Price Dispersion and Vanilla Options in a Financial Market Game

Abstract

We construct a game-theoretic model characterised by perfect information, no transaction costs, in which agents can borrow at the risk-free interest rate and engage in short selling. Traders can freely and instantaneously eliminate any unexploited profit opportunities through pure arbitrage just as the efficient market hypothesis postulates they should. Yet, in this work we put forth, in a frictionless framework, a counterexample in which the Law of One Price fails in the underlying assets markets at equilibrium. At a theoretical level, this leads both the Binomial Option Pricing Model (BOPM) and in the limit, the Black–Scholes–Merton Model (BSMM), to misprice the vanilla options written on these assets. This compelling result is pregnant with far-reaching ramifications: (i) theoretically, it establishes that no-arbitrage, while necessary, is not sufficient for any of the BOPM and BSMM to yield consistent results; (ii) practically, it crystallises the need for practitioners to rely on additional, more data-adaptive, methods of option pricing.

JEL Classification: G12, G14, C72, D43, D50.

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1 Introduction

The absence of arbitrage in financial markets is a subtle yet paramount tenet which underlies most of the financial economics machinery today. The no-arbitrage condition (or requirement, even) lies at the core of corporate capital structure, modern international finance, and option pricing, amongst others. In the complete absence of economic and financial frictions—the most important of which are transaction costs, imperfect information, transportation costs and liquidity constraints—conventional wisdom hypothesises that no-arbitrage is intimately related to the so-called Law of One Price (LOOP). In a financial context, the LOOP posits that at equilibrium identical stocks/assets/commodities[1] need to be priced exactly the same (see, e.g., Lamont and Thaler, 2003). If not, traders could make riskless profits by buying an asset at a lower price and immediately selling it a higher tariff. Interestingly, these traders need not all be rational and sophisticated. They need only be able to identify an arbitrage opportunity and have the means to exploit it. This way, the forces of demand and supply work in tandem to drive prices upward and downward until prices reach a state of repose. And indeed, the nature of stocks means that the LOOP obtains instantaneously as stocks can be bought and sold immediately to wipe out any arbitrage opportunity which presents itself. Accordingly, in a ***risk-neutral*** world, option pricing methods such as the Binomial Option Pricing Model (BOPM) and the Black–Scholes–Merton Model (BSMM) rely heavily on the prevalence of no-arbitrage to calculate a *unique* price for options yielding identical state-specific payoffs. An interesting question in this regard is: provided the prerequisite conditions for the applicability of both the BOPM and BSMM are met, could these models theoretically misprice the relevant derivatives in a frictionless context?

In this work, we demonstrate via a counterexample the failure of the LOOP in the option and underlying asset markets. More precisely, we present an ***equilibrium*** scenario where the same commodity trades at different prices across different platforms. We put forth a frictionless game-theoretic general equilibrium framework in which agents can, and do, freely and strategically trade in commodities using a system of trading platforms (or markets). On these platforms, agents may place bids for the assets they wish to buy, and also offer these assets for sale. In line with how trade takes place on real-world platforms, traders can simultaneously enter both sides (buy side and sell side) of each market to manipulate prices in their favour. We show that in such a setup, which closely resembles a call auction (see §2.1), it is possible for an identical commodity to trade at different prices across different markets at equilibrium. Since all the conditions for the successful applicability of the BOPM and BSMM also hold,[2] we then use the latter to derive the prices of European vanilla options across different platforms. The BOPM—as does the BSMM—as opposed to yielding a unique risk-neutral price, instead generates different (equilibrium) prices, for identical options! Intriguingly, even if these prices were to give rise to

[1]Throughout the rest of this paper, the terms stocks, assets, and commodities will be used interchangeably.

[2]Traders can trade in both the asset and options markets, there are no transaction costs, short sales of commodities are allowed, and at the equilibrium situation we exhibit, agents are not liquidity-constrained such that arbitrage is feasible. In particular, for the type of market game we consider, the ability for traders to take short positions in assets is novel and part of our contribution.

arbitrage opportunities in the option markets, the resultant equilibrium option price would still lie strictly between the prices given by the BOPM and BSMM, i.e., the latter ***fail*** no matter what. This result is a meaningful and opportune addition to the literature, informing both academics and practitioners. Many important papers and theories are founded on the BOPM and BSMM (see, e.g., Baldwin and Alhalboni, 2020, and the references therein). Our findings thus suggest a rethinking of the pervasive use of classical option pricing models which are well-known to have several shortcomings not only from a computational viewpoint, but from economic and statistical perspectives as well. Our claim is bolstered by recent research into this issue. Indeed, for option valuation purposes, Ivascu (2021) finds that Machine Learning methods perform better than the BSMM and the Corrado–Su Model in practice.

This compelling phenomenon is driven by the fact that individual agents wield a non-negligible degree of market power in the asset markets. Hence, even though the price disparity seems to suggest the existence of a profitable deviation, this is in fact not so. By adjusting their bids and offers, traders influence the market-clearing prices of the commodities, ironically through the aforementioned equilibrating forces of demand and supply themselves. Any deviation from this unequal-price equilibrium situation, no matter how little, nudges prices in the "wrong" direction for the traders, such that any gains made in one market are, at best, exactly cancelled by losses made in the other markets. Hence, traders make no unilateral move, and equilibrium is maintained. In a risk-neutral world, pricing vanilla options in terms of the price of the underlying means that these unequal asset prices also migrate into the derivatives markets.

2 The market game Γ with numéraire and finitely many agents

The set of agents is N, where $|N| < \infty$. The set of stocks/assets/commodities traded (bought and sold) in the economy is denoted by $\mathcal{C} = \{1, 2, \dots, K\}$. There is also a $(K+1)^{th}$ asset, m, which acts as money and yields utility in consumption—i.e., a numéraire. Each $k \in \mathcal{C}$ is traded for money at a different trading platform, and for each k, there are $T_k < \infty$ trading platforms. Every $k \in \{m\} \cup \mathcal{C}$ is perfectly divisible. For ease of exposition, let $\mathbf{T} = \sum_{k=1}^{K} T_k$.

Each agent $n \in N$ is endowed with a strictly positive amount of every asset k, i.e., $e_{n,k} \gg 0$ $\forall n \in N$ and $\forall k \in \{m\} \cup \mathcal{C}$. The consumption set of each agent is therefore identified with $\mathbb{R}^{K+1}_{+}$. In this light, each $n \in N$ may be described by a preference relation representable by a utility function $u_n : \mathbb{R}^{K+1}_{+} \to \mathbb{R}$, and an initial endowment of commodities, $e_n \in \mathbb{R}^{K+1}_{++}$. Rationality is common knowledge and there is perfect information.

Formally, the strategy set of each $n \in N$ is defined as:

$$S_n = \Big\{ \big(b_n, q_n\big) \in \mathbb{R}_+^{\mathbf{T}} \times \mathbb{R}_+^{\mathbf{T}} : \sum_{k=1}^{K} \sum_{i=1}^{T_k} b_{n,k}^i \le e_{n,m}; \sum_{n=1}^{|N|} \sum_{i=1}^{T_k} q_{n,k}^i \le \sum_{n=1}^{|N|} e_{n,k}, k \in \mathcal{C} \Big\}.$$

Let us explicate the meaning of the individual elements of S_n. Each $n \in N$ may place bids for (make purchases of) commodities $1, 2, ..., K$ by distributing amounts of money, m, across the various trading platforms, with $b_{n,k}^i$ representing the bid placed by agent n for asset k at trading platform i, $i \in \{1, \ldots, T_k\}$. Likewise, agent n may simultaneously offer commodities $1, 2, ..., K$ for sale by allocating amounts of $k \in \mathcal{C}$ across platforms, with $q_{n,k}^i$ standing for the amount of commodity k offered for sale by agent n on market i.

Note, in particular, that while agents are not allowed to bid a total amount exceeding their endowment of m ($\sum_{k=1}^{K} \sum_{i=1}^{T_k} b_{n,k}^i \le e_{n,m}$), we impose a different type of constraint on offers thereby departing from the extant literature in two important respects. First, observe that our restriction means agents can engage in ***short-selling***, such that $\sum_{i=1}^{T_k} q_{n,k}^i > e_{n,k}$ is allowed for any $k \in \mathcal{C}$—i.e., a trader may put up for sale an amount greater than his endowment of k. Of course, while agents can take on leverage, the nature of the trading mechanism means they must also close their short positions by buying back (placing bids for) the relevant commodities. Second, since our system is a closed one, the total amount of short sales of an asset at any time cannot exceed the total amount of the asset in existence ($\sum_{n=1}^{|N|} \sum_{i=1}^{T_k} q_{n,k}^i \le \sum_{n=1}^{|N|} e_{n,k}$). This is akin to preventing agents from making *naked* short sales.

Throughout the rest of the paper, we will rely on the following assumption:

Assumption 1. *Utility functions for all agents are concave, smooth, differentiably strictly monotone,*[3] *and the closure in $\mathbb{R}_+^{K+1}$ of each indifference surface in $\mathbb{R}_{++}^{K+1}$ is contained in $\mathbb{R}_{++}^{K+1}$.*

2.1 Interactions in the economy

Given a strategy profile $(b, q) \in S := \prod_{n \in N} S_n$, we define $B_k^i = \sum_{n \in N} b_{n,k}^i$, $Q_k^i = \sum_{n \in N} q_{n,k}^i$, $B_{-n,k}^i = \sum_{h \in N \setminus \{n\}} b_{h,k}^i$, and $Q_{-n,k}^i = \sum_{h \in N \setminus \{n\}} q_{h,k}^i$. Final allocations of commodity $k \in \{m\} \cup \mathcal{C}$

[3] i.e., for all $x_n \in \mathbb{R}_{++}^{K+1}$, $\frac{\partial u_n(x_n)}{\partial x_{n,k}} > 0 \;\; \forall k \in \{m\} \cup \mathcal{C}$.

for any $n \in N$ are then determined as follows:

$$\begin{aligned} x_{n,k} &= e_{n,k} + \sum_{i=1}^{T_k} \frac{b_{n,k}^i}{B_k^i} Q_k^i - \sum_{i=1}^{T_k} q_{n,k}^i, \quad k \in \mathcal{C}; \\ x_{n,m} &= e_{n,m} - \sum_{k=1}^{K} \sum_{i=1}^{T_k} b_{n,k}^i + \sum_{k=1}^{K} \sum_{i=1}^{T_k} \frac{q_{n,k}^i}{Q_k^i} B_k^i, \end{aligned} \tag{1}$$

where the market game convention that any division by zero, including $\frac{0}{0}$, is equal to zero whenever it appears in any of the expressions above, has been adopted.

The allocation rule in (1) is very intuitive. Let us analyse its component parts. Q_k^i is the total amount of asset k offered for sale at market i, while B_k^i is the total amount of money placed (bidden) at platform i to purchase k. When $B_k^i \cdot Q_k^i > 0$, trader n, having bidden $b_{n,k}^i$ at i, then receives asset k in proportion to his bids. Likewise, trader n, having offered $q_{n,k}^i$ units of k for sale at i, then receives money (m) in proportion to his offers. Moreover, observe that as B_k^i increases (decreases), $\frac{B_k^i}{Q_k^i}$ rises (falls), while $\frac{B_k^i}{Q_k^i}$ rises (falls) as Q_k^i decreases (increases). As such, whenever $B_k^i \cdot Q_k^i > 0$, the fraction $\frac{B_k^i}{Q_k^i} := p_k^i$ can be naturally interpreted as the *market-clearing* price of asset k at trading platform i. Henceforth, the price of k at platform i will be denoted by any of p_k^i and $\frac{B_k^i}{Q_k^i}$.[4] We mention in passing that the price formation mechanism hereby described is perfectly aligned with the fundamental premise of the efficient market hypothesis: all publicly available information is fully embedded in the prevailing market-clearing price. As such, any increase or decrease in demand and supply for the underlying asset is *instantaneously* reflected in its price, price which is then used to value options written on that commodity. Hence, regardless of the frequency at which these options are traded, one can simply run an iteration of the market game $\mathit{\Gamma}$, derive the commodity price(s), and then use an appropriate option valuation technique to price the relevant options. This is particularly useful for scalpers who execute option trades

[4] A simple numerical example will help to cast light on how (1) works. Let there be three agents, A, B, C, and pick a commodity k and a trading platform i. Let the bids of the agents be $(b_{A,k}^i, b_{B,k}^i, b_{C,k}^i) = (10, 3, 5)$ and their offers be $(q_{A,k}^i, q_{B,k}^i, q_{C,k}^i) = (1, 2, 3)$. Equation (1) then stipulates that the price, $p_k^i = \frac{B_k^i}{Q_k^i} = \frac{10+3+5}{1+2+3} = 3$. Accordingly, on trading platform i, agent A, having bid 10 units of m, receives $b_{A,k}^i \frac{Q_k^i}{B_k^i} = 10 \times \frac{1}{3} = \frac{10}{3}$ units of k—B and C obtain $\frac{3}{3}$ and $\frac{5}{3}$ units of k, respectively. Since A also sells 1 unit of k, his *net trade* of k on market i is $\frac{10}{3} - 1 = \frac{7}{3}$. Using this same procedure, we can derive A's net trade of k on every other platform (given his bids and offers there). Hence, A's final allocation of k, $x_{A,k}$, is just his endowment of k, plus his total net trades across all T_k trading platforms.

Now, on platform i for commodity k, trader A disburses 10 units of m. However, given A also sells 1 unit of k and $p_k^i = 3$, his *net revenue* on market i is $(1 \times 3) - 10 = -7$. Repeating this argument, we can derive A's net revenue on every other platform, for every commodity (given his bids and offers there). So, A's final allocation of m, $x_{A,m}$, is just his endowment of m plus his total net revenue across all $\sum_{k=1}^{K} T_k$ trading platforms.

on a minute-by-minute basis. Ergo, our model, though simple, has been executed with great minuteness.

It is noteworthy that the cash-in-advance constraint that we use here allows us to consider a model that is truly decentralised, such that no bankruptcy rules are required. This is in stark contrast to Peck and Shell (1990), who, in order to deal with any trader who goes bankrupt, impose extremely harsh punishments such as a referee shutting down the game altogether, resulting in autarky. We also remark our treatment of short sales is different from Peck and Shell's (1990), who allow agents to engage in unrestricted, arbitrarily large short-selling.

Definition 1. A Nash equilibrium (NE) of Γ consists of agents' bids for, and sales of, commodities such that

(i) Every agent's actions are best-responses given the expectations of other agents' actions;

(ii) The best-responses are consistent with all agents' expectations of other agents' actions.

At an NE, any trader $n \in N$ is viewed as solving the following programme:

$$\max_{(b_n, q_n) \in S_n} \left\{ u_n \left(\left(x_{n,k}\big(b_{n,k}, l_{n,k}, B_{-n,k}, Q_{-n,k}\big) \right)_{k=1}^{K}, x_{n,m}\Big(\big(b_{n,k}, l_{n,k}, B_{-n,k}, Q_{-n,k}\big)_{k=1}^{K} \Big) \right) \right\}. \tag{2}$$

Definition 2. An NE is termed "interior" if every $n \in N$ solves (2) in the interior of S_n, i.e., no constraint binds.

Before closing this section, we deem it important to highlight a few features of our model, and in particular, of the trading mechanism that could be of interest to financial analysts. First, our model is a simultaneous-move (or static) one such that our trading mechanism closely resembles a (periodic) ***call auction***. Verily, just like with a call auction, at any given trading platform for any given commodity, traders' bid-and-offer orders are pooled together and processed *simultaneously* at a *single market-clearing* price, i.e., the system is well-behaved even out of equilibrium. Second, notice that trade takes place *costlessly* in our setup—cf. Toraubally (2018) where traders incur a service charge whenever their net trade is non-zero. This is aligned with how call auctions, by gathering many transactions in one place, improve liquidity thereby dramatically reducing transaction costs for traders.

2.2 Equilibrium analysis

Our first result delineates the equilibrium relationship between the price for asset k on any trading platform i and the price for k on any other platform f.

Proposition 1. *At an interior NE, the prices for commodity k between any two trading platforms i and f must satisfy the following no-arbitrage condition for any $n \in N$:*

$$(p_k^f)^2 = \frac{B^f_{-n,k} Q^i_{-n,k}}{Q^f_{-n,k} B^i_{-n,k}} (p_k^i)^2.$$

Proof. See Appendix. □

We now lay the groundwork for our next result. In particular, by augmenting the number of trading platforms per commodity as we have done in this paper, no equilibrium is lost. More precisely, any equilibrium of the market game with a *single* trading platform per commodity, is also an equilibrium of the market game with multiple trading platforms per commodity (see, e.g., Koutsougeras, 1999; Toraubally, 2018). And of course, if there is a single platform for each commodity, it is trivial that at any NE, the LOOP should obtain. It follows that such equilibria will also constitute equilibria of our model. We collect this result in the next proposition.

Proposition 2. *There (always) exists an NE of Γ at which the Law of One Price holds.*

Proof. See Dubey and Shubik (1978), and Toraubally (2022). □

For the sake of completeness, we remark that Dubey and Shubik (1978) prove the existence of NE which may not be nontrivial. However, this apparent limitation is but benign for our purposes since in our proposed counterexample, we do indeed construct a nontrivial NE.

Before moving on to our counterexample, we deem it important to remind the reader that the result in Proposition 2 does not preclude the possibility of there (simultaneously) being a no-arbitrage equilibrium, at which the LOOP fails, a fact which we set forth below:

Corollary 1. *By Proposition 1, at an interior NE, if $\frac{B^f_{-n,k} Q^i_{-n,k}}{Q^f_{-n,k} B^i_{-n,k}} \neq 1$, the Law of One Price fails.*

The foregoing result is crucial in that it dispels the misconception of no-arbitrage being synonymous with the LOOP. Indeed, Proposition 1 and Corollary 1 demonstrate that at equilibrium, the LOOP fails insofar as one is able to find a profile of strategies such that $B^f_{-n,k} Q^i_{-n,k} \neq$

$Q^f_{-n,k}B^i_{-n,k}$; yet, this is perfectly consistent with the complete absence of arbitrage. That is to say, no-arbitrage is a necessary, but not sufficient, condition for the prevalence of the LOOP.

3 The counterexample

For simplicity, let us price a European call and a European put options in discrete time. Time, t, is measured in years. Let $p^i_{k,t}$ represent the price of commodity k at trading platform i prevailing at time t. Let the strike prices of the call and put options on commodity k be X_k and Y_k, respectively. Assume that $p^i_{k,t}$ evolves according to a binomial model, with constant time steps of length Δt. The time to maturity of each option is $\mathcal{T} < \infty$, such that the number of time steps is $g = \mathcal{T}/\Delta t$. Following Cox et al. (1979), over each Δt, $p^i_{k,t}$ goes up by a factor $u := \exp(\sigma_k\sqrt{\Delta t})$ with probability $p_u := \frac{\exp(r\Delta t)-d}{u-d}$, or down by a factor $d := 1/u$ with probability $p_d = 1 - p_u$, where σ_k represents the volatility of commodity k's price, and r is the risk-free rate of interest. Let the number of upward movements over $\mathcal{T}$ be ℓ. The prices of our call and put options can then be derived, respectively, as follows:

$$\begin{aligned}
\mathscr{P}^{C,i}_{k,t} &= \exp\left(-r\mathcal{T}\right) \cdot \sum_{\ell=0}^{g} \frac{g!}{(g-\ell)!\ell!} p_u^{\ell} p_d^{g-\ell} \max\left\{p^i_{k,t} u^{\ell} d^{g-\ell} - X_k, 0\right\}; \\
\mathscr{P}^{P,i}_{k,t} &= \exp\left(-r\mathcal{T}\right) \cdot \sum_{\ell=0}^{g} \frac{g!}{(g-\ell)!\ell!} p_u^{\ell} p_d^{g-\ell} \max\left\{Y_k - p^i_{k,t} u^{\ell} d^{g-\ell}, 0\right\}.
\end{aligned} \tag{3}$$

The prices of the underlying assets at time t will be determined by traders interacting as per the rules detailed in §2. Let the set of agents be $N = \{A, B, C\}$, the set of assets be $\{m\} \cup \mathcal{C}$, where $\mathcal{C} = \{1, 2\}$, and let there be two trading platforms i and f for each asset. The consumption set of each agent is thus a subset of $\mathbb{R}^3_+$. The endowments of the traders are:

$$\begin{aligned}
\left(e_{A,1}, e_{A,2}, e_{A,m}\right) &= \left(\tfrac{3777}{700}, \tfrac{254}{25}, \tfrac{407}{70}\right), \\
\left(e_{B,1}, e_{B,2}, e_{B,m}\right) &= \left(\tfrac{9411}{700}, \tfrac{13611}{700}, \tfrac{153}{35}\right), \\
\left(e_{C,1}, e_{C,2}, e_{C,m}\right) &= \left(\tfrac{304}{25}, \tfrac{277}{700}, \tfrac{687}{70}\right).
\end{aligned}$$

The traders' preferences are represented by the following utility functions:

$$u_A(x_A) = 15\ln(x_{A,1}) + 4\ln(x_{A,2}) + 9\ln(x_{A,m}),$$
$$u_B(x_B) = \ln(x_{B,1}) + 2\ln(x_{B,2}) + 9\ln(x_{B,m}),$$
$$u_C(x_C) = 5\ln(x_{C,1}) + 10\ln(x_{C,2}) + 18\ln(x_{C,m}).$$

It may be verified that the following profile of strategies constitutes an N.E:

$$\text{Asset 1}: \begin{cases} \left(b^i_{A,1}, q^i_{A,1}, b^f_{A,1}, q^f_{A,1}\right) & = \left(\frac{10}{3}, \frac{243}{50}, \frac{1}{2}, \frac{9}{7}\right), \\ \left(b^i_{B,1}, q^i_{B,1}, b^f_{B,1}, q^f_{B,1}\right) & = \left(2, \frac{729}{50}, \frac{1}{14}, \frac{9}{7}\right), \\ \left(b^i_{C,1}, q^i_{C,1}, b^f_{C,1}, q^f_{C,1}\right) & = \left(\frac{2}{3}, \frac{243}{50}, \frac{1}{7}, \frac{9}{14}\right). \end{cases}$$
$$\text{Asset 2}: \begin{cases} \left(b^i_{A,2}, q^i_{A,2}, b^f_{A,2}, q^f_{A,2}\right) & = \left(\frac{1}{7}, \frac{9}{14}, \frac{2}{3}, \frac{243}{50}\right), \\ \left(b^i_{B,2}, q^i_{B,2}, b^f_{B,2}, q^f_{B,2}\right) & = \left(\frac{1}{14}, \frac{9}{7}, 2, \frac{729}{50}\right), \\ \left(b^i_{C,2}, q^i_{C,2}, b^f_{C,2}, q^f_{C,2}\right) & = \left(\frac{1}{2}, \frac{9}{7}, \frac{10}{3}, \frac{243}{50}\right). \end{cases}$$

The corresponding market-clearing prices to the strategies above are:

$$\text{Asset 1: } (p^i_1, p^f_1) = \left(\tfrac{20}{81}, \tfrac{2}{9}\right),$$
$$\text{Asset 2: } (p^i_2, p^f_2) = \left(\tfrac{2}{9}, \tfrac{20}{81}\right),$$

and each trader ends up with consumption:

$$\left(x_{A,1}, x_{A,2}, x_{A,m}\right) = (15, 8, 4),$$
$$\left(x_{B,1}, x_{B,2}, x_{B,m}\right) = (6, 12, 8),$$
$$\left(x_{C,1}, x_{C,2}, x_{C,m}\right) = (10, 10, 8).$$

Now, let $X^i_1 = X^f_1 = X^i_2 = X^f_2 = \frac{11}{50}$, $Y^i_1 = Y^f_1 = Y^i_2 = Y^f_2 = \frac{1}{4}$, and $r = 4\%$. Next, recall from Proposition 2 that there always exists an NE at which the LOOP holds. Recall further that when the LOOP holds, the number of trading platforms for a commodity is irrelevant as these can all be consolidated into a unique trading platform. Hence, assume from previous iterations of the game[5] that we had $p^i_{1,t-1} = p^f_{1,t-1} = \frac{243}{1000}$, and $p^i_{2,t-1} = p^f_{2,t-1} = \frac{47}{200}$. Define

[5]The NE allocations from the last iteration of Γ constitute traders' endowments for the present round of trading. As explained in Toraubally (2022), as long as the endowment point one starts from is not Pareto optimal, traders will

$\varsigma^i_{k,j} = \varsigma^f_{k,j} = \ln\!\left(\frac{p^i_{k,j+1}}{p^i_{k,j}}\right)$, and let $\sum_{j=1}^{250} \varsigma^i_{1,j} = \sum_{j=1}^{250} \varsigma^f_{1,j} = \frac{583}{500}$, $\sum_{j=1}^{250}(\varsigma^i_{1,j})^2 = \sum_{j=1}^{250}(\varsigma^f_{1,j})^2 = \frac{249}{6250}$, and $\sum_{j=1}^{250} \varsigma^i_{2,j} = \sum_{j=1}^{250} \varsigma^f_{2,j} = \frac{1153}{1000}$, $\sum_{j=1}^{250}(\varsigma^i_{2,j})^2 = \sum_{j=1}^{250}(\varsigma^f_{2,j})^2 = \frac{411}{10000}$.[6] We can derive the annualised volatilities of commodities 1 and 2 across trading platforms i and f as follows:

$$\text{Asset 1: } (\sigma^i_1, \sigma^f_1) = (0.187, 0.209),$$
$$\text{Asset 2: } (\sigma^i_2, \sigma^f_2) = (0.199, 0.195).$$

For simplicity, let $\mathcal{T} = \frac{1}{6}$ and $\Delta t = \frac{1}{12}$, such that $g = 2$. We thus have a two-step Binomial tree, and may compute $\mathscr{P}^C_t$ and $\mathscr{P}^P_t$ using (3) to get:

$$\text{Asset 1: } (\mathscr{P}^{C,i}_{1,t}, \mathscr{P}^{C,f}_{1,t}, \mathscr{P}^{P,i}_{1,t}, \mathscr{P}^{P,f}_{1,t}) = (0.0284, 0.0091, 0.0081, 0.0263),$$
$$\text{Asset 2: } (\mathscr{P}^{C,i}_{2,t}, \mathscr{P}^{C,f}_{2,t}, \mathscr{P}^{P,i}_{2,t}, \mathscr{P}^{P,f}_{2,t}) = (0.0088, 0.0284, 0.0261, 0.0084).$$

The BOPM thus generates a pair of prices for identical call options on the very same commodity, and likewise for put options. Note that this conclusion holds good regardless of the number of time steps considered—one, two, or more. In particular, it can be verified that even in the limit, the BSMM will still give unequal prices as we report below, for concreteness:

$$\text{Asset 1: } (\mathscr{P}^{C,i}_{1,t}, \mathscr{P}^{C,f}_{1,t}, \mathscr{P}^{P,i}_{1,t}, \mathscr{P}^{P,f}_{1,t}) = (0.0288, 0.0095, 0.0083, 0.0270),$$
$$\text{Asset 2: } (\mathscr{P}^{C,i}_{2,t}, \mathscr{P}^{C,f}_{2,t}, \mathscr{P}^{P,i}_{2,t}, \mathscr{P}^{P,f}_{2,t}) = (0.0091, 0.0289, 0.0269, 0.0086).$$

In other words, both the BOPM and BSMM fail. We scrutinise the intricacies underlying this critical statement in the next section. ♦

4 Discussion, implications, and conclusion

Before discussing how the unequal asset-price situation in §3 can be supported as an equilibrium, let us first expound some of its important features. This endeavour will help enhance the reader's understanding of the mechanism at play.

1. First, note that all traders engage in short-selling. A and B engage in short-selling for com-

be better off reallocating resources.

[6] As is commonplace in the literature, we assume there are 252 trading days per year, and hence 251 return observations. Here, time t represents the 252^{nd} trading day and $\varsigma^i_{k,251}$ will constitute the 251^{st} observation.

modity 1 $\left(\frac{243}{50}+\frac{9}{7}>\frac{3777}{700}\text{ and }\frac{729}{50}+\frac{9}{7}>\frac{9411}{700},\text{ respectively}\right)$, while C shorts commodity 2 $\left(\frac{243}{50}+\frac{9}{7}>\frac{277}{700}\right)$.

2. The NE exhibited is interior. Importantly, we have chosen these numbers purposely to prove to the reader that no trader is financially constrained, such that every $n \in N$ has the possibility to move his bids (or offers) from one market to another if he so wishes.

3. Trade is costless, and every trader actually enters both sides of each market for each commodity.

4. The set of numbers we have chosen for the market game are all rational, such that our unequal-price equilibrium is most assuredly ***not*** the outcome of any rounding errors. Additionally, our example is robust in endowment and utility spaces. That is, given our equilibrium profile of strategies, we may, for each $n \in N$, pick *any* utility functions and endowments, such that Assumption 1 and the first-order conditions (see Appendix) are satisfied.

Now, to see why the same asset selling at unequal prices is a legitimate equilibrium, pick, say, Commodity 1. Consider a trader, say, A, who tries to profit from the price difference in the capital markets by reorganising his bids and offers across platforms i and f in a seemingly astute manner. For example, A could consider simultaneously moving some of his bids from platform i (where the price is higher) to platform f, and shifting some of his offers from f to i. However, in moving his bids from i to f, he increases the amount demanded of the asset at f, thereby driving up p_1^f immediately. To exacerbate things, it is not only the marginal price of the asset that soars, but its average price as well, i.e., *every* unit at f now costs more. A's purchasing power at platform f falls, such that his move fails to have the desired effect. Additionally, in shifting his offers from f to i, he increases the amount sold of the asset at i, thereby depressing p_1^i immediately. Again, it is not only the marginal price of the commodity that falls, but the average price as well—*every* unit at i is now cheaper. This deviation only serves to reduce the revenue that A obtains from selling Commodity 1 at platform i. These effects will combine in such a way as to make A's move unprofitable. The same argument can be made for any other trader and commodity, such that equilibrium prevails.

Since we assume a risk-neutral framework, the unequal asset prices then permeate the option markets. Consequently, given the volatility of Commodity 1 and the risk-free rate of return, the BOPM yields two prices for the exact same option, as does the BSMM. This is, in and of itself, already a blatant violation of the fundamental prediction of both models, which is a unique price for identical options. But this is not all. Even if this price disparity gave rise to arbitrage opportunities in the option markets and traders were to track down and risklessly exploit this mispricing, the resultant equilibrium no-arbitrage option price would lie (strictly) between $\mathscr{P}_{1,t}^{C,f}$ and $\mathscr{P}_{1,t}^{C,i}$ thanks to the workings of demand and supply—$\mathscr{P}_{1,t}^{C,f}$ will be revised upward thanks to increased demand for the call option, while $\mathscr{P}_{1,t}^{C,i}$ will fall due to traders selling more of the instrument. This process continues until traders can no longer arbitrage prices. Ultimately, this implies an option price that is in accordance, neither with the BOPM nor the BSMM. That is to say, the failure of each option pricing technique to deliver consistent prices is a ***robust***, and by extension, material issue which must not be overlooked by finance theorists and practitioners alike.

The equilibrating mechanism described above goes through similarly for put options, and if Commodity 2 were considered as well. We have therefore shown that when agents have market power, as opposed to being individually negligible, one can derive equilibria at which the LOOP fails, and with it, traditional methods of option pricing like the BOPM and BSMM. Crucially, our model makes the significant contribution of proving that for the BOPM and BSMM to yield meaningful results at a theoretical level, an important qualification is required: not only should there be no-arbitrage, but the LOOP must also obtain.

At a practical level, our findings underscore the need to increase our reliance on data-adaptive approaches founded on nonparametric models (e.g., artificial neural networks), which process structural changes in data through courses of action that classical parametric ones (BOPM, BSMM, trinomial tree models, finite difference, and Monte Carlo methods) cannot—e.g., Liang et al. (2009) use data from the Hong Kong option market and they show, for a sample comprising 122 different options, that the use of nonparametric approaches (such as Multilayer Perceptron and Support Vector Machines (SVMs)) attained forecast errors twice as low as those generated by parametric methods. Ivascu (2021) uses a suite of nonparametric (Machine Learning) methods to price 1465 European call options on WTI crude oil, with crude oil futures contracts traded on the Chicago Mercantile Exchange between 03.01.2017 and 14.11.2018 as the underlying assets. He also finds that Machine Learning models outdistance conventional parametric ones. As Liang et al. (2009) and Pagnottoni (2019) show, a comprehensive approach would entail using classical parametric models in the first instance to *roughly* forecast the option price—this is aligned with our findings. Neural Networks and SVMs can subsequently be used to refine these forecasts. This strategy allows for a substantial minimisation of forecast error.

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Appendix

Proof of Proposition 1. For all $n \in N$, utility functions are concave and the budget sets convex (see, e.g., Toraubally, 2018). Hence, solving for the first-order necessary and sufficient conditions, we have at an interior NE that $\forall n \in N$ and $\forall k \in \mathcal{C}$:

$$\frac{\partial u_n(x_n)}{\partial x_{n,k}} \frac{B^i_{-n,k} Q^i_k}{(B^i_k)^2} - \frac{\partial u_n(x_n)}{\partial x_{n,m}} \frac{Q^i_{-n,k}}{Q^i_k} = 0; \tag{4}$$

$$-\frac{\partial u_n(x_n)}{\partial x_{n,k}} \frac{B^i_{-n,k}}{B^i_k} + \frac{\partial u_n(x_n)}{\partial x_{n,m}} \frac{Q^i_{-n,k} B^i_k}{(Q^i_k)^2} = 0; \tag{5}$$

$$\frac{\partial u_n(x_n)}{\partial x_{n,k}} \frac{B^f_{-n,k} Q^f_k}{(B^f_k)^2} - \frac{\partial u_n(x_n)}{\partial x_{n,m}} \frac{Q^f_{-n,k}}{Q^f_k} = 0; \tag{6}$$

$$-\frac{\partial u_n(x_n)}{\partial x_{n,k}} \frac{B^f_{-n,k}}{B^f_k} + \frac{\partial u_n(x_n)}{\partial x_{n,m}} \frac{Q^f_{-n,k} B^f_k}{(Q^f_k)^2} = 0. \tag{7}$$

Note that (4) is equivalent to (5), as is (6) to (7). Our conclusion can therefore be obtained by combining (4) and (6), and rearranging appropriately. □

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