{t}[\rho(\tau,x(\tau))]^{-1}\frac{\partial\rho(\tau,x(\tau))}{\partial x}\frac{\partial S(\tau,t_0,x(\tau),x_0)}{\partial x}\,d\tau\right\}. \qquad (5.27)$$

Recall that $x(\tau)$ is given by (4.48).

Lemma 5.2.4. *The function $\lambda=\lambda(t,x,x_0)$, $0\leqslant t_0<t<T$ satisfies the equation*

$$\rho\, d_t\lambda+\rho\frac{\partial\lambda}{\partial x}\frac{\partial S}{\partial x}\,dt+\lambda\frac{\partial\rho}{\partial x}\frac{\partial S}{\partial x}\,dt=0. \qquad (5.28)$$

Proof. Making the change of the variable $x=X(t,t_0,x_0,p_0)$ we find from (5.27)

$$\lambda(t,X(t),x_0)=\exp\left\{-\int_{t_0}^{t}[\rho(\tau,X(\tau))]^{-1}\frac{\partial\rho}{\partial x}(\tau,X(\tau))\frac{\partial S}{\partial x}(\tau,t_0,X(\tau),x_0)\,d\tau\right\}.$$

Hence

$$d_t\lambda(t, X(t), x_0) = -\lambda\rho^{-1}\frac{\partial\rho}{\partial x}\frac{\partial S}{\partial x}\,dt, \tag{5.29}$$

where in the right-hand side of (5.29) we omit arguments of ρ, λ and S.

On the other hand, using (4.52) gives

$$d_t\lambda(t, X(t), x_0) = d_t\lambda + \frac{\partial\lambda}{\partial x}dX(t) = d_t\lambda + \frac{\partial\lambda}{\partial x}\frac{\partial S}{\partial x}\,dt \tag{5.30}$$

Combining (5.29), (5.30) and making again the change of the variables $p_0 = p_0(t, t_0, x, x_0)$ give the proof. □

Lemma 5.2.5. *One can rewrite formula* (5.27) *in the form*

$$\lambda(t, x, x_0) = \prod_{t_0<\tau\leqslant t}\frac{\widetilde{\rho}(\tau, x(\tau))}{\widetilde{\rho}(\tau, x)}, \tag{5.31}$$

where

$$\widetilde{\rho}(\tau, x) = \exp\{-c(x)\Delta\xi_\tau\}(1 + c(x)\Delta\xi_\tau). \tag{5.32}$$

Proof. We deduce from (5.21)

$$[\rho(\tau, x)]^{-1}\frac{\partial\rho(\tau, x)}{\partial x} = \sum_{t_0<s\leqslant\tau}\frac{\partial\ln\widetilde{\rho}(s, x)}{\partial x}$$

and so, by (4.52), we find

$$\ln\lambda = -\int_{t_0}^{t}\sum_{t_0<s\leqslant\tau}\frac{\partial\ln\widetilde{\rho}(s, x(\tau))}{\partial x}p(\tau)\,d\tau. \tag{5.33}$$

Changing the order of the integration and the summation in (5.33) and using $p(\tau)\,d\tau = dx(\tau)$ we have

$$\ln\lambda = -\sum_{t_0<s\leqslant t}\int_{s}^{t}\frac{\partial\ln\widetilde{\rho}(s, x(\tau))}{\partial x}\,dx(\tau) = -\sum_{t_0<s\leqslant t}\ln\frac{\widetilde{\rho}(s, x)}{\widetilde{\rho}(s, x(s))}.$$

Using again (5.21) we complete the proof. □

An application of Corollary 5.2.1 and Lemmas 5.2.2, 5.2.5 to (5.26) gives

Corollary 5.2.2. *For $0 \leqslant t_0 < t < T$ one has*

$$\phi(t, t_0, x, x_0) = \frac{\exp\{-c(x_0)\Delta\xi_{t_0}\}}{\left(\sqrt{2\pi h(t-t_0)}\right)^d}(1+o(1)) \qquad as \quad t \to t_0$$

for all $x, x_0 \in \mathbb{R}^d$.

We put

$$\psi_G^{as}(t, t_0, x, x_0) = \phi(t, t_0, x, x_0)\exp\left\{-\frac{S(t, t_0, x, x_0)}{h}\right\}. \tag{5.34}$$

Lemma 5.2.6. *The function $\phi(t, t_0, x, x_0)$, $0 \leqslant t_0 < t < T$ satisfies the transport equation*

$$\int_{t_0}^{t} \exp\left\{-\frac{S(\tau-)}{h}\right\}\left[d\phi + \frac{\partial\phi}{\partial x}\frac{\partial S}{\partial x}d\tau + \frac{1}{2}\phi\,\mathrm{tr}\left(\frac{\partial^2 S}{\partial x^2}\right)d\tau\right] + \Sigma = 0, \tag{5.35}$$

where

$$\Sigma = \sum_{t_0 \leqslant \tau \leqslant t}\Bigg(\Delta\psi_G^{as}(\tau) - \exp\left\{-\frac{S(\tau-)}{h}\right\}\Delta\phi(\tau) + \frac{1}{h}\phi(\tau)\exp\left\{-\frac{S(\tau-)}{h}\right\}\Delta S(\tau)\Bigg). \tag{5.36}$$

Here we write for short $\psi_G^{as}(\tau) = \psi_G^{as}(\tau, t_0, x, x_0)$, $S(\tau) = S(\tau, t_0, x, x_0)$, $\phi(\tau) = \phi(\tau, t_0, x, x_0)$, $\rho(\tau) = \rho(\tau, x)$.

Proof. Using Itô's formula and the fact that $[J^{\frac{1}{2}}, J^{\frac{1}{2}}] = 0$, $[\lambda, \lambda] = 0$ we have

$$\begin{aligned}
& d\phi + \frac{\partial\phi}{\partial x}\frac{\partial S}{\partial x}d\tau + \frac{1}{2}\phi\,\mathrm{tr}\left(\frac{\partial^2 S}{\partial x^2}\right)d\tau \\
= \; & \frac{\exp\{-c(x_0)\Delta\xi_{t_0}\}}{\left(\sqrt{2\pi h}\right)^d}\rho\lambda\left(dJ^{-\frac{1}{2}} + \frac{\partial J^{-\frac{1}{2}}}{\partial x}\frac{\partial S}{\partial x}d\tau + \frac{1}{2}J^{-\frac{1}{2}}\,\mathrm{tr}\left(\frac{\partial^2 S}{\partial x^2}\right)d\tau\right) \\
& + \frac{\exp\{-c(x_0)\Delta\xi_{t_0}\}}{\left(\sqrt{2\pi h}\right)^d}J^{-\frac{1}{2}}\left(d(\rho\lambda) + \frac{\partial(\rho\lambda)}{\partial x}\frac{\partial S}{\partial x}d\tau\right) \\
= \; & \frac{\exp\{-c(x_0)\Delta\xi_{t_0}\}}{\left(\sqrt{2\pi h}\right)^d}\rho\lambda\,\mathbf{I} + \frac{\exp\{-c(x_0)\Delta\xi_{t_0}\}}{\left(\sqrt{2\pi h}\right)^d}J^{-\frac{1}{2}}\,\mathbf{II}.
\end{aligned}$$

An application of Lemma 5.2.1 (resp. 5.2.4) shows $\mathbf{I} = 0$ (resp. $\mathbf{II} = \lambda d\rho$)) and so it is enough to prove that

$$\frac{\exp\{-c(x_0)\Delta\xi_{t_0}\}}{(\sqrt{2\pi h})^d}\int_{t_0}^{t}\lambda J^{-\frac{1}{2}}\exp\left\{-\frac{S(\tau-)}{h}\right\}d\rho + \Sigma = 0. \tag{5.37}$$

Using Lemma 5.2.3 and continuity of $J^{-\frac{1}{2}}$, λ, we obtain

$$\sum_{t_0\leqslant\tau\leqslant t}\left(\Delta\psi_G^{as}(\tau) + \frac{1}{h}\phi(\tau)\exp\left\{-\frac{S(\tau-)}{h}\right\}\Delta S(\tau)\right) = 0$$

and so

$$\Sigma = -\frac{\exp\{-c(x_0)\Delta\xi_{t_0}\}}{(\sqrt{2\pi h})^d}\sum_{t_0\leqslant\tau\leqslant t}\lambda J^{-\frac{1}{2}}\exp\left\{-\frac{S(\tau-)}{h}\right\}\Delta\rho(\tau). \tag{5.38}$$

We have

$$\begin{aligned}\sum_{t_0\leqslant\tau\leqslant t}|\Delta\rho(\tau)| &= \sum_{t_0\leqslant\tau\leqslant t}[1-\exp\{-c(x)\Delta\xi_\tau\}(1+c(x)\Delta\xi_\tau)]\,\rho(\tau-)\\ &\leqslant \sum_{t_0\leqslant\tau\leqslant t}(c(x))^2|\Delta\xi_\tau|^2 = O(1)\sum_{t_0\leqslant\tau\leqslant t}|\Delta\xi_\tau|^2 < \infty.\end{aligned}$$

Corollary 5.1.1 and the condition $V(x) + ha(x) \geqslant C_0$ for some constant $C_0 \in \mathbb{R}$ imply that, by (4.50) S is bounded below. Consequently $\exp\{-S/h\}$ is bounded. Thus we have proved the existence of the integral

$$\int_{t_0}^{t}\lambda J^{-\frac{1}{2}}\exp\left\{-\frac{S(\tau-)}{h}\right\}d\rho = \sum_{t_0\leqslant\tau\leqslant t}\lambda J^{-\frac{1}{2}}\exp\left\{-\frac{S(\tau-)}{h}\right\}\Delta\rho(\tau). \tag{5.39}$$

Piecing together (5.38) and (5.39) we arrive at (5.37). □

Now we can prove that ψ_G^{as} is a formal asymptotics for the Green function of equation (5.10). More precisely, the following result holds.

Proposition 5.2.1. *The function $\psi_G^{as}(t) = \psi_G^{as}(t, t_0, x, x_0)$, $0 \leqslant t_0 < t < T$ satisfies the equation (5.10) up to a remainder $O(h^2)$. Namely,*

$$\begin{aligned}h\,d\psi_G^{as} + \left(-\frac{h^2}{2}tr\frac{\partial^2}{\partial x^2} + V + ha\right)\psi_{G-}^{as}\,dt - h\psi_{G-}^{as}c\,d\xi_t\\ = -\frac{h^2}{2}tr\frac{\partial^2\phi}{\partial x^2}\exp\left\{-\frac{S}{h}\right\}dt.\end{aligned} \tag{5.40}$$

Here $\psi^{as}_{G-} = \psi^{as}_G(t-, t_0, x, x_0)$. *Moreover,*

$$\psi^{as}_G(t, t_0, x, x_0)\big|_{t=t_0} = \delta(x - x_0). \tag{5.41}$$

Proof. Write for short $S_- = S(\tau-, t_0, x, x_0)$. We multiply equation (5.13) by $-\exp\{-S_-/h\}\phi$, integrate this equation over $[t_0, t]$, and then add equation (5.35) multiplied by h. This gives

$$\begin{aligned}
&\int_{t_0}^{t} \exp\left\{-\frac{S_-}{h}\right\}(h\,d\phi - \phi\,dS) + h\Sigma \\
&= \frac{h^2}{2}\int_{t_0}^{t} \exp\left\{-\frac{S_-}{h}\right\}\left[-\frac{2}{h}\frac{\partial\phi}{\partial x}\frac{\partial S}{\partial x} + \phi\left(\frac{1}{h^2}\left(\frac{\partial S}{\partial x}\right)^2 - \frac{1}{h}\mathrm{tr}\left(\frac{\partial^2 S}{\partial x^2}\right)\right)\right] d\tau \\
&\quad - \int_{t_0}^{t} \exp\left\{-\frac{S_-}{h}\right\}\phi\,(V\,d\tau + ha\,d\tau - hc\,d\xi_\tau).
\end{aligned} \tag{5.42}$$

Recall that $\psi^{as}_G = \phi\exp\{-S/h\}$ and note that the expression in square brackets is equal to

$$\exp\left\{\frac{S}{h}\right\}\mathrm{tr}\frac{\partial^2\psi^{as}_G}{\partial x^2} - \mathrm{tr}\frac{\partial^2\phi}{\partial x^2}.$$

An application of Itô's formula yields

$$\begin{aligned}
\psi^{as}_G(t) = \psi^{as}_G(t_0) &+ \int_{t_0}^{t}\exp\left\{-\frac{S_-}{h}\right\}d\phi - \frac{1}{h}\int_{t_0}^{t}\exp\left\{-\frac{S_-}{h}\right\}\phi\,dS \\
&- \frac{1}{h}\int_{t_0}^{t}\exp\left\{-\frac{S_-}{h}\right\}d[\phi, S]^c + \frac{1}{2h^2}\int_{t_0}^{t}\phi\exp\left\{-\frac{S_-}{h}\right\}d[S, S]^c + \Sigma
\end{aligned}$$

with Σ as in (5.36). Equation (5.13) and the fact that $[\xi, \xi]^c = 0$ imply that $[S, S]^c = 0$. By its definition ρ satisfies $[\rho, \rho]^c = 0$. This and formula (5.26) imply that $[\phi, S]^c = \lambda J^{-\frac{1}{2}}[\rho, S]^c = 0$. Therefore, the corresponding terms in Itô's formula vanish and we get

$$\psi^{as}_G(t) = \psi^{as}_G(t_0) + \int_{t_0}^{t}\exp\left\{-\frac{S_-}{h}\right\}d\phi - \frac{1}{h}\int_{t_0}^{t}\exp\left\{-\frac{S_-}{h}\right\}\phi\,dS + \Sigma.$$

This, (5.42) and a lengthy but elementary calculation show (5.40).

It suffices to show (5.41). An application of Corollaries 5.1.1, 5.2.2 to definition (5.34) yields

$$\begin{aligned}
&\psi_G^{as}(t,t_0,x,x_0)\\
&= \frac{(1+o(1))}{(\sqrt{2\pi h(t-t_0)})^d}\exp\left\{-\frac{1}{h}[V(x_0)+ha(x_0)](t-t_0)+o(1)\right\}\times\\
&\times\exp\left\{\frac{1}{h}O(|x-x_0|)-\frac{(x-x_0)^2}{2h(t-t_0)}(1+O(t-t_0))\right\}, \qquad (5.43)
\end{aligned}$$

as $t \to t_0$, which implies (5.41). □

5.3 Multiplicative asymptotics, two-sided estimates and a large deviation principle for the Green function

5.3.1 Asymptotics for the Green function

We rewrite equation (5.40) in the integral form

$$\begin{aligned}
h\psi_G^{as}(t,t_0,x,x_0) &= h\delta(x-x_0)-\int_{t_0}^{t}L_0\psi_G^{as}(s,t_0,x,x_0)\,ds \qquad (5.44)\\
&-\sum_{i=1}^{d}\int_{t_0}^{t}L_i\psi_G^{as}(s,t_0,x,x_0)\,d\xi_{i,s}-h^2\int_{t_0}^{t}\mathcal{K}(s,t_0,x,x_0)\,ds,
\end{aligned}$$

where $\mathcal{K}(t,t_0,x,x_0)$ is given by (5.50),

$$L_0\psi_G^{as} = \left(-\frac{h^2}{2}\mathrm{tr}\frac{\partial^2}{\partial x^2}+V+ha\right)\psi_G^{as}, \qquad (5.45)$$

$$L_i\psi_G^{as} = -hc_i\psi_G^{as}, \quad i=1,\dots,d \qquad (5.46)$$

and put

$$\psi_G(t,t_0,x,x_0) = (1+h\mathcal{F}+h^2\mathcal{F}^2+\dots)\psi_G^{as}(t,t_0,x,x_0), \qquad (5.47)$$

where the integral operator defined in appendix 6.3.3 by formula (C.26). We now prove two auxiliary results which we need later on.

Corollary 5.3.1. *There exists a constant $K_4 > 0$ such that*

$$\sum_{k=1}^{\infty} h^k \mathcal{F}^k \frac{\partial^{|L|}\psi_G^{as}}{\partial x^L}(t, t_0, x, x_0)$$
$$= (h(t-t_0))^{-|L|} O(h(t-t_0)t^{(1-\varepsilon)})(1+|x-x_0|)^{|L|}\psi_G^{as}(t, t_0, x, x_0)$$

holds for $0 \leqslant t_0 < t < T_{1,\varepsilon} \wedge K_4$, $|L| = 0, \ldots, q$ with $T_{1,\varepsilon}$ being a stopping time defined in (C.15).

Proof. By Lemma $C.5$ $\partial^{|L|}\psi/\partial x^L$ has the form (C.27) with $\alpha(t,\tau) = (h(t-\tau))^{|L|}$ and $m = |L|$. Using induction one can easily deduce from Proposition 6.3.1

$$\left| h^k \mathcal{F}^k \frac{\partial^{|L|}\psi_G^{as}}{\partial x^L} \right| \leqslant h^k C^k t^{(1-\varepsilon)k} (1+|x-x_0|)^{|L|} \, \alpha_k \, \psi_G^{as}(t, t_0, x, x_0) \qquad (5.48)$$

for some constant $C = C(d, K) > 0$ and $0 \leqslant t_0 < t < T_{1,\varepsilon}$, where

$$\alpha_k = \int_{t_0}^{t}\int_{\tau_k}^{t} \ldots \int_{\tau_2}^{t} \alpha(t, \tau_1)\, d\tau_1 \ldots d\tau_k = h^{|L|}\, \frac{(t-t_0)^{k+|L|}}{k!} \leqslant h^{|L|}\, (t-t_0)^{|L|+1}.$$

Summing inequalities (5.48) over $k \in \mathbb{N}$ and using that $\sum_{k=1}^{\infty} h^k C^k t^{(1-\varepsilon)k} = O(t^{1-\varepsilon})$ for $t < K_4 = (C^{-(1-\varepsilon)^{-1}}/2)$, give the proof. □

We set

$$T_\varepsilon = T_{1,\varepsilon} \wedge K_4. \qquad (5.49)$$

Lemma 5.3.1. *Let $\zeta_t = t$ or $\zeta_t = \xi_{i,t}$ for some $i = 1, \ldots, d$. Then for any predictable process $b_0(t) = b_0(t, \tau, x, x_0)$ (w.r.t. the natural filtration of $\{\zeta_t\}_{t \geqslant \tau}$) and the process*

$$b_1(t, \tau, x, x_0) = \int_{\tau}^{t} b_0(s, \tau, x, x_0)\, d\zeta_s$$

we have

$$\int_{\tau}^{t} (\mathcal{F}b_0)(s, \tau, x, x_0) d\zeta_s = (\mathcal{F}b_1)(t, \tau, x, x_0).$$

for $0 \leqslant \tau < t < T$.

Proof. Changing the order of the integration we have

$$\int_\tau^t \int_\tau^s b_0(s,l,x,\eta)\mathcal{K}(l,\tau,\eta,x_0)\,dl d\zeta_s = \int_\tau^t \int_l^t b_0(s,l,x,\eta)\mathcal{K}(l,\tau,\eta,x_0)\,d\zeta_s dl,$$

where

$$\mathcal{K}(t,t_0,x,x_0) = \frac{1}{2}\operatorname{tr}\frac{\partial^2\phi(t,t_0,x,x_0)}{\partial x^2}\exp\left\{-\frac{S(t,t_0,x,x_0)}{h}\right\} \tag{5.50}$$

and so

$$\begin{aligned}\int_\tau^t (\mathcal{F}b_0)(s,\tau,x,x_0)\,d\zeta_s &= \int_\tau^t \int_\tau^s \int_{\mathbb{R}^d} b_0(s,l,x,\eta)\mathcal{K}(l,\tau,\eta,x_0)\,d\eta dl d\zeta_s \\ &= \int_\tau^t \int_{\mathbb{R}^d} b_1(t,l,x,\eta)\mathcal{K}(l,\tau,\eta,x_0)\,d\eta dl = (\mathcal{F}b_1)(t,\tau,x,x_0).\end{aligned}$$

□

Theorem 5.3.1. *Let the assumptions given at the beginning of section 5.2 hold. Series* (5.47) *converges and*

$$\begin{aligned}\psi_G(t,t_0,x,x_0) &= \psi_G^{as}(t,t_0,x,x_0)\left(1+O(h(t-t_0)t^{(1-\varepsilon)})\right) \\ &= \phi(t,t_0,x,x_0)\exp\left\{-\frac{1}{h}S(t,t_0,x,x_0)\right\}\left(1+O(h(t-t_0)t^{(1-\varepsilon)})\right).\end{aligned} \tag{5.51}$$

Moreover, $\psi_G(t,\tau,x,x_0)$ satisfies the equation

$$\begin{aligned}&h\psi_G(t,t_0,x,x_0) \\ &= h\delta(x-x_0) - \int_{t_0}^t L_0\psi_G(s,t_0,x,x_0)\,ds - \sum_{i=1}^d \int_{t_0}^t L_i\psi_G(s,t_0,x,x_0)\,d\xi_{i,s}\end{aligned} \tag{5.52}$$

for $0 \leqslant t_0 < t < T_\varepsilon$ *with* T_ε *given by* (5.49).

Proof. An application of Corollary 5.3.1 with $L=0$ implies the convergence of series (5.47) and gives asymptotic formula (5.51).

Now we are going to show that $\psi_G(t,t_0,x,x_0)$ is a Green function for equation (5.52). Definitions (C.26), (5.45), (5.46) imply $\mathcal{F}^k L_i = L_i \mathcal{F}^k \quad i = 0,\dots,d$. It follows

$$\sum_{k=0}^{\infty} h^k \mathcal{F}^k L_i \psi_G^{as}(t,t_0,x,x_0) = L_i \sum_{k=0}^{\infty} h^k \mathcal{F}^k \psi_G^{as}(t,t_0,x,x_0). \tag{5.53}$$

Notice that Corollary 5.3.1 gives convergence in (5.53). Applying $h^k\mathcal{F}^k$ to the both sides of equation (5.44) and using Lemma 5.3.1 we have

$$\begin{aligned} h^{k+1}&[\mathcal{F}^k\psi^{as}](t,t_0,x,x_0) \\ &= h^{k+1}[\mathcal{F}^k\chi_1](t,t_0,x,x_0) - h^k \int_{t_0}^{t} [\mathcal{F}^k L_0 \psi^{as}](s,t_0,x,x_0)\,ds \\ &\quad -h^k \sum_{i=1}^{d} \int_{t_0}^{t} [\mathcal{F}^k L_i \psi^{as}](s,t_0,x,x_0)\,d\xi_{i,s} - h^{k+2}[\mathcal{F}^k\chi_2](t,t_0,x,x_0), \end{aligned} \tag{5.54}$$

where $\chi_1(t,t_0,x,x_0) = \delta(x-x_0)$, $\chi_2(t,t_0,x,x_0) = \int_{t_0}^{t} \mathcal{K}(s,t_0,x,x_0)\,ds$. Since $[\mathcal{F}\chi_1] = \chi_2$, it follows

$$\sum_{k=0}^{\infty} h^{k+1}[\mathcal{F}^k\chi_1](t,t_0,x,x_0) - \sum_{k=0}^{\infty} h^{k+2}[\mathcal{F}^k\chi_2](t,t_0,x,x_0) = h\delta(x-x_0).$$

Summarising (5.54) over $k = 0,1,\dots$ we arrive at (5.52). □

5.3.2 Applications

Now we deduce some direct important consequences of Theorem 5.3.1.

Proposition 5.3.1 (Two-sided estimates for heat kernels). *Under the assumptions given in section 5.2 there exist constants $K_5, h_0 > 0$ such that*

$$\begin{aligned} &\frac{K_5^{-1}}{\left(\sqrt{h(t-t_0)}\right)^d} \exp\left\{-\frac{(x-x_0)^2}{h(t-t_0)}\right\} \\ &\qquad \leqslant \psi_G(t,t_0,x,x_0) \leqslant \frac{K_5}{\left(\sqrt{h(t-t_0)}\right)^d} \exp\left\{-\frac{(x-x_0)^2}{3h(t-t_0)}\right\}. \end{aligned} \tag{5.55}$$

for $t_0 \leqslant t < T_\varepsilon$, $h \leqslant h_0$.

Proof. Corollary 5.1.1, (ii) Lemma $C.4$ and (5.51) give the proof. □

Proposition 5.3.2 (Large deviation principle). *Under the assumptions given in section* (5.2) *we have*

(i)

$$\lim_{h\to 0} h \ln \psi_G(t, t_0, x, x_0) = -S(t, t_0, x, x_0).$$

(ii)

$$\lim_{t\to t_0} (t - t_0) \ln \psi_G(t, t_0, x, x_0) = -\frac{|x - x_0|^2}{2h}.$$

5.4 The Cauchy problem and global asymptotics for stochastic heat equations

5.4.1 Well posedness of the Cauchy problem for heat equation

Theorem 5.4.1. *We assume that* $\psi_0(x) \in C_\infty(\mathbb{R}^d)$, *i.e.* $\psi_0(x)$ *is continuous and vanishing at infinity. Then the formula*

$$[R_t\psi_0](x) = \psi(t, t_0, x) = \int_{\mathbb{R}^d} \psi_G(t, t_0, x, x_0)\psi_0(x_0)\, dx_0 \tag{5.56}$$

for $0 \leqslant t_0 < t < T_\varepsilon$ *gives the unique solution* $\psi(t, t_0, \cdot) \in C_\infty(\mathbb{R}^d)$ *of the equation*

$$h\psi(t, t_0, x) = h\psi_0(x) - \int_{t_0}^{t} L_0\psi(s, t_0, x)\, ds - \sum_{i=1}^{d} \int_{t_0}^{t} L_i\psi(s, t_0, x)\, d\xi_{i,s} \tag{5.57}$$

with L_i $i = 0, \ldots, d$ *given by* (5.45), (5.46).

Proof. Theorem 5.3.1 implies that a solution of equation (5.57) is given by formula (5.56). It remains to prove the uniqueness. Let $\psi(t, t_0, x)$ be a solution of (5.57) such that $\psi(t_0, t_0, x) = 0$. For short we write $\psi = \psi(\tau, t_0, x)$, $\psi_- = \psi(\tau-, t_0, x)$, $c = c(x)$, $V = V(x)$, $a = a(x)$.

Let us first assume that

$$V + ha - h \sum_{i=1}^{d} c_i \, \mathbb{E}\xi_{i,1} + \frac{h}{2} \left[\sum_{i,j=1}^{d} c_i c_j \int_{\mathbb{R}^d \setminus 0} y_i y_j \, \nu(dy) \right] \geqslant 1 \tag{5.58}$$

for any $x \in \mathbb{R}^d$. An application of Itô's formula shows

$$d\psi^2 = 2\,\psi_- \,d\psi + d[\psi,\psi] = 2\psi_- \,d\psi + \psi_-^2 \,d < \mathcal{Z}_\tau \, c, c >$$

where $(\mathcal{Z}_\tau)_{i,j} = [\xi_i, \xi_j]_\tau$, $\mathcal{Z}_\tau \in \mathbb{R}^{d \times d}$, and as

$$L_0 \psi^2 = 2\psi L_0 \psi - (V + ha)\psi^2 - h^2 \sum_{i=1}^{d} \left(\frac{\partial \psi}{\partial x_i} \right)^2 ,$$

we obtain

$$\begin{aligned} h\,d\psi^2 + L_0\psi^2 \,d\tau + \sum_{i=1}^{d} L_i \psi_-^2 \,d\xi_{i,\tau} \\ = \; 2\psi \left[h\,d\psi + L_0 \psi \,d\tau + \sum_{i=1}^{d} L_i \psi_- \,d\xi_{i,\tau} \right] - (V + ha)\psi^2 \,d\tau \\ -h^2 \sum_{i=1}^{d} \left(\frac{\partial \psi}{\partial x_i} \right)^2 d\tau - \sum_{i=1}^{d} L_i \psi_-^2 d\xi_{i,\tau} + h\psi_-^2 \,d < \mathcal{Z}_\tau \, c, c > . \end{aligned} \tag{5.59}$$

Using the fact that $\psi(t, t_0, x)$ is a solution of (5.57) and so the expression in square brackets vanishes, we rewrite (5.59) as

$$\begin{aligned} h\,d\psi^2 + h^2 \sum_{i=1}^{d} \left(\frac{\partial \psi}{\partial x_i} \right)^2 d\tau \;&=\; \frac{h^2}{2} \operatorname{tr} \left(\frac{\partial^2 \psi^2}{\partial x^2} \right) d\tau - 2(V + ha)\psi^2 \,d\tau \\ & -2 \sum_{i=1}^{d} L_i \psi_-^2 d\xi_{i,\tau} + h\psi_-^2 \,d < \mathcal{Z}_\tau \, c, c > . \end{aligned} \tag{5.60}$$

Since $\psi \in C_\infty(\mathbb{R}^d)$, it follows

$$\int_{\mathbb{R}^d} \operatorname{tr} \left(\frac{\partial^2 \psi^2}{\partial x^2} \right) dx = 2 \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} \left(\psi \frac{\partial \psi}{\partial x_i} \right) dx = 0.$$

Integrate (5.60) over $\mathbb{R}^d \times [t_0, t]$ to get

$$h \int_{\mathbb{R}^d} \psi^2(t, t_0, x)dx + h^2 \int_{\mathbb{R}^d} \int_{t_0}^{t} \sum_{i=1}^{d} \left(\frac{\partial\psi}{\partial x_i}\right)^2 d\tau dx \tag{5.61}$$

$$= -2 \int_{\mathbb{R}^d} \int_{t_0}^{t} (V + ha)\psi^2 \, d\tau dx + 2 \sum_{i=1}^{d} \int_{\mathbb{R}^d} \int_{t_0}^{t} hc_i\psi_-^2 \, d\xi_{i,\tau} dx$$

$$+h \int_{\mathbb{R}^d} \int_{t_0}^{t} \psi_-^2 \, d < \mathcal{Z}_\tau \, c, c > dx.$$

Denote the right-hand side of (5.61) by $\mathbf{I}$. Since the left-hand side of (5.61) is non-negative it follows $\mathbf{I} \geqslant 0$. On the other hand

$$\mathbb{E}\mathbf{I} = -2 \int_{\mathbb{R}^d} \int_{t_0}^{t} (V + ha)(\mathbb{E}\,\psi^2) \, d\tau dx + 2h \sum_{i=1}^{d} \int_{\mathbb{R}^d} \int_{t_0}^{t} c_i \left(\mathbb{E}\,\psi_-^2\right) \, d\left(\mathbb{E}\,\xi_{i,\tau}\right) dx$$

$$+h \int_{\mathbb{R}^d} \int_{t_0}^{t} (\mathbb{E}\,\psi_-^2) \, d(\mathbb{E} < \mathcal{Z}_\tau \, c, c >) \, dx.$$

Using assumption (5.58) and the fact that

$$\mathbb{E}\,\xi_{i,\tau} = \tau \mathbb{E}\,\xi_{i,1}, \quad \mathbb{E}\,[\xi_i, \xi_j]_\tau = \tau \int_{\mathbb{R}^d \setminus 0} y_i y_j \, \nu(dy)$$

we obtain

$$\mathbb{E}\mathbf{I} = -2 \int_{\mathbb{R}^d} \int_{t_0}^{t} C\mathbb{E}\,\psi^2 \, d\tau dx \leqslant -2 \int_{\mathbb{R}^d} \int_{t_0}^{t} \mathbb{E}\,\psi^2 \, d\tau dx \leqslant 0,$$

where $C = C(x)$ is the left-hand side of (5.58), and so $\mathbb{E}\,\psi^2 \equiv 0$ that is $\psi \equiv 0$ a.s.

We proceed with the case $C(x) < 1$ for some $x \in \mathbb{R}^d$. Recall that $\partial c/\partial x$ has compact support and V, a are bounded below (see section 5.2), and so $C(x) \geqslant C_0$ for some constant C_0. Let

$$\widetilde{\psi}(\tau, t_0, x) = \exp\{(1 - C_0)\tau\}\psi(\tau, t_0, x),$$

where C is given by (5.58). One readily sees that the function $\widetilde{\psi}$ satisfies the equation (5.57) with the coefficients $\widetilde{V} = V - C_0 + 1$, $\widetilde{a} = a$, $\widetilde{c} = c$. Clearly, $\widetilde{V}$, $\widetilde{a}$, $\widetilde{c}$ satisfy condition (5.58) and so $\widetilde{\psi} \equiv 0$ a.s. Then $\psi \equiv 0$ a.s. $\square$

As trivial consequence of formulae (5.43), (5.51), (5.56) we obtain

Corollary 5.4.1. *We assume that the conditions of Theorem* 5.4.1 *are satisfied and $\psi_0(x) \geqslant 0$. Then the solution of the Cauchy problem $\psi(t, t_0, x)$ is also nonnegative.*

The results obtained on the Green function allows us to get easily the following qualitative properties of the solution of the Cauchy problem.

Theorem 5.4.2. *We assume that the conditions of the Theorem* 5.4.1 *are satisfied. Then*

(i) $[R_t\psi_0](x)$ tend to $\psi_0(x)$ as $t \to t_0$ for each x and any $\psi_0 \in C_\infty(\mathbb{R}^d)$; moreover, if $\psi_0 \in C_0(\mathbb{R}^d)$, then $[R_t\psi]$ tend to ψ_0 uniformly, as $t \to t_0$.

(ii) R_t is a continuous operator $C(\mathbb{R}^d) \to C^m(\mathbb{R}^d)$ with the norm of order $((t - t_0)h)^{-m}$ for all $m \leqslant q$. Here q is from (5.2).

Proof. Using the trivial fact that

$$\frac{1}{\left(\sqrt{2\pi h(t - t_0)}\right)^d} \exp\left\{-\frac{(x_0 - x)^2}{2h(t - t_0)}\right\} \xrightarrow{weakly} \delta(x_0 - x) \quad \text{as} \quad t \to t_0 \tag{5.62}$$

and formulae (5.43), (5.51), (5.56) we have $|R_t\psi_0(x) - \psi_0(x)| \to 0$ as $t \to t_0$. Since convergence in (5.62) is uniform in $x \in K$ for any compact set $K \subset \mathbb{R}^d$, we establish (i).

Formula (5.56) and Corollary 5.3.1 imply

$$\begin{aligned}
\frac{\partial^{|L|} R_t\psi_0}{\partial x^L} &= \int\limits_{\mathbb{R}^d} \frac{\partial^L \psi_G(t, t_0, x, x_0)}{\partial x^L} \psi_0(x_0)\, dx_0 \\
&= \int\limits_{\mathbb{R}^d} \sum_{k=0}^{\infty} \mathcal{F}^k \frac{\partial^L \psi_G^{as}(t, t_0, x, x_0)}{\partial x^L} \psi_0(x_0)\, dx_0 \\
&= O((h(t - t_0))^{-|L|}) \int\limits_{\mathbb{R}^d} (1 + |x - x_0|)^{|L|} \psi_G^{as}(t, t_0, x, x_0) \psi_0(x_0)\, dx_0.
\end{aligned}$$

By (5.43) we see

$$\int_{\mathbb{R}^d} (1+|x-x_0|)^{|L|}\psi_G^{as}(t,t_0,x,x_0)\,dx_0 = O(1)$$

and (ii) follows. □

Theorem 5.4.3. *We assume that the conditions of Theorem* 5.4.1 *are fulfilled and* $V(x) \geqslant -V_0$ *with some* $V_0 \geqslant 0$. *Let*

$$a(x) \geqslant (\mathbb{E}\,\xi_1)\,c(x) = \sum_{i=1}^{d} (\mathbb{E}\,\xi_{i,1})\,c_i(x). \tag{5.63}$$

Then

(i)

$$\mathbb{E}\left(\|\psi(t,t_0,x)\|_{L^1(dx)}\mathbf{1}_{\{t<T_\varepsilon\}}\right) \leqslant \|\psi_0(x)\|_{L^1(dx)} \exp\left\{\frac{1}{h}V_0 t\right\}.$$

In particular, if $V(x) \geqslant 0$ *then the solution of equation* (5.52) *is dissipative, that is*

$$\mathbb{E}\left(\|\psi(t,t_0,x)\|_{L^1(dx)}\mathbf{1}_{\{t<T_\varepsilon\}}\right) \leqslant \|\psi_0(x)\|_{L^1(dx)}.$$

(ii) In the case of vanishing potential $V(x) = 0$ *we get*

$$\mathbb{E}\left(\|\psi(t,t_0,x)\|_{L^1(dx)}\mathbf{1}_{\{t<T_\varepsilon\}}\right) = \|\psi_0(x)\|_{L^1(dx)}$$

provided that

$$a(x) = -(\mathbb{E}\,\xi_1)\,c(x) = -\sum_{i=1}^{d} (\mathbb{E}\,\xi_{i,1})\,c_i(x). \tag{5.64}$$

Proof. We first assume that $\psi_0(x) \geqslant 0$ for any $x \in \mathbb{R}^d$. Then by Corollary 5.4.1 $\psi(t,t_0,x) \geqslant 0$. Integrating equation (5.57) over x gives

$$h\mathbf{1}_{t<T_\varepsilon}\int_{\mathbb{R}^d}\psi(t,t_0,x)\,dx = h\mathbf{1}_{t<T_\varepsilon}\int_{\mathbb{R}^d}\psi_0(x)\,dx \tag{5.65}$$

$$-\mathbf{1}_{t<T_\varepsilon}\int_{\mathbb{R}^d}\int_{t_0}^{t} L_0\psi(\tau,t_0,x)\,d\tau dx - \mathbf{1}_{t<T_\varepsilon}\sum_{i=1}^{d}\int_{\mathbb{R}^d}\int_{t_0}^{t} L_i\psi(\tau,t_0,x)\,d\xi_{i,\tau}dx.$$

It follows by (5.43), (5.51), (5.56) that for any $l = 1, \dots, d$ $(\partial^2 \psi(t, t_0, \cdot)/\partial x_l^2), (\partial \psi(t, t_0, \cdot)/\partial x_l) \in L^1(\mathbb{R}^d)$ and

$$\frac{\partial \psi(t, t_0, x)}{\partial x_l} \to 0 \qquad \text{as} \qquad |x_l| \to +\infty.$$

From which we see

$$\int_{\mathbb{R}^d} \operatorname{tr} \frac{\partial^2 \psi(\tau, t_0, x)}{\partial x^2} \, dx = 0.$$

We deduce from this and definition (5.46)

$$\int_{\mathbb{R}^d} L_0 \psi(\tau, t_0, x) \, dx = \int_{\mathbb{R}^d} V(x) \psi(\tau, t_0, x) \, dx + h \int_{\mathbb{R}^d} a(x) \psi(\tau, t_0, x) \, dx. \quad (5.66)$$

On the other hand, since

$$\mathbb{E} \left[\mathbf{1}_{t<T_\varepsilon} \int_{t_0}^{t} \psi(\tau, t_0, x) \, d\xi_{i,\tau} \right] = (\mathbb{E} \xi_{i,1}) \int_{t_0}^{t} [\mathbb{E} \psi(\tau, t_0, x) \, \mathbf{1}_{t<T_\varepsilon}] \, d\tau,$$

it follows from (5.63) and the assumption $\psi(\tau, t_0, x) \geqslant 0$ that

$$\begin{aligned} h\mathbb{E} \left[\mathbf{1}_{t<T_\varepsilon} \int_{t_0}^{t} a(x) \psi(\tau, t_0, x) \, d\tau \right] + \sum_{i=1}^{d} \mathbb{E} \left[\mathbf{1}_{t<T_\varepsilon} \int_{t_0}^{t} L_i \psi(\tau, t_0, x) \, d\xi_{i,\tau} \right] \quad (5.67) \\ = h \left(a(x) - \sum_{i=1}^{d} \mathbb{E} \xi_{i,1} c_i(x) \right) \int_{t_0}^{t} [\mathbb{E} \psi(\tau, t_0, x) \, \mathbf{1}_{t<T_\varepsilon}] \, d\tau \geqslant 0. \end{aligned}$$

Thus, taking mathematical expectation from both sides of (5.65) and using estimates (5.66), (5.67), yield

$$\begin{aligned} h\mathbb{E} \left[\int_{\mathbb{R}^d} \psi(t, t_0, x) \, dx \, \mathbf{1}_{t<T_\varepsilon} \right] \quad (5.68) \\ \leqslant h \int_{\mathbb{R}^d} \psi_0(x) \, dx - \int_{t_0}^{t} \mathbb{E} \left[\int_{\mathbb{R}^d} V(x) \psi(\tau, t_0, x) \, dx \, \mathbf{1}_{t<T_\varepsilon} \right] d\tau. \end{aligned}$$

Since $V(x) \geqslant -V_0$, it follows

$$h\mathbb{E}\left[\|\psi(t,t_0,\cdot)\|_{L^1(dx)}\,\mathbf{1}_{t<T_\varepsilon}\right] \leqslant h\|\psi_0(\cdot)\|_{L^1(dx)} + V_0\int_{t_0}^{t}\mathbb{E}\left[\|\psi(\tau,t_0,\cdot)\|_{L^1(dx)}\,\mathbf{1}_{\tau<T_\varepsilon}\right]d\tau,$$

and an application of Gronwall Lemma gives (i).

We now proceed with a general case. Let $\psi_0 = \psi_0^+ - \psi_0^-$, where $\psi_0^+, \psi_0^- \geqslant 0$. Then

$$\|\psi(t,t_0,x)\|_{L^1(dx)} = \|\psi^+(t,t_0,x)\|_{L^1(dx)} + \|\psi^-(t,t_0,x)\|_{L^1(dx)}.$$

Here ψ^+ and ψ^- are solutions of equation (5.52) with initial conditions ψ_0^+ and ψ_0^- respectively. Since (i) is proven for ψ^+ and ψ^-, it holds for ψ.

Condition (5.64) implies that the left-hand side of (5.67) vanishes. Consequently (5.68) turns into equality. Using $V(x) = 0$ we establish (ii). □

Corollary 5.4.2. *The Green function $\psi_G(t,t_0,x,x_0)$ satisfies the non-homogeneous Chapman-Kolmogorov equation*

$$\psi_G(t,t_0,x,x_0) = \int_{\mathbb{R}^n}\psi_G(t,\tau,x,\eta)\psi_G(\tau,t_0,\eta,x_0)\,d\eta,$$

where $0 \leqslant t_0 \leqslant \tau \leqslant t < T_\varepsilon$.

This simple fact is important by different reasons. First of all this property together with the positivity of the Green function allows to interpret this Green function (after some normalisations if necessary) as a transition probability density of a certain stochastic process. Moreover, using Corollary 5.4.2 we can extend the asymptotic for the Green function to large times t, i.e. to obtain global semiclassical asymptotics, and therefore to get a corresponding extension of the result of section 5.3.

5.4.2 Generalised solutions for Hamilton-Jacobi equation

In this section we discuss briefly a construction of generalised solutions of the Hamilton-Jacobi equation which leads to the well-posedness theorem for the

Cauchy problem with rather general (even discontinuous) initial data. This construction is quite similar to the case of deterministic Hamilton-Jacobi equations (see [KMa]) or to the case of stochastic equations driven by a Wiener process (see [K1]) and therefore will be only sketched here.

Notice now that formula (4.67) in preliminaries still makes sense if S_0 is merely bounded below and lower semicontinuous. Therefore one can expect that a reasonable definition of a generalised solution of the Cauchy problem for equation (4.13) is given by

$$[R_t S_0](x) = \inf_{\zeta}(S_0(\zeta) + S(t, t_0, x, \zeta)), \quad t \geqslant t_0 \tag{5.69}$$

for this solution.

One way to come to such a definition is based on the method of vanishing viscosity (see e.g. [Kr] and [GL] for the case of the deterministic Hamilton-Jacobi equations). An alternative approach comes from the ideas of idempotent analysis ([KMa]). This later approach is based on the simple observation that the operators R_t, $t > 0$, are linear operators on the space of functions which take values in a metric semiring $\mathbb{R} \cup \{+\infty\}$ with the metric $\rho(a, b) = |e^{-a} - e^{-b}|$ and with the commutative binary operations $\oplus = \min$ and $\odot = +$. One can show (cf. [KMa]) that convex smooth functions form a basis for the semimodule of continuous functions with values in the semiring $\mathbb{R} \cup \{+\infty\}$. Thus formula (5.69) can be considered as the natural extension (by continuity and linearity) of the operator R_t defined initially on convex smooth functions where it gives (at least for small times) a classical solution. In this set-up one can also introduce a notion of duality which gives the analogue of the usual L^2 inner product and thus define the generalised solutions in the sense of distributions similar to the standard Sobolev construction for the case of linear equations. This leads again to formula (5.69). For details we refer to the paper [K2]. The same formula (5.69) can be justified by the method of viscosity solutions. For a thorough comparison of these two approaches to the construction of generalised solution to HJB equations in the deterministic case see recent papers [DeMDo], [McCB].

5.4.3 Asymptotics of the solutions of the Cauchy problem for heat equation

The next result is a direct consequence of Lemmas $D.1$, $D.2$ and formulae (5.56) and (5.69).

Theorem 5.4.4. *Let $\psi(t, t_0, x)$ be a solution of Cauchy problem for equation (5.57) with initial condition $\psi_0(x) = \chi_D$ at $t = t_0$, where $D \subset \mathbb{R}^d$, $S(t, t_0, x)$ be a generalised solution of Hamilton-Jacobi equation with initial condition*

$$S_0(x) = \begin{cases} 0 & \text{if } x \in D, \\ +\infty & \text{otherwise} \end{cases}$$

at $t = t_0$. We define $D_t = D_t(\omega) \subset \mathbb{R}^d$ by saying that

$$x \in D_t \quad \text{iff there exits} \quad \hat{x}_0 \in D \quad \text{such that} \quad x = X(t, t_0, \hat{x}_0, 0).$$

1. *If $x \in D_t$ then*

$$\begin{aligned} \psi(t, t_0, x) &= (\sqrt{2\pi h})^d \phi(t, t_0, x, \hat{x}_0) \left(\det \frac{\partial^2 S(t, t_0, x, \hat{x}_0)}{\partial x_0^2} \right)^{-\frac{1}{2}} \\ &\quad \times \exp\{-\frac{S(t, t_0, x)}{h}\}(1 + O(h)) \end{aligned}$$

   for some $\hat{x}_0 \in D$.

2. *If $x \notin D_t$ then*

$$\begin{aligned} \psi(t, t_0, x) &= (\sqrt{2\pi h})^{d-1} \frac{h}{|b|} \phi(t, t_0, x, \hat{x}_0) \\ &\quad \times \left(\det[|b| \, G^{(2)}(0) + \Lambda_b] \right)^{-\frac{1}{2}} \exp\{-\frac{S(t, t_0, x)}{h}\}(1 + O(h)) \end{aligned}$$

   for some $\hat{x}_0 \in \partial D$. Here $b = (\partial S(t, t_0, x, \hat{x}_0)/\partial x_0)$, $\Lambda = (\partial^2 S(t, t_0, x, \hat{x}_0)/\partial x_0^2)$, Λ_b is given by (D.1) and ∂D is given by (D.2) with $e = b/|b|$ in some neighbourhood of $\hat{x}_0$.

Chapter 6

Appendices

6.1 Appendix A

Lemma A.1. *Let us denote by*

$$I_s = \int_{-1}^{+1} (1-\delta^2)^s \, d\delta, \tag{A.1}$$

$s > -1$, $s \in \mathbb{R}$. *Then*

$$I_s = \frac{2s+3}{2s+2} I_{s+1} \tag{A.2}$$

for any $s > -1$.

Proof. Chosen any $0 < r < 1$ and $n \in \mathbb{N}_0$ we find from (A.1)

$$\sum_{m=0}^{n} I_{m+r} = \int_{-1}^{1} \frac{1-(1-\delta^2)^{n+1}}{\delta^2}(1-\delta^2)^r \, d\delta. \tag{A.3}$$

We integrate (A.3) by parts (with $u = (1-(1-\delta^2)^{n+1})(1-\delta^2)^r$ and $v = -(1/\delta)$) to get

$$\sum_{m=0}^{n} I_{m+r} = \int_{-1}^{1} \frac{2(n+1)\delta(1-\delta^2)^n}{\delta}(1-\delta^2)^r \, d\delta$$

$$-\int_{-1}^{1} \frac{1-(1-\delta^2)^{n+1}}{\delta}(1-\delta^2)^{r-1}(2r\delta)\,d\delta$$
$$= 2(n+1)I_{n+r} - 2rI_{r-1} + 2rI_{n+r}.$$

Consequently we obtain the recurrent formula

$$I_{n+r} = \frac{1}{2(n+r)+1}\,(2rI_{r-1} + \sum_{m=0}^{n-1} I_{m+r}). \tag{A.4}$$

Performing elementary calculations and using (A.4), we prove (A.2) for all non-integer $s > -1$. By continuity we obtain (A.2) for $s \in \mathbb{N}_0$. □

Lemma A.2. *For any $v \in \mathbb{R} \setminus \{0\}$, $a \geqslant 1$, $d \geqslant 3$ we have*

$$J(v) = \int_{-1}^{1} (1-\delta^2)^{\frac{d-3}{2}} \ln(v^2 + 2v\delta + a)\,d\delta > (\ln a)\, I_{\frac{d-3}{2}}. \tag{A.5}$$

Proof. One can calculate $J(v)$ explicitly for $d = 3$ and check the statement. We proceed with $d \geqslant 4$. Clearly $J(v) = J(-v)$ and $J(v)$ is increasing for $v \geqslant 1$, since $v^2 + 2v\delta + a$ is increasing for all $|\delta| < 1$, $a \geqslant 1$. Using the fact that $J(0) = (\ln a)\, I_{\frac{d-3}{2}}$, it suffices to show that $J(v)$ is increasing for $0 < v < 1$. Taking the derivative

$$J'(v) = 2\int_{-1}^{1} (1-\delta^2)^{\frac{d-3}{2}} \frac{v+\delta}{v^2 + 2v\delta + a}\,d\delta$$

and using the decomposition

$$\frac{v+\delta}{v^2+2v\delta+a} = \frac{v+\delta}{v^2+a}\sum_{n=0}^{\infty}(-1)^n \left(\frac{2v}{v^2+a}\right)^n \delta^n = \phi_1(v,\delta) + \phi_2(v,\delta),$$

where

$$\phi_1(v,\delta) = \frac{v}{v^2+a} + \frac{v^2-a}{v^2+a}\sum_{m=1}^{\infty}\left(\frac{2v}{v^2+a}\right)^{2m-1}\delta^{2m}, \tag{A.6}$$

and $\phi_2(v,\delta) = -\phi_2(v,-\delta)$, we obtain

$$J'(v) = 2\int_{-1}^{1} (1-\delta^2)^{\frac{d-3}{2}} \phi_1(v,\delta)\,d\delta. \tag{A.7}$$

Since

$$\sum_{m=1}^{\infty}\left(\frac{2v}{v^2+a}\right)^{2m-1}\delta^{2m} \leqslant \frac{2v}{v^2+a}\,\delta^2 + 8\left(\frac{v}{v^2+a}\right)^3\frac{\delta^4}{1-\delta^2},$$

we find from (A.6)

$$\phi_1(v,\delta) \geqslant \frac{v}{v^2+a}\left[(1-A_1+2A_2)+(A_1-A_2)(1-\delta^2)-A_2\,\frac{1}{1-\delta^2}\right], \tag{A.8}$$

where

$$A_1 = 2\,\frac{a-v^2}{a+v^2}, \quad A_2 = 8\,\frac{a-v^2}{a+v^2}\left(\frac{v}{v^2+a}\right)^2. \tag{A.9}$$

Substituting (A.8) into (A.7) and using the notations of Lemma $A.1$ we get

$$\begin{aligned}\frac{v^2+a}{2v}\,J'(v) &= (1-A_1+2A_2)\,I_{\frac{d-3}{2}} + (A_1-A_2)I_{\frac{d-1}{2}} - A_2 I_{\frac{d-5}{2}} \\ &= I_{\frac{d-3}{2}} - (A_1-A_2)[I_{\frac{d-3}{2}} - I_{\frac{d-1}{2}}] - A_2[I_{\frac{d-5}{2}} - I_{\frac{d-3}{2}}].\end{aligned}$$

An application of Lemma $A.1$ shows

$$\frac{v^2+a}{2v}\,J'(v) = I_{\frac{d-3}{2}}(1-\frac{1}{d}\,(A_1-A_2)-\frac{1}{d-3}\,A_2). \tag{A.10}$$

From (A.9) we find

$$0 \leqslant \frac{1}{d}\,A_1 + \frac{3}{d(d-3)}\,A_2 < 1 \tag{A.11}$$

for $0 \leqslant v \leqslant \sqrt{a}$, $d \geqslant 4$. Combining (A.10) and (A.11) gives $J'(v) > 0$ for $0 < v < 1$. $\square$

Lemma A.3. *Let $d \geqslant 3$, $0 < \alpha < 2$. There exists $\gamma = \gamma(\alpha, d) > 0$ such that*

$$\int_{\mathbb{R}^d} f(p,\zeta)\,\frac{1}{|\zeta|^{d+\alpha}}\,d\zeta \leqslant 0 \qquad \forall p \in \mathbb{R}^d, \lambda \in \mathbb{R}, \tag{A.12}$$

where

$$f(p,\zeta) = \frac{1}{(|p+\lambda\zeta|^2+1)^\gamma} - \frac{1}{(|p|^2+1)^\gamma}.$$

Proof. Without loss of generality we prove the lemma for $\lambda = 1$. Denote the left-hand side of (A.12) as $I(\gamma)$. Changing the coordinates to the polar coordinates we get

$$I(\gamma) = \int_{S^{d-2}} \int_0^{+\infty} Z\, r^{-1-\alpha}\, dr d\theta = |S^{d-2}| \int_0^{+\infty} Z\, r^{-1-\alpha}\, dr,$$

where

$$Z = \int_{-1}^{1} (1-\delta^2)^{\frac{d-3}{2}} \left[\frac{1}{(r^2 + |p|^2 + 2r|p|\delta + 1)^\gamma} - \frac{1}{(|p|^2+1)^\gamma} \right] d\delta,$$

or $Z = |p|^{-2\gamma} Z_1$,

$$Z_1 = \int_{-1}^{1} (1-\delta^2)^{\frac{d-3}{2}} \left[\frac{1}{(v^2 + 1 + 2v\delta + |p|^{-2})^\gamma} - \frac{1}{(1+|p|^{-2})^\gamma} \right] d\delta, \qquad v = \frac{r}{|p|}.$$

An application of Lemma $A.2$ with $a = 1 + |p|^{-2}$ implies

$$\begin{aligned} \left.\frac{dZ_1}{d\gamma}\right|_{\gamma=0} &= -\int_{-1}^{1} (1-\delta^2)^{\frac{d-3}{2}} \left[\ln(v^2 + 2v\delta + a) - \ln(a)\right] d\delta \\ &= -\left[J(v) - (\ln a)\, I_{\frac{d-3}{2}}\right] < 0 \end{aligned}$$

and so

$$I'(0) = -|p|^{-\alpha-2\gamma} \int_0^{+\infty} \left[J(v) - (\ln a)\, I_{\frac{d-3}{2}}\right] v^{-1-\alpha}\, dv < 0.$$

This and the fact that $I(0) = 0$ give the proof. $\square$

Corollary A.1. *Let $d \geqslant 3$, $0 < \alpha < 2$. There exists $\gamma > 0$ such that for any $B \in \mathbb{R}^{d\times d}$, $b > 0$*

$$\int_{\mathbb{R}^d} f_B(p,\zeta) \frac{1}{|\zeta|^{d+\alpha}}\, d\zeta \leqslant 0 \qquad \forall p \in \mathbb{R}^d,$$

where

$$f_B(p,\zeta) = \frac{1}{(|p + B\zeta|^2 + b)^\gamma} - \frac{1}{(|p|^2+b)^\gamma}.$$

Proof. Without loss of generality we give the proof for $b = 1$. We take $\lambda = \lambda(p, B) > 0$ such that

$$\int_{\mathbb{R}^d} \frac{1}{(|p + B\zeta|^2 + 1)^\gamma} \frac{1}{|\zeta|^{d+\alpha}} d\zeta = \lambda^\alpha \int_{\mathbb{R}^d} \frac{1}{(|p + \zeta|^2 + 1)^\gamma} \frac{1}{|\zeta|^{d+\alpha}} d\zeta. \tag{A.13}$$

Then

$$\begin{aligned} \int_{\mathbb{R}^d} f_B(p, \zeta) \frac{1}{|\zeta|^{d+\alpha}} d\zeta &= \int_{\mathbb{R}^d} \left(\frac{1}{(|p + B\zeta|^2 + 1)^\gamma} - \frac{1}{(|p + \lambda\zeta|^2 + 1)^\gamma} \right) \frac{1}{|\zeta|^{d+\alpha}} d\zeta \\ &\quad + \int_{\mathbb{R}^d} \left(\frac{1}{(|p + \lambda\zeta|^2 + 1)^\gamma} - \frac{1}{(|p|^2 + 1)^\gamma} \right) \frac{1}{|\zeta|^{d+\alpha}} d\zeta \\ &= \mathbf{I} + \mathbf{II}. \end{aligned}$$

From (A.13) we find $\mathbf{I} = 0$. An application of Lemma $A.3$ shows $\mathbf{II} \leqslant 0$, which gives the proof. $\square$

6.2 Appendix B

For the proof of Proposition 3.0.1 we used some technical estimates which are not directly related to the arguments of this section.

Lemma B.1. *For $y_1, \ldots, y_n \geqslant 1$ one has*

$$\prod_{l=1}^{n} (y_1 + \ldots + y_l) \Gamma(y_l) > \frac{1}{2^n} \Gamma(n+1) \left(\Gamma\left(\frac{y_1 + \ldots + y_n}{n} + 1 \right) \right)^n.$$

Proof. Since $y_1 + \ldots + y_l > l - 1$ we easily see

$$\prod_{l=1}^{n} (y_1 + \ldots + y_l) > \frac{n!}{2^n} \prod_{l=1}^{n} \frac{y_1 + \ldots + y_l + l - 1}{l}. \tag{B.1}$$

We are going to prove by induction that

$$\prod_{l=1}^{n} \frac{y_1 + \ldots + y_l + l - 1}{l} \geqslant \prod{}' \prod_{s=1}^{i_l} (y_l + s - 1) \tag{B.2}$$

for some $i_1, \ldots, i_n \geqslant 0$ such that $i_1 + \ldots + i_n = n$. Here the product $\prod'$ ranges over all $1 \leqslant l \leqslant n$ with $i_l > 0$. Indeed, for $n = 1$ we take $i_1 = 1$. Assume that (B.2) is true for $n-1$. Let $B = \min\{y_1 + i_1, \ldots, y_{n-1} + i_{n-1}, y_n\}$. Then

$$\frac{y_1 + \ldots + y_n + n - 1}{n} = \frac{(y_1 + i_1) + \ldots + (y_{n-1} + i_{n-1}) + y_n}{n} \geqslant B.$$

If $B = y_r + i_r$ for some $1 \leqslant r \leqslant n-1$ we take $j_r = i_r + 1$, $j_s = i_s$ for $s \neq r$ and $j_n = 0$. Otherwise we put $j_n = 1$, $j_s = i_s$, $s = 1, \ldots, n-1$. Then the inequality (B.2) holds for $(j_1, \ldots, j_n)$, and all n.

Combining (B.1) and (B.2) we get

$$\prod_{l=1}^{n} (y_1 + \ldots + y_l)\Gamma(y_l) > \frac{n!}{2^n} \prod_{l=1}^{n} \Gamma(y_l + i_l).$$

The Lemma follows since the Gamma-function is a log-convex, cf. [A]. □

Lemma B.2. *For $n = 1, \ldots, k$ one has*

$$n^{n-k} k^k \geqslant \frac{1}{2^k} (\ln(k+1))^k. \tag{B.3}$$

Proof. For $f(x) = (x-k)\ln x + k \ln k$ we have

$$f'(x) = \frac{x-k}{x} + \ln x. \tag{B.4}$$

Let $f'(x_0) = 0$. Clearly, $1 < x_0 < k$. From (B.4) we get

$$x_0(\ln x_0 + 1) = k, \tag{B.5}$$

so

$$\ln k = \ln x_0 + \ln(\ln x_0 + 1) \leqslant 2 \ln x_0. \tag{B.6}$$

Thus

$$n^{n-k} k^k \geqslant x_0^{x_0 - k} k^k \geqslant \left(\frac{k}{x_0}\right)^k. \tag{B.7}$$

Inequalities (B.5) and (B.6) imply

$$\frac{k}{x_0} = (\ln x_0 + 1) \geqslant \frac{1}{2} \ln k + 1 \geqslant \frac{1}{2} \ln(k+1). \tag{B.8}$$

Combining (B.7) and (B.8) we arrive at (B.3). □

Lemma B.3. *We define q_k by the following recursion formula*

$$q_0 = 1, \qquad q_k = \sum_{m=0}^{k-1} \frac{3}{(k-m)\, m!}\, q_{k-m-1}, \qquad k \in \mathbb{N}. \tag{B.9}$$

Then

$$q_k \leqslant \frac{(2^6)^k}{(\ln(k+1))^{\frac{k}{2}}}, \qquad k \in \mathbb{N}. \tag{B.10}$$

Proof. With $a_{k,l} = 3/[(l+1)(k-l-1)!]$ formula (B.9) reads

$$q_k = \sum_{l=0}^{k-1} a_{k,l} q_l, \quad k \geqslant 1.$$

Clearly,

$$q_k = \sum{}' a_{i_0,i_1} \ldots a_{i_{n-1},i_n}, \tag{B.11}$$

where the sum $\sum'$ ranges over all $(i_0, \ldots, i_n)$ with $k = i_0 > \ldots i_n = 0$. Notice that

$$a_{i_0,i_1} \ldots a_{i_{n-1},i_n} = \prod_{s=0}^{n-1} \frac{3}{(i_{s+1}+1)\Gamma(i_s - i_{s+1})}. \tag{B.12}$$

Lemma $B.1$ implies that for $b_1, \ldots, b_r \geqslant 1$

$$\begin{aligned} &\Gamma(b_n) \prod_{l=1}^{n-1} (b_1 + \ldots + b_l)\Gamma(b_l) \\ &\geqslant \frac{1}{2^n} \frac{\Gamma(n+1)}{b_1 + \ldots + b_n} \left(\Gamma\left(\frac{b_1 + \ldots + b_n}{n} + 1\right)\right)^n. \end{aligned}$$

If $b_l = i_{n-l} - i_{n-l+1}$, $l = 2, \ldots, n$, $b_1 = i_{n-1} - i_n + 1$, then

$$b_1 + \ldots + b_n = i_0 - i_n + 1 = k + 1,$$

and it follows that

$$\begin{aligned} B &:= \Gamma(i_0 - i_1) \prod_{s=1}^{n-1} (i_s + 1)\Gamma(i_s - i_{s+1}) \\ &\geqslant \frac{1}{2^n} \frac{\Gamma(n+1)}{k+1} \left(\Gamma\left(\frac{k+1}{n} + 1\right)\right)^n. \end{aligned}$$

Recall that $n \leqslant k-1$. Since the inequalities $\Gamma(b+1) \geqslant b^{\frac{b}{2}}$ for $b \geqslant 2$, $\Gamma(b+1) \geqslant 2^{-1} b^{\frac{b}{2}}$ for $1 \leqslant b \leqslant 2$ and, by (B.3), $n^{n-k}k^k \geqslant 2^{-k}\,(\ln(k+1))^k$ we have

$$B \geqslant \frac{1}{4^n}\frac{n^{\frac{n}{2}}}{k+1}\left(\frac{k}{n}\right)^{\frac{k}{2}} \geqslant \frac{1}{k4^k}\left(n^{n-k}k^k\right)^{\frac{1}{2}} \geqslant \frac{1}{k8^k}\,(\ln(k+1))^{\frac{k}{2}}.$$

Consequently, the right-hand side of (B.12) does not exceed $3^n B^{-1}$ and so

$$q_k \leqslant 2^k \max_{k=i_0>\ldots>i_n=0} a_{i_0,i_1}\ldots a_{i_{n-1},i_n} \leqslant \frac{k(48)^k}{(\ln(k+1))^{\frac{k}{2}}},$$

where we used that the number of terms in (B.11) is equal to

$$\sum_{n=1}^{k}\binom{k-1}{n-1} = 2^{k-1}.$$

□

Lemma B.4. *For any $m, k, \alpha_1, \ldots, \alpha_n \in \mathbb{N}$ with*

$$\alpha_1 + \ldots + \alpha_n + m = k, \qquad m \geqslant n-1 \tag{B.13}$$

we have

$$(\ln\{\alpha_1+1\})^{\frac{\alpha_1}{2}}\ldots(\ln\{\alpha_n+1\})^{\frac{\alpha_n}{2}} m! \geqslant 2^{-k}\,(\ln\{\ln\{k+2\}\})^{\frac{k}{16}}. \tag{B.14}$$

Proof. Denote the left-hand side of (B.14) by $\mathbf{I}$ and observe that

$$\mathbf{I} \geqslant (\sqrt{\ln 2})^n \geqslant 2^{-k}.$$

Since $(\ln\{\ln\{k+2\}\})^{\frac{k}{8}} \leqslant 1$ for $k \leqslant 10$, (B.14) holds for $k \leqslant 10$. If $k > 10$ we get from (B.13)

$$\frac{\alpha_1+\ldots+\alpha_n}{n} \geqslant \frac{k-m}{m+1}$$

and so, using the log-convexity of $f(x) = (\ln\{x+1\})^{\frac{x}{2}}$,

$$\begin{aligned}\mathbf{I} &\geqslant \left(\ln\left\{\frac{\alpha_1+\ldots+\alpha_n}{n}+1\right\}\right)^{\frac{\alpha_1+\ldots+\alpha_n}{2}} m! \\ &\geqslant \left(\ln\left\{\frac{k-m}{m+1}+1\right\}\right)^{\frac{k-m}{2}} m!\end{aligned}$$

Applying lemma $B.5$, completes the proof. □

Lemma B.5. *For* $1 \leqslant m < k$, $k \geqslant 10$, $m, k \in \mathbb{N}$ *one has*

$$\left(\ln\left(\frac{k+1}{m+1}\right)\right)^{\frac{k-m}{2}} \Gamma(m+1) \geqslant 2^{-k}\left(\ln\left(\ln(k+2)\right)\right)^{\frac{k}{16}}. \tag{B.15}$$

Proof. We write $\mathbf{I}$ for the left-hand side of (B.15) and set $x_0 = (\ln k)^{-1}k$.

Case 1. $x_0 < m < k$. We split the proof into three steps. Take $x_0 < x < k$ $x \in \mathbb{R}$.

Step1. Since $(k+1)/(x+1) < k/x < \ln k$ we get

$$\ln\left(\frac{k+1}{x+1}\right) < \ln\left(\frac{k}{x}\right) < \ln\ln k < \frac{k}{\ln k} < x,$$

and so

$$\frac{1}{8}\ln x > \frac{1}{8}\ln\left(\ln\left(\frac{k+1}{x+1}\right)\right). \tag{B.16}$$

Step 2. Using the elementary inequality

$$\ln(1+a) \geqslant ab \qquad \text{for} \quad 0 < b < 1 \quad \text{and} \quad 0 < a < \frac{1}{b} - 1$$

with $a = (k-x)/(x+1)$, $b = 1/(2\ln x)$, and

$$0 < a = \frac{k-x}{x+1} \leqslant \frac{k-x_0}{x_0+1} < \ln k - 1 < 2\ln x_0 - 1 < 2\ln x - 1 = \frac{1}{b} - 1.$$

we find that

$$2\ln x \ln\left(\frac{k+1}{x+1}\right) = 2\ln x \ln\left(1 + \frac{k-x}{x+1}\right) \geqslant \frac{k-x}{x+1},$$

and therefore

$$\frac{1}{4}\ln x \geqslant \frac{k-x}{8}\frac{1}{\ln(\frac{k+1}{x+1})}\frac{1}{x+1}. \tag{B.17}$$

Step 3. Set

$$f(x) = \frac{k-x}{8}\ln\left(\ln\left(\frac{k+1}{x+1}\right)\right) + \frac{1}{2}x\ln x.$$

Clearly,

$$f'(x) = -\frac{1}{8}\ln\left(\ln\left(\frac{k+1}{x+1}\right)\right) - \frac{k-x}{8}\frac{1}{\ln(\frac{k+1}{x+1})}\frac{1}{x+1} + \frac{1}{2}(\ln x + 1).$$

Adding (B.16) and (B.17) we obtain $f'(x) \geqslant 0$ and so

$$\mathbf{I} \geqslant \exp\{f(x)\} \geqslant \exp\{f(x_0)\}. \tag{B.18}$$

Since $(k+1)(x_0+1)^{-1} \geqslant (3/4)\ln k$, we find

$$\ln\left(\frac{k+1}{x_0+1}\right) \geqslant \ln\left(\frac{3\ln k}{4}\right) \geqslant \frac{1}{2}\left(\ln\ln(k+2)\right)^{\frac{1}{2}}, \qquad k \geqslant 10, \tag{B.19}$$

and so

$$\begin{aligned} f(x_0) &\geqslant \frac{1}{8}(k-x_0)\left(\frac{1}{2}\ln\ln\ln(k+2)-1\right)+\frac{1}{2}x_0\ln x_0 \\ &= \frac{k}{16}\ln\ln\ln(k+2)+Z, \end{aligned} \tag{B.20}$$

where

$$\frac{2}{k}Z = -\frac{1}{8}\frac{\ln\ln\ln(k+2)}{\ln k}+\frac{1}{4\ln k}+\frac{3}{4}-\frac{\ln\ln k}{\ln k} \geqslant \frac{3}{4}-\frac{9}{8}\frac{\ln\ln k}{\ln k} \geqslant 0 \qquad k \geqslant 10.$$

Combining (B.18) and (B.20) gives (B.15).

Case 2. Let $1 < x \leqslant (k/\ln k)$. Using (B.19) we have

$$\mathbf{I} \geqslant \left(\ln\left(\frac{k+1}{x+1}\right)\right)^{\frac{k-x}{2}} \geqslant \left(\frac{\ln\ln(k+2)}{2}\right)^{\frac{k}{4}\left(1-\frac{1}{\ln k}\right)} \geqslant 2^{-k}\left(\ln\ln(k+2)\right)^{\frac{k}{8}}.$$

□

6.3 Appendix C

6.3.1 Estimates for Newton systems

Recall that p_0 is defined by formula (4.46) and denote by $\|M\|_\infty = \max_{1\leqslant i,j\leqslant d} |(M_{ij})|$, $M \in \mathbb{R}^{d\times d}$.

Lemma C.1. *Under the assumptions of Lemma* 4.1.2

$$\left\|(t-t_0)\frac{\partial p_0(t,t_0,x,x_0)}{\partial x}-E_d\right\|_\infty = O(t-t_0), \tag{C.1}$$

$$\left|\frac{\partial^{|L|}p_0(t,t_0,x,x_0)}{\partial x^L}\right| = O(1) \quad |L| = 2,\dots,q \tag{C.2}$$

hold for $0 \leqslant t_0 < t < T$.

Proof. Since X and p_0 are inverse functions, it implies

$$\begin{aligned}\left\|(t-t_0)\frac{\partial p_0}{\partial x}-E_d\right\|_\infty &= \left\|\left((t-t_0)E_d-\frac{\partial X}{\partial p_0}\right)\left(\frac{\partial X}{\partial p_0}\right)^{-1}\right\|_\infty\\ &\leqslant \|M\|_\infty\sum_{n=0}^{\infty}(t-t_0)^{-(n+1)}d^n\|M\|_\infty^n,\end{aligned}$$

where $(\partial X/\partial p_0)=(t-t_0)E_d+M$. By (4.6) $\|M\|_\infty=O((t-t_0)^2)$ and the first formula in Lemma $C.1$ follows.

Now we choose $k,j_1,\ldots,j_m\in\{1,\ldots,d\}$, $m\in\mathbb{N}$. Differentiating the identity

$$p_{0,k}(t,t_0,X(t,t_0,x_0,p_0),x_0)=p_{0,k}$$

over $p_{0,j_1},\ldots,p_{0,j_m}$ we get

$$\begin{aligned}\sum_{i+\ldots+j=L}\frac{\partial^{|L|}p_{0,k}}{\partial x^L}\frac{\partial X_i}{\partial p_{0,j_1}}\cdots\frac{\partial X_j}{\partial p_{0,j_m}}+\sum_{r=1}^{d}\frac{\partial p_{0,k}}{\partial x_r}\frac{\partial^m X_r}{\partial p_{0,j_m}\ldots\partial p_{0,j_1}}&\\ +\sum{}'O(1)\frac{\partial^{|R|}p_{0,k}}{\partial x^R}\frac{\partial^{|A|}X_i}{\partial p_0^A}\cdots\frac{\partial^{|B|}X_j}{\partial p_0^B}&=0, \qquad (C.3)\end{aligned}$$

where the sum $\sum'$ is taken over all is taken over all $R,i,\ldots,j,A,\ldots,B$, such that

$$i+\ldots+j=R,\quad 1<|R|<m,\qquad A+\ldots+B=j_1+\ldots+j_m. \qquad (C.4)$$

Note that here $i+\ldots+j$ and $j_1+\ldots+j_m$ are understood as sums of multi-indices.

We are going to prove (C.2) by induction in $m=|L|$. If $m=2$, then from (C.3) we get

$$\sum_{i,j=1}^{d}\frac{\partial^2 p_{0,k}}{\partial x_j\partial x_i}\frac{\partial X_i}{\partial p_{0,j_1}}\frac{\partial X_j}{\partial p_{0,j_2}}+\sum_{r=1}^{d}\frac{\partial p_{0,k}}{\partial x_r}\frac{\partial^2 X_r}{\partial p_{0,j_2}\partial p_{0,j_1}}=0.$$

Using formulae (4.5), (4.31) and, by (C.1), the fact that $\partial p_0/\partial x=O((t-t_0)^{-1})$ we arrive at

$$\sum_{i,j=1}^{d}\frac{\partial^2 p_{0,k}}{\partial x_j\partial x_i}\left((t-t_0)^2\delta_{i,j_1}\delta_{j,j_2}+O((t-t_0)^3)\right)=O((t-t_0)^2). \qquad (C.5)$$

Divide (C.5) by $(t-t_0)^2$ to get $MY = O(1)$, where

$$M = E_{d^2} + O(t-t_0),\ M \in \mathbb{R}^{d^2\times d^2} \qquad \text{and} \quad (Y)_{(i,j)} = \frac{\partial^2 p_{0,k}}{\partial x_j \partial x_i},\ Y \in \mathbb{R}^{d^2}.$$

The fact that $M^{-1} = E_{d^2} + O(t-t_0)$ implies $Y = O(1)$ and formula (C.2) for $|L| = 2$ follows.

Let us assume that formula (C.2) holds for all L such that $|L| < m$. We denote the left-hand side of (C.3) by **I** + **II** + **III** . Formulae (4.31), (C.1) imply **II** $= O((t-t_0)^m)$.

We proceed with **III** . From (4.31), (C.4) we find

$$\frac{\partial^{|A|} X_i}{\partial p_0^A} \cdots \frac{\partial^{|B|} X_j}{\partial p_0^B} = O((t-t_0)^{|A|+\ldots+|B|}) = O((t-t_0)^m).$$

Using the induction assumption $(\partial^{|R|} p_{0,k}/\partial x^R) = O(1)$ we arrive at **III** $= O((t-t_0)^m)$.

Substituting the estimates above to (C.3) we have the system of d^m linear equations with respect to $(\partial^{|L|} p_{0,k}/\partial x^L)$

$$\sum_{i+\ldots+j=L} b_{i,\ldots,j}^{j_1,\ldots,j_m} \frac{\partial^{|L|} p_{0,k}}{\partial x^L} = O(1), \tag{C.6}$$

where

$$b_{i,\ldots,j}^{j_1,\ldots,j_m} = \frac{1}{(t-t_0)^m} \frac{\partial X_i}{\partial p_{0,j_1}} \cdots \frac{\partial X_j}{\partial p_{0,j_m}} = \delta_{i,j_1} \ldots \delta_{j,j_m} + O(t-t_0).$$

We rewrite (C.6) in the form $MY = O(1)$, where

$$M = E_{d^m} + O(t-t_0), \quad M \in \mathbb{R}^{d^m \times d^m}$$

and

$$(Y)_{(i,\ldots,j)} = \frac{\partial^2 p_{0,k}}{\partial x_j \ldots \partial x_i}, \quad Y \in \mathbb{R}^{d^m},$$

which implies $Y = O(1)$ and gives (C.2) for $|L| = m$. □

Corollary C.1. *For* $0 \leqslant t_0 < t < T$ *one has*

$$\frac{\partial^{|M|} S}{\partial x^M}(t, t_0, x, x_0) = O(1), \quad |M| \geqslant 3. \tag{C.7}$$

Proof. Using the chain rule we find from (4.47) that

$$\frac{\partial^{|A|}p}{\partial x^A} = \sum_{r=1}^{d} \frac{\partial P}{\partial p_{0,r}} \frac{\partial^{|A|}p_{0,r}}{\partial x^A} + \mathbf{I}$$

for $|A| \geqslant 2$, where $\mathbf{I}$ is a sum of the terms of the type

$$Q_{i,\ldots,j;B,\ldots,F} = \frac{\partial^{|L|}P}{\partial p_0^L} \frac{\partial^{|B|}p_{0,i}}{\partial x^B} \times \ldots \times \frac{\partial^{|F|}p_{0,j}}{\partial x^F}$$

such that $i + \ldots + j = L$, $|L| \geqslant 2$, $B + \ldots + F = A$. By Lemma $C.1$, $\partial^{|B|}p_0/\partial x^B = O((t-t_0)^{(|B|-2)\wedge 0})$, for any $B \in \mathbb{N}_0^d$. Since, by (4.32), $\partial^{|L|}P/\partial p_0^L = O((t-t_0)^{|L|})$ for $|L| \geqslant 2$, it follows that $Q_{i,\ldots,j;B,\ldots,F} = O((t-t_0)^\gamma)$, where

$$\begin{aligned} \gamma &= |L| + [(|B|-2) \wedge 0] + \ldots + [(|F|-2) \wedge 0] \\ &= [(|B|-1) \wedge 1] + \ldots + [(|F|-1) \wedge 1] \geqslant 0 \end{aligned}$$

and so $\mathbf{I} = O(1)$. Using formulae (4.5), (C.2) we obtain $(\partial P/\partial p_{0,r})(\partial^{|A|}p_{0,r}/\partial x^A) = O(1)$, $r = 1, \ldots, d$. Hence

$$\frac{\partial^{|A|}p}{\partial x^A}(t, t_0, x, x_0) = O(1), \quad |A| \geqslant 2. \tag{C.8}$$

Applying Theorem 1.1.1 we complete the proof. □

Corollary C.2. *For $0 \leqslant t_0 < t < T$ and $|A| = 1, \ldots, q$ we have*

$$\frac{\partial^{|A|}x(\tau)}{\partial x^A} = O(t - t_0)$$

with $x(\tau)$ given by (4.48).

Proof. Applying the chain rule to (4.48) we represent $\partial^{|A|}x(\tau)/\partial x^A$ as a sum of the terms of the type

$$Q_{i,\ldots,j;B,\ldots,F} = \frac{\partial^{|L|}X}{\partial p_0^L} \frac{\partial^{|B|}p_{0,i}}{\partial x^B} \times \ldots \times \frac{\partial^{|F|}p_{0,j}}{\partial x^F}.$$

such that $i + \ldots + j = L$, $B + \ldots + F = A$. Applying formulae (4.6), (4.31) and Lemma $C.1$ give $Q_{i,\ldots,j;B,\ldots,F} = O((t-t_0)^\gamma)$, where

$$\gamma = |L| + 1 + [(|B|-2) \wedge 0] + \ldots + [(|F|-2) \wedge 0] \geqslant 2$$

for $2 \leqslant |L| \leqslant |A| - 1$ and $\gamma = 1$ for $|L| = 1$, $|L| = |A|$, which gives the proof. □

6.3.2 Estimates for the transport and heat equations

Lemma C.2. *Let $f = f(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be $q+1$ times differentiable in x function with bounded derivatives up to order q. Then one can find a constant $K_f > 0$ such that for $0 < t < \mathcal{R}_\varepsilon \wedge K_f$, $1 \leqslant |L| \leqslant q$*

$$\frac{\partial^{|L|}\rho_f(t,x)}{\partial x^L} = O(t^{1-\varepsilon})\rho_f(t,x), \tag{C.9}$$

where

$$\rho_f(t,x) = \prod_{t_0<\tau\leqslant t} \widetilde{\rho}(\tau, f(\tau,x)) \tag{C.10}$$

and $\widetilde{\rho}(\tau,x)$ is given by (5.32). In particular,

$$\frac{\partial^{|L|}\rho(t,x)}{\partial x^L} = O(t^{1-\varepsilon})\rho(t,x), \tag{C.11}$$

$$\frac{\partial^{|L|}\lambda(t,x,x_0)}{\partial x^L} = O(t^{1-\varepsilon})\lambda(t,x,x_0). \tag{C.12}$$

hold for $0 \leqslant t_0 < t < \mathcal{R}_\varepsilon \wedge K_1$, $1 \leqslant |L| \leqslant q$ for some constant $K_1 > 0$.

Proof. From (C.10) we find

$$\begin{aligned}
&\frac{\partial^{|L|}\rho_f(t,x)}{\partial x^L} \qquad\qquad (C.13)\\
&= O(1)\sum_{k=1}^{|L|}{\sum}' \frac{\partial^{|i_1|}\widetilde{\rho}(\tau_1, f(\tau_1,x))}{\partial x^{i_1}} \cdots \frac{\partial^{|i_k|}\widetilde{\rho}(\tau_k, f(\tau_k,x))}{\partial x^{i_k}} \times \\
&\quad \times \prod_{\tau\neq\tau_1,\ldots,\tau_k} \widetilde{\rho}(\tau, f(\tau,x)).
\end{aligned}$$

Here the sum $\sum'$ is taken over all $\tau_1,\ldots,\tau_k \in (t_0,t)$ and $i_1,\ldots,i_k$ such that $i_1+\ldots+i_k = L$.

Clearly

$$\frac{\partial^{|A|}\widetilde{\rho}(\tau, f(\tau,x))}{\partial x^A} = Q_A(\tau,x)\exp\{-c(f(\tau,x))\Delta\xi_\tau\}, \qquad |A| = 1,\ldots,q,$$

where Q_A is polynomial with respect to $(\partial^{|B|}c(f(\tau,x))/\partial x^B)\Delta\xi_\tau$, $|B| = 0,\ldots,|A|$. One can check that $Q_A = |\Delta\xi_\tau|^2 O(1)$ if $|A| = 1$. Using conditions

(4.3), (5.2) and the boundedness of $(\partial^{|B|} f(\tau,x)/\partial x^B)$, $|B| = 0,\dots,q$, it follows that

$$\left|\frac{\partial^{|A|}\widetilde{\rho}(\tau, f(\tau,x))}{\partial x^A}\right| \leqslant C|\Delta\xi_\tau|^2 \exp\{-c(f(\tau,x))\Delta\xi_\tau\}, \tag{C.14}$$

for some constant $C = C(q,d,K,f) > 0$. Together (C.13), (C.14) gives

$$\begin{aligned}
\left|\frac{\partial^{|L|}\rho_f(t,x)}{\partial x^L}\right| &= O(1)\sum_{k=1}^{|L|}{\sum}' C^k|\Delta\xi_{\tau_1}|^2\dots|\Delta\xi_{\tau_k}|^2 \times \\
&\quad\times \prod_{n=1}^{k}\exp\{-c(f(\tau_n,x))\Delta\xi_{\tau_n}\} \prod_{\tau\neq\tau_1,\dots,\tau_k}\widetilde{\rho}(\tau, f(\tau,x)) \\
&= O(1)\rho_f(t,x)\sum_{k=1}^{|L|}{\sum}' C^k|\Delta\xi_{\tau_1}|^2\dots|\Delta\xi_{\tau_k}|^2,
\end{aligned}$$

where we used that, by (5.12), $1 + c(f(\tau,x))\Delta\xi_\tau \geqslant 1$ and so

$$\prod_{n=1}^{k}\exp\{-c(f(\tau_n,x))\Delta\xi_{\tau_n}\} \prod_{\tau\neq\tau_1,\dots,\tau_k}\widetilde{\rho}(\tau, f(\tau,x)) \leqslant \rho_f(t,x).$$

Consequently, for $t < \mathcal{R}_\varepsilon$

$$\begin{aligned}
\frac{\partial^{|L|}\rho(t,x)}{\partial x^L} &= O(1)\rho_f(t,x)\sum_{k=1}^{|L|}\left(C\sum_{t_0<\tau\leqslant t}|\Delta\xi_\tau|^2\right)^k \\
&= O(1)\rho(t,x)\sum_{k=1}^{|L|}\left(Ct^{1-2\varepsilon}\right)^k.
\end{aligned}$$

Using $\sum_{k=1}^{|L|}(Ct^{1-2\varepsilon})^k = O(t^{1-2\varepsilon})$ for $t < K_1 = (2C)^{-(1-2\varepsilon)^{-1}}$, proves (C.9).

Formula (C.11) is a particular case of (C.9) with $f(t,x) = x$. An application of Corollary C.2, Lemma 5.2.5 and formula (C.9) with $f(\tau,x) = x(\tau)$ give (C.12). □

We set

$$T_{1,\varepsilon} = T \wedge \mathcal{R}_\varepsilon \wedge K_1. \tag{C.15}$$

Lemma C.3. *For* $0 \leqslant t_0 < t < T$, $|L| = 1, \dots, q$ *we get*

$$\frac{\partial^{|L|} J^{-\frac{1}{2}}(t, t_0, x, x_0)}{\partial x^L} = O(t - t_0) J^{-\frac{1}{2}}(t, t_0, x, x_0), \tag{C.16}$$

with $J = J(t, t_0, x, x_0)$ *being given by* (5.16).

Proof. An application of the chain rule yields that $\partial^{|L|} J^{-\frac{1}{2}} / \partial x^L$ is a sum of the terms of the type

$$R_{i,\dots,j;B,\dots,F} = \frac{\partial^{|A|} I^{-\frac{1}{2}}}{\partial p_0^A} \frac{\partial^{|B|} p_{0,i}}{\partial x^B} \times \dots \times \frac{\partial^{|F|} p_{0,j}}{\partial x^F}, \tag{C.17}$$

where $i + \dots + j = A$, $B + \dots + F = L$. One readily sees

$$\frac{\partial^{|A|} I^{-\frac{1}{2}}}{\partial p_0^A} = \frac{\partial^{|A|}}{\partial p_0^A} \exp\left\{-\frac{1}{2} \mathrm{tr}\ \ln \left\{\frac{\partial X}{\partial p_0}\right\}\right\} = \frac{\partial^{|A|}}{\partial p_0^A} \exp\{Q\} = Q_A I^{-\frac{1}{2}}, \tag{C.18}$$

where Q_A a sum of the terms of the type

$$\frac{\partial^{|E|} Q}{\partial p_0^E} \cdots \frac{\partial^{|G|} Q}{\partial p_0^G}, \qquad E + \dots + G = A.$$

The elementary formulae $(\mathrm{tr}\ \ln M)' = \mathrm{tr}\ M^{-1} M'$, $(M^{-1})' = -M^{-1} M' M^{-1}$, where M is a positive definite matrix, imply

$$\frac{\partial^{|E|}}{\partial p_0^E} \mathrm{tr}\ \ln M = O(1) \sum \left| \mathrm{tr} \left(M^{-1} \frac{\partial^{|B|} M}{\partial p_0^B} \cdots M^{-1} \frac{\partial^{|F|} M}{\partial p_0^F} \right) \right|, \tag{C.19}$$

where the sum is taken over all $B, \dots, F$ such that $B + \dots + F = E$. We put $M = (\partial X / \partial p_0) \in \mathbb{R}^{d \times d}$. Using (4.6), (4.31) give

$$M^{-1} \frac{\partial^{|B|} M}{\partial p_0^B} = O((t - t_0)^{|B|+1}).$$

Hence

$$\frac{\partial^{|E|} Q}{\partial p_0^E} = -\frac{1}{2} \frac{\partial^{|E|}}{\partial p_0^E} \mathrm{tr}\ \ln M = O((t - t_0)^{|E|+1})$$

and $Q_A = O((t - t_0)^{|A|+1})$. Consequently

$$\frac{\partial^{|A|} I^{-\frac{1}{2}}}{\partial p_0^A} = O((t - t_0)^{|A|+1}) I^{-\frac{1}{2}}. \tag{C.20}$$

Using (C.20) and Lemma $C.1$, give the proof. □

Lemma C.4.

(i) For $0 \leqslant t_0 < t < T_{1,\varepsilon}$ *we have*

$$\frac{\partial^{|L|}\phi(t,t_0,x,x_0)}{\partial x^L} = O(t^{1-\varepsilon})\phi(t,t_0,x,x_0), \quad |L| = 1,\ldots,q.$$

(ii) There exist constants $K_2, K_3 > 0$ *such that*

$$\frac{K_2}{[h(t-t_0)]^{\frac{d}{2}}} \leqslant \phi(t,t_0,x,x_0) \leqslant \frac{K_3}{[h(t-t_0)]^{\frac{d}{2}}} \tag{C.21}$$

holds for $0 \leqslant t_0 < t < T_{1,\varepsilon}$, $\forall x, x_0 \in \mathbb{R}^d$.

Proof. An application of Lemmas $C.2$, $C.3$ to definition (5.21) show (i).
Applying (5.22) and (5.23) to (C.10) we find

$$C_f^{-1} \leqslant \rho_f(t,x) \leqslant C_f$$

for some constant $C_f > 0$. In particular,

$$C_1^{-1} \leqslant \rho(t,x) \leqslant C_1, \qquad C_2^{-1} \leqslant \lambda(t,x,x_0) \leqslant C_2 \tag{C.22}$$

for some constants $C_1, C_2 > 0$. Combining (5.17) and (C.22) imply (ii). □

Lemma C.5. *The derivatives of the asymptotic Green function* ψ_G^{as} *given by* (5.34) *satisfy*

$$\frac{\partial^{|L|}\psi_G^{as}(t,t_0,x,x_0)}{\partial x^L} = (h(t-t_0))^{-|L|}(1+|x-x_0|)^{|L|}\psi_G^{as}(t,t_0,x,x_0)O(1), \tag{C.23}$$

for $0 \leqslant t_0 < t < T$, $|L| = 1,\ldots,q$.

Proof. One readily sees that

$$\frac{\partial^{|B|}}{\partial x^B}\exp\left\{-\frac{1}{h}S\right\} = Q_B \exp\left\{-\frac{1}{h}S\right\}, \tag{C.24}$$

where $S = S(t,t_0,x,x_0)$ and

$$Q_B = O(1) \max_{E+\ldots+F=B}\left|\frac{1}{h}\frac{\partial^{|E|}S(x)}{\partial x^E}\cdots\frac{1}{h}\frac{\partial^{|F|}S(x)}{\partial x^F}\right|.$$

Using formulae (4.52), (5.4) we obtain

$$\left|\frac{\partial S(t,t_0,x,x_0)}{\partial x}\right| = |p(t,t_0,x,x_0)| \leqslant |p_0(t,t_0,x,x_0)| + O(1).$$

Expanding $p_0(t,t_0,x,x_0)$ into Taylor series in x and using, by (C.2), $\partial p_0/\partial x = O((t-t_0)^{-1})$, give

$$\begin{aligned} p_0(t,t_0,x,x_0) &= p_0(t,t_0,x_0,x_0) + O((t-t_0)^{-1})(x-x_0) \\ &= O(1) + O((t-t_0)^{-1})(x-x_0), \end{aligned}$$

where we used (5.5). Thus

$$\left|\frac{\partial S(t,t_0,x,x_0)}{\partial x}\right| = \frac{1}{t-t_0}(1+|x-x_0|)O(1).$$

Combining this estimate, (C.7) and Corollary 4.2.1 we get

$$Q_B = (h(t-t_0))^{-|B|}(1+|x-x_0|^{|B|})O(1). \tag{C.25}$$

Applying the chain rule to (5.34) yields

$$\frac{\partial^{|L|}\psi_G^{as}(t,t_0,x,x_0)}{\partial x^L} = \sum_{A+B=L} O(1)\frac{\partial^{|A|}\phi}{\partial x^A}\frac{\partial^{|B|}}{\partial x^B}\exp\left\{-\frac{1}{h}S\right\}.$$

Using formulae (C.24), (C.25) and (i) Lemma $C.4$ we arrive at (C.23). □

6.3.3 Properties of the integral operator $\mathcal{F}$

Let us recall that $S(t,t_0,x,x_0)$ is a two point function for equation (5.13) and $\phi(t,t_0,x,x_0)$ is given by formula (5.26). For a function $\chi(t,t_0,x,x_0)$ we define

$$\begin{aligned} [\mathcal{F}\chi] &= [\mathcal{F}\chi](t,t_0,x,x_0) \\ &= \frac{1}{2}\int_{t_0}^{t}\int_{\mathbb{R}^d}\chi(t,\tau,x,\eta)\,\mathrm{tr}\frac{\partial^2\phi(\tau,t_0,\eta,x_0)}{\partial\eta^2}\exp\left\{-\frac{S(\tau,t_0,\eta,x_0)}{h}\right\}d\tau d\eta. \end{aligned} \tag{C.26}$$

Proposition 6.3.1. *Let*

$$\chi(\tau_1,\tau_2,x,x_0)=\alpha(\tau_1,\tau_2)(1+|x-x_0|)^m\psi_G^{as}(\tau_1,\tau_2,x,x_0)O(1) \tag{C.27}$$

for some function $\alpha:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, $m\in\mathbb{N}_0$. *Then*

$$[\mathcal{F}\chi]=O(t^{1-\varepsilon})\left(\int_{t_0}^{t}\alpha(t,\tau)\,d\tau\right)(1+|x-x_0|)^m\psi_G^{as}(t,t_0,x,x_0) \tag{C.28}$$

holds for $0\leqslant t_0<t<T_{1,\varepsilon}$ *with* $T_{1,\varepsilon}$ *being given by* (C.15).

Proof. Using (C.27) rewrite (C.26) in the form

$$[\mathcal{F}\chi](t,t_0,x,x_0)=\frac{1}{2}\int_{t_0}^{t}\int_{\mathbb{R}^d}B(\eta,\tau)\exp\left\{-\frac{\Phi(\eta)}{h}\right\}d\tau d\eta \tag{C.29}$$

with $\Phi=\Phi(\eta)=S(t,\tau,x,\eta)+S(\tau,t_0,\eta,x_0)$ and

$$B(\eta,\tau)=\alpha(t,\tau)(1+|x-\eta|)^m\phi(t,\tau,x,\eta)\,\mathrm{tr}\frac{\partial^2\phi(\tau,t_0,\eta,x_0)}{\partial\eta^2}O(1).$$

Step 1. Clearly $\Phi(\hat\eta)=\min\limits_{\eta\in\mathbb{R}^d}\Phi(\eta)=S(t,t_0,x,x_0)$, where $\hat\eta=x(\tau)$ is given by (4.48), and

$$\Phi(\eta)\geqslant\Phi(\hat\eta)+\frac{1}{2}(\Lambda(\eta-\hat\eta),\eta-\hat\eta)$$

provided that $(\partial^2\Phi/\partial\eta^2)\geqslant\Lambda$ for some $\Lambda\in\mathbb{R}^{d\times d}$. One has

$$\frac{\partial^2\Phi}{\partial\eta^2}=\frac{\partial^2S(t,\tau,x,\eta)}{\partial\eta^2}+\frac{\partial^2S(\tau,t_0,\eta,x_0)}{\partial\eta^2}.$$

By Corollary 4.2.1 we see

$$\frac{\partial^2\Phi}{\partial\eta^2}\geqslant\frac{1}{2}\left(\frac{1}{t-\tau}+\frac{1}{\tau-t_0}\right)E_d=\Lambda$$

and so

$$\Phi\geqslant S(t,t_0,x,x_0)+\frac{1}{2}\left(\frac{1}{t-\tau}+\frac{1}{\tau-t_0}\right)(\eta-\hat\eta)^2. \tag{C.30}$$

Step 2. By (i) Lemma $C.4$

$$\operatorname{tr} \frac{\partial^2 \phi(\tau, t_0, \eta, x_0)}{\partial \eta^2} = O(\tau^{1-\varepsilon}) \phi(\tau, t_0, \eta, x_0)$$

and so

$$B(\eta, \tau) = O(t^{1-\varepsilon})\, \alpha(t, \tau)(1 + |x - \eta|)^m \phi(t, \tau, x, \eta)\phi(\tau, t_0, \eta, x_0)$$

Using (ii) Lemma $C.4$ we have

$$B(\eta, \tau) \quad = \quad O(t^{1-\varepsilon})\, \frac{\alpha(t, \tau)}{(2\pi h)^d} \frac{(1 + |x - \eta|)^m}{[(t - \tau)(\tau - t_0)]^{\frac{d}{2}}}. \tag{C.31}$$

An application of formula (5.6) shows that

$$|x - \hat{\eta}| \leqslant |x - x_f| + |x_f - \hat{\eta}| = |x - x_f| + O(1),$$

where $x_f = x_f(\tau)$ is defined by (5.7), and so, using $|x - x_f| \leqslant |x - x_0|$, we get

$$|\eta - x| \leqslant |x - \hat{\eta}| + |\eta - \hat{\eta}| \leqslant |x - x_0| + |\eta - \hat{\eta}| + C_1 \tag{C.32}$$

for some $C_1 = C_1(K, d)$. Substituting (C.32) to (C.31) we arrive at

$$B(\eta, \tau) = O(t^{1-\varepsilon})\, \frac{\alpha(t, \tau)}{(2\pi h)^d} (1 + |x - x_0| + C_1)^m \frac{(1 + |\eta - \hat{\eta}|)^m}{[(t - \tau)(\tau - t_0)]^{\frac{d}{2}}}. \tag{C.33}$$

Step 3. Substituting (C.30) and (C.33) to (C.29) and making the change of the variables $\eta := \eta - \hat{\eta}$ we obtain

$$\mathcal{F}\chi = \frac{O(t^{1-\varepsilon})}{h^d} (1 + |x - x_0|)^m \exp\left\{-\frac{1}{h} S(t, t_0, x, x_0)\right\} Z(t, t_0), \tag{C.34}$$

where

$$\begin{aligned} Z(t, t_0) \quad &= \quad \int_{t_0}^{t} \frac{\alpha(t, \tau)}{[(t - \tau)(\tau - t_0)]^{\frac{d}{2}}}\, z(\tau)\, d\tau, \\ z(\tau) \quad &= \quad \int_{\mathbb{R}^d} (1 + |\eta|)^m \exp\left\{-\frac{1}{2h}\left(\frac{1}{t - \tau} + \frac{1}{\tau - t_0}\right)|\eta|^2\right\} d\eta. \end{aligned}$$

An application of the elementary formula

$$\int\limits_{\mathbb{R}^d} (1+|y|)^m \exp\{-by^2\}\, dy = b^{-\frac{d}{2}} \left(1 + b^{-\frac{m}{2}}\right) O(1), \quad b > 0,$$

with $b = (1/2h)\left((t-\tau)^{-1} + (\tau - t_0)^{-1}\right)$ shows

$$z(\tau) = h^{\frac{d}{2}} \left[\frac{(t-\tau)(\tau - t_0)}{t - t_0}\right]^{\frac{d}{2}} O(1)$$

and so

$$Z(t,t_0) \quad = \quad \frac{h^{\frac{d}{2}}}{(t-t_0)^{\frac{d}{2}}} \int\limits_{t_0}^{t} \alpha(t,\tau)\, d\tau. \tag{C.35}$$

Together (C.34) and (C.35) give

$$\mathcal{F}\chi = \frac{O(t^{1-\varepsilon})}{[h(t-t_0)]^{\frac{d}{2}}} \left(\int\limits_{t_0}^{t} \alpha(t,\tau)\, d\tau\right) \times$$
$$\times (1 + |x - x_0|)^m \exp\left\{-\frac{1}{h} S(t,t_0,x,x_0)\right\}.$$

By (ii) Lemma $C.4$ we get

$$\frac{1}{[h(t-t_0)]^{\frac{d}{2}}} = O(1)\phi(t,t_0,x,x_0),$$

which completes the proof. □

6.4 Appendix D

The following two lemmas give the well-known Laplace method (see e.g. [Fe]) in a convenient way.

Lemma D.1. *Let $D \subset \mathbb{R}^d$ be an open set, $f \in C_b^1(D,\mathbb{R})$, $\Phi \in C^3(D,\mathbb{R})$, $h > 0$. We assume that Φ has unique global minimum at $\hat{x}_0 \in D$, $\Phi(\hat{x}_0) \geqslant 0$, $\Phi^{(2)} \geqslant M > 0$ for some $M \in \mathbb{R}^{d\times d}$. Then*

$$\int\limits_D f(x) \exp\left\{-\frac{1}{h}\Phi(x)\right\} dx$$
$$= (\sqrt{2\pi h})^d f(\hat{x}_0) (\det \Phi^{(2)}(\hat{x}_0))^{-\frac{1}{2}} \exp\left\{-\frac{1}{h}\Phi(\hat{x}_0)\right\} (1 + O(h)).$$

For any $\Lambda \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$ we define $\Lambda_b \in \mathbb{R}^{(d-1)\times(d-1)}$ by the formula

$$(\Lambda_b y, y) = \left[(\Lambda y, y)(\Lambda b, b) - (\Lambda b, y)^2\right](\Lambda b, b)^{-1}, \quad \forall y \perp b. \tag{D.1}$$

Lemma D.2. *As before $D \subset \mathbb{R}^d$ is an open set, $f \in C_b^1(D, \mathbb{R})$, $\Phi \in C^3(D, \mathbb{R})$, $h > 0$. We assume that Φ has unique global minimum at $\hat{x}_0 \in \partial D$, $\Phi(\hat{x}_0) \geqslant 0$, $\Phi^{(2)} \geqslant M > 0$ for some $M \in \mathbb{R}^{d\times d}$ and*

$$(x - \hat{x}_0, e) = G(x - \hat{x}_0 - (x - \hat{x}_0, e)e) \qquad \forall x \in \partial D \cap \mathcal{O}(\hat{x}_0), \tag{D.2}$$

where $\mathcal{O}(\hat{x}_0)$ is some neighbourhood of $x = \hat{x}_0$, $G : T_{\hat{x}_0}\partial D \to \mathbb{R}_+$, $G \in C^3(\mathbb{R}^d)$, $G^{(2)} > 0$. Then

$$\int_D f(x) \exp\left\{-\frac{1}{h}\Phi(x)\right\} dx \tag{D.3}$$

$$= \frac{(\sqrt{2\pi h})^{d-1} h}{|b|} f(\hat{x}_0) \left(\det\left(|b|\, G^{(2)}(0) + \Lambda_b\right)\right)^{-\frac{1}{2}} \times$$
$$\times \exp\left\{-\frac{1}{h}\Phi(\hat{x}_0)\right\}(1 + O(h)),$$

where $b = \Phi^{(1)}(\hat{x}_0)$, Λ_b is given by (D.1) *with $\Lambda = \Phi^{(2)}(\hat{x}_0)$.*

For completeness we give a proof of Lemma *D*.2.

Proof. Let us take $a > 0$ such that

$$D_a = \{x \in \mathbb{R}^d : G(x - \hat{x}_0 - (x - \hat{x}_0, e)e) \leqslant x \leqslant a\} \subset D$$

and $R = R(a, |b|, M) > 0$ such that

$$(x - \hat{x}_0, b) + \frac{1}{2}(M(x - \hat{x}_0), x - \hat{x}_0) \geqslant C|x - \hat{x}_0|^2 \qquad \forall x \in \mathbb{R}^d \setminus B_R(\hat{x}_0) \tag{D.4}$$

for some constant $C = C(|b|, M) > 0$, $D_a \subset B_R(\hat{x}_0)$, where $B_R(\hat{x}_0)$ is the ball of radius R with centre $\hat{x}_0$. We split the left-hand side of (D.3) into the sum

$$\int_{D\setminus B_R(\hat{x}_0)} + \int_{B_R(\hat{x}_0)\cap(D\setminus D_a)} + \int_{D_a} = \mathbf{I} + \mathbf{II} + \mathbf{III}.$$

Step 1. Since $f \in C_b^1$ and using Taylor's decomposition

$$\Phi(x) \geqslant \Phi(\hat{x}_0) + (x - \hat{x}_0, b) + \frac{1}{2}\,(M(x-\hat{x}_0), x-\hat{x}_0)$$

we have

$$\mathbf{I} = O\Big(\exp\Big\{-\frac{1}{h}\,\Phi(\hat{x}_0)\Big\}\Big)\times$$
$$\times \int\limits_{D\setminus B_R(\hat{x}_0)} \exp\Big\{-\frac{1}{h}\,(x-\hat{x}_0, b) - \frac{1}{2h}\,(M(x-\hat{x}_0), x-\hat{x}_0)\Big\}\,dx.$$

Applying (D.4) we get

$$\mathbf{I} = O\Big(\exp\Big\{-\frac{1}{h}\,\Phi(\hat{x}_0)\Big\}\Big) \int\limits_{D\setminus B_R(\hat{x}_0)} \exp\Big\{-\frac{C}{h}\,|x-\hat{x}_0|^2\Big\}\,dx. \tag{D.5}$$

The elementary formula

$$\int\limits_1^\infty \exp\Big\{-\frac{1}{\alpha}\,z\Big\} z^{d-1}\,dz \sim \alpha \exp\Big\{-\frac{1}{\alpha}\Big\} \qquad \text{as} \quad \alpha \to 0+.$$

gives

$$\begin{aligned}\int\limits_R^\infty \exp\Big\{-\frac{C}{h}\,z^2\Big\} z^{d-1}\,dz &= R^d \int\limits_1^\infty \exp\Big\{-\frac{CR^2}{h}\,z^2\Big\} z^{d-1}\,dz \\ &= \frac{hR^{d-2}}{C}\,O\Big(\exp\Big\{-\frac{CR^2}{h}\Big\}\Big).\end{aligned} \tag{D.6}$$

Combining (D.5) and (D.6) we have

$$\mathbf{I} = h \exp\Big\{-\frac{1}{h}\,\Phi(\hat{x}_0)\Big\} \exp\Big\{-\frac{CR^2}{h}\Big\} O(1). \tag{D.7}$$

Step 2. We take $\varepsilon > 0$ such that $\Phi(x) \geqslant \Phi(\hat{x}_0) + \varepsilon$ for $x \in B_R(\hat{x}_0) \cap (D \setminus D_a)$. Then

$$\mathbf{II} = \exp\Big\{-\frac{1}{h}\,(\Phi(\hat{x}_0) + \varepsilon)\Big\} O(1). \tag{D.8}$$

Step 3. We proceed with **III**. By Taylor's formula

$$\Phi(x) = \Phi(\hat{x}_0) + (b, x - \hat{x}_0) + \frac{1}{2}\,(\Lambda(x - \hat{x}_0), x - \hat{x}_0) + O(|x - \hat{x}_0|^3) \qquad x \in D_a.$$

Using decomposition $x - \hat{x}_0 = ze + y,\quad y \in T_{\hat{x}_0}\partial D,\ e = b/|b|,\ z \in \mathbb{R}$ and definition (D.1) we get

$$\begin{aligned}(\Lambda(x - \hat{x}_0), (x - \hat{x}_0)) &= z^2(\Lambda e, e) + 2z(\Lambda e, y) + (\Lambda y, y)\\ &= (\Lambda e, e)\left(z + (\Lambda e, y)(\Lambda e, e)^{-1}\right)^2 + (\Lambda_b y, y).\end{aligned}$$

Hence

$$\Phi(x) = \Phi(\hat{x}_0) + Z_1 + Z_2,$$

where

$$Z_1 = \frac{1}{2}(\Lambda_b y, y) + O(|y|^3)$$

and

$$Z_2 = |b|\,z + \frac{1}{2}\,(\Lambda e, e)\left(z + (\Lambda e, y)(\Lambda e, e)^{-1}\right)^2 + O(|z|^3).$$

Consequently

$$\mathbf{III} = \exp\left\{-\frac{1}{h}\,\Phi(\hat{x}_0)\right\} \int\limits_{\mathbb{R}^{d-1}} \int\limits_{G(y)}^{a} \widetilde{f}(y, z) \exp\left\{-\frac{1}{h}\,Z_1\right\} \exp\left\{-\frac{1}{h}\,Z_2\right\} dzdy, \tag{D.9}$$

where

$$\widetilde{f}(y, z) = f(\hat{x}_0 + ze + y).$$

The formula

$$\int\limits_0^{+\infty} g(z) \exp\left\{-\frac{A_1}{h}\,z - \frac{A_2}{h}\,(z - A_3)^2 + \frac{A_4}{h}\,(z^3 \wedge 1)\right\} dz = g(0)\,\frac{h}{A_1}\,(1 + O(h))$$

for any $g \in C_b(\mathbb{R})$, $A_1, \ldots, A_4 \in \mathbb{R}$, $A_1, A_2 > 0$ implies

$$\int\limits_{G(y)}^{a} \widetilde{f}(y, z) \exp\left\{-\frac{1}{h}\,Z_2\right\} dz = \sum_{i=1}^{2} (-1)^i \frac{h}{|b|}\,\widetilde{f}(y, \alpha_i) \exp\left\{-\frac{|b|}{h}\,\alpha_i\right\} (1 + O(h))$$

where $\alpha_1 = a$, $\alpha_2 = G(y)$. From (D.9) we find

$$\mathbf{III} = \exp\Big\{-\frac{1}{h}\,\Phi(\hat{x}_0)\Big\}(I_1 + I_2),$$

where

$$I_i = \frac{h}{|b|}\,(1 + O(h)) \int\limits_{\mathbb{R}^{d-1}} \widetilde{f}(y, \alpha_i) \exp\Big\{-\frac{|b|}{h}\,\alpha_i\Big\} \exp\Big\{-\frac{1}{h}\,Z_1\Big\}\,dy.$$

An application of Lemma D.1 shows

$$I_1 = \exp\Big\{-\frac{a}{h}\Big\}(\sqrt{2\pi h})^{d-1}\frac{h}{|b|}\,\widetilde{f}(0, a)\,(\det\Lambda_b)^{-\frac{1}{2}}\,(1 + O(h)).$$

and

$$I_2 = (\sqrt{2\pi h})^{d-1}\frac{h}{|b|}\,\widetilde{f}(0, 0)\,\big(\det(|b|\,G^{(2)}(0) + \Lambda_b)\big)^{-\frac{1}{2}}\,(1 + O(h)).$$

Hence

$$\mathbf{III} = (\sqrt{2\pi h})^{d-1}\frac{h}{|b|}\,\widetilde{f}(0, 0)\,\big(\det(|b|\,G^{(2)}(0) + \Lambda_b)\big)^{-\frac{1}{2}} \times \\ \times \exp\Big\{-\frac{1}{h}\,\Phi(\hat{x}_0)\Big\}(1 + O(h)). \tag{D.10}$$

Combining (D.7), (D.8), (D.10) and using $\widetilde{f}(0,0) = f(\hat{x}_0)$, we complete the proof. $\square$

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