# Chapter 1

# Modelling high-frequency waves using the radiative transfer equation

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**Abstract**. *Modelling high-frequency mechanical or electromagnetic waves is important for a wide range of applications, including noise and vibration of lightweight electric vehicles and mobile network coverage. We will explain the connection between high-frequency asymptotic solutions to frequency domain wave equations and the radiative transfer equation (RTE) via a kinetic formulation of the classical Hamiltonian ray dynamics. The numerical solution of the arising RTE requires a discretisation in both the position and direction coordinates. We will introduce a computational framework based on the Discontinuous Galerkin (DG) method in space, and compare the commonly used discrete ordinate method in direction with a Fourier based approach. The DG method can be viewed as a boundary integral method since we reformulate the PDE in a weak form including both domain and boundary integrals, the latter of which describe inter-elemental fluxes. Finally, we discuss the potential of this approach for analysing complex built-up structures, highlighting its promise for addressing challenges in high-frequency wave propagation in the future.*

## 1.1 Introduction

High-frequency wave phenomena arise in a myriad of applications, from seismology [1] and acoustics [2] to optics [3] and electromagnetism [4], and occur whenever the wavelength is significantly smaller than the characteristic length scales of the medium. This provides challenges in complex media/structures where the requirement to adequately resolve the rapid wave oscillations, as well as the complex geometric features can lead to very large and computationally costly numerical schemes as the frequency is increased [5, 6, 7]. In this paper, we explore the connection between high-frequency asymptotic solutions of frequency domain wave equations and the radiative transfer equation (RTE) through a kinetic formulation rooted in classical Hamiltonian ray dynamics. This is beneficial as it offers a framework that captures the detailed

local wave dynamics accurately and efficiently, which is crucial for modelling high-frequency waves, and thereby enabling the development of simulation techniques that are both more robust and computationally efficient in complex media.
During the presentation, we will explain in detail how the RTE can be numerically solved using a discretisation framework based on the Discontinuous Galerkin (DG) method. I will also present simulation results that underscore the potential of this integrated framework in addressing challenges in high-frequency wave modelling.

## 1.2 Monochromatic wave problems at high-frequencies

We begin by considering frequency domain wave equations of the form:

$$\Delta^{\alpha}\phi(x,y) - (ik)^{2\alpha}\phi(x,y) = 0, \tag{1.1}$$

for a given $\alpha = 1, 2$, where $k = \omega/c$ with $\omega$ being the angular frequency and $c > 0$ the wave speed in the medium [7]. When $\alpha = 1$, Eqn. (1.1) is the Helmholtz equation which models, for example, acoustic waves or in-plane waves in thin plates. When $\alpha = 2$ and $c \propto \sqrt{\omega}$, Eqn. (1.1) is the biharmonic wave equation describing the flexural motion of thin plates [6]. In the high-frequency regime for the Helmholtz equation, we adopt the usual ansatz [8]:

$$\phi(\boldsymbol{x}) = A(\boldsymbol{x})e^{i\omega S(\boldsymbol{x})} \tag{1.2}$$

with $\boldsymbol{x} = (x, y)$ representing the two-dimensional spatial coordinates, $A(\boldsymbol{x})$ denoting the amplitude function, and S($\boldsymbol{x}$) the phase function [7]. The idea is to replace the task of approximating the rapidly oscillating function $\phi$, with instead approximating the more slowly varying functions $A$ and $S$. Substituting (1.2) into the Helmholtz equation, we obtain:

$$\Delta^{\alpha}\left(A(\boldsymbol{x})e^{i\omega S(\boldsymbol{x})}\right) + k^2 A(\boldsymbol{x})e^{i\omega S(\boldsymbol{x})} = 0. \tag{1.3}$$

We can then expand in terms of $x$ and $y$ to obtain:

$$\frac{\partial^2(A(\boldsymbol{x})e^{i\omega S})}{\partial x^2} + \frac{\partial^2(A(\boldsymbol{x})e^{i\omega S})}{\partial y^2} + k^2 A e^{i\omega S} = 0. \tag{1.4}$$

Evaluating the second order partial derivatives and summing relevant terms we arrive at:

$$\begin{aligned}\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right)e^{i\omega S} + 2i\omega\left(\frac{\partial A}{\partial x}\frac{\partial S}{\partial x} + \frac{\partial A}{\partial y}\frac{\partial S}{\partial y}\right)e^{i\omega S} \\ + A\left[i\omega\left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2}\right) - \omega^2\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right)\right]e^{i\omega S} + k^2 A e^{i\omega S} = 0,\end{aligned} \tag{1.5}$$

which simplifies to

$$\Delta A + 2i\omega\left(\nabla A \cdot \nabla S\right) + i\omega A\Delta S - \omega^2 A|\nabla S|^2 + k^2 A = 0. \tag{1.6}$$

In the high-frequency regime $\omega \gg 1$, equation (1.6) is predominantly influenced by the highest order $\omega$ terms. Therefore, considering only the $\mathcal{O}(\omega^2)$ terms and noting that $k = \omega/c$ we obtain:

$$\frac{\omega^2}{c^2}A - \omega^2\left|\nabla S\right|^2 A = 0. \tag{1.7}$$

This can be rearranged to yield the standard form of the eikonal equation [6]

$$|\nabla S| = \frac{1}{c} = \eta, \tag{1.8}$$

where $\eta$ is referred to as the slowness. Similarly for the biharmonic wave equation, where $\alpha = 2$ in equation (1.1), the solution ansatz is modified to

$$\phi(\boldsymbol{x}) = A(\boldsymbol{x})e^{i\sqrt{\omega}S(\boldsymbol{x})}, \tag{1.9}$$

in order to ensure that $|\nabla S|$ is independent of $\omega$. Following a similar procedure to the one outlined for the Helmholtz equation, we again arrive at the eikonal equation

$$|\nabla S| = \frac{\sqrt{\omega}}{c} = \eta. \tag{1.10}$$

Note that since $c \propto \sqrt{\omega}$, then $|\nabla S|$ is frequency independent and thus $S$ may be assumed to be slowly varying. In general, the eikonal equation for the phase $S$ can be written as:

$$|\nabla S| = \frac{\omega^{\frac{\alpha-1}{2}}}{c} = \eta, \tag{1.11}$$

where $\alpha$ is either 1 or 2 for the Helmholtz or the biharmonic equations, respectively.

## 1.3 Derivation of the ray equations

We now shall seek to transform the eikonal equation (1.11) into an ODE system, by using the method of characteristics [9]. We introduce the notation

$$p_1 = \frac{\partial S}{\partial x}, \quad p_2 = \frac{\partial S}{\partial y}, \tag{1.12}$$

so that the square of the eikonal equation (1.11) becomes

$$p_1^2 + p_2^2 = \eta^2. \tag{1.13}$$

Also as an aside, since

$$p_1 = \frac{\partial S}{\partial x} \quad \text{then} \quad \frac{\partial p_1}{\partial y} = \frac{\partial^2 S}{\partial y \partial x} \tag{1.14}$$

and since

$$p_2 = \frac{\partial S}{\partial y} \quad \text{then} \quad \frac{\partial p_2}{\partial x} = \frac{\partial^2 S}{\partial x \partial y}. \tag{1.15}$$

Clairaut's theorem ensures that $\frac{\partial^2 S}{\partial y \partial x} = \frac{\partial^2 S}{\partial x \partial y}$ [11], so that we obtain

$$\frac{\partial p_1}{\partial y} = \frac{\partial p_2}{\partial x}, \tag{1.16}$$

which we shall use later. Now, we differentiate the eikonal equation (1.13) with respect to $x$ to obtain

$$2p_1 \frac{\partial p_1}{\partial x} + 2p_2 \frac{\partial p_2}{\partial x} = 2\eta \frac{\partial \eta}{\partial x}, \tag{1.17}$$

and similarly with respect to $y$ to give

$$2p_1\frac{\partial p_1}{\partial y} + 2p_2\frac{\partial p_2}{\partial y} = 2\eta\,\frac{\partial \eta}{\partial y}. \tag{1.18}$$

Dividing the equations (1.17) and (1.18) by $2\eta^2$ yields

$$\frac{p_1}{\eta^2}\frac{\partial p_1}{\partial x} + \frac{p_2}{\eta^2}\frac{\partial p_2}{\partial x} = \frac{1}{\eta}\frac{\partial \eta}{\partial x}, \tag{1.19}$$

$$\frac{p_1}{\eta^2}\frac{\partial p_1}{\partial y} + \frac{p_2}{\eta^2}\frac{\partial p_2}{\partial y} = \frac{1}{\eta}\frac{\partial \eta}{\partial y}. \tag{1.20}$$

We now introduce a parameter $t$ along the characteristic curves. The total derivative of $p_1$ along these curves is given by the chain rule:

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial p_1}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial p_1}{\partial y} = \frac{1}{\eta}\frac{\partial \eta}{\partial x}, \tag{1.21}$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial p_2}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial p_2}{\partial y} = \frac{1}{\eta}\frac{\partial \eta}{\partial y}. \tag{1.22}$$

Now by applying (1.16) and comparing equations (1.19) and (1.20) with equations (1.21) and (1.22) gives us

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p_1}{\eta^2}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{p_2}{\eta^2}. \tag{1.23}$$

Therefore, the full system of ODEs is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p_1}{\eta^2}, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{p_2}{\eta^2}, \quad \frac{\mathrm{d}p_1}{\mathrm{d}t} = \frac{1}{\eta}\frac{\partial \eta}{\partial x}, \quad \frac{\mathrm{d}p_2}{\mathrm{d}t} = \frac{1}{\eta}\frac{\partial \eta}{\partial y}. \tag{1.24}$$

We denote the momentum vector by $\boldsymbol{p} = (p_1, p_2)$ and the position by $\boldsymbol{x} = (x, y)$, as before, to give

$$\frac{d\boldsymbol{p}}{dt} = \frac{1}{\eta}\nabla_{\boldsymbol{x}}\eta \quad \text{and} \quad \frac{d\boldsymbol{x}}{dt} = \frac{1}{\eta^2}\boldsymbol{p}. \tag{1.25}$$

These equations, which describe the evolution of both $\boldsymbol{x}$ and $\boldsymbol{p}$ along the ray trajectories, are known as the *ray equations*. They can be recast in Hamiltonian form as

$$\frac{d\boldsymbol{x}}{dt} = \nabla_{\boldsymbol{p}}H(\boldsymbol{x}, \boldsymbol{p}), \quad \frac{d\boldsymbol{p}}{dt} = -\nabla_{\boldsymbol{x}}H(\boldsymbol{x}, \boldsymbol{p}), \tag{1.26}$$

with the Hamiltonian defined by

$$H(\boldsymbol{x}, \boldsymbol{p}) = \frac{|\boldsymbol{p}|}{\eta(\boldsymbol{x})}. \tag{1.27}$$

Therefore, we find that the ray dynamics follow Hamiltonian mechanics [12].

## 1.4 The Liouville equation and radiative transfer

Adopting a kinetic interpretation of the rays as trajectories of particles following the Hamiltonian dynamics of (1.26) and (1.27), we can introduce a phase-space particle density function $\rho(t, \boldsymbol{x}, \boldsymbol{p})$ that will satisfy the Liouville equation [7]:

$$\rho_t + \nabla_{\boldsymbol{p}}H \cdot \nabla_{\boldsymbol{x}}\rho - \nabla_{\boldsymbol{x}}H \cdot \nabla_{\boldsymbol{p}}\rho = 0. \tag{1.28}$$

Since we are solving frequency domain wave equations (1.1), which are considered independently of their harmonic time-dependence until after a solution of (1.1) has been found, we assume that in the high-frequency regime the particle density has reached a steady state. Therefore, we consider the stationary Liouville equation, with $\rho_t = 0$. Now let $u(\boldsymbol{x}, \boldsymbol{p})$ satisfy the stationary Liouville equation as follows

$$\nabla_{\boldsymbol{p}} H \cdot \nabla_{\boldsymbol{x}} u = \nabla_{\boldsymbol{x}} H \cdot \nabla_{\boldsymbol{p}} u. \tag{1.29}$$

Using equation (1.26), the stationary Liouville equation (1.29) can be re-written as

$$\frac{1}{\eta^2} \boldsymbol{p} \cdot \nabla_{\boldsymbol{x}} u = -\frac{1}{\eta} \nabla_{\boldsymbol{x}} \eta \cdot \nabla_{\boldsymbol{p}} u. \tag{1.30}$$

For a spatially constant slowness function $\eta$ we have $\nabla_{\boldsymbol{x}} \eta = 0$, and thus the stationary Liouville equation can be simplified again to give:

$$\frac{1}{\eta^2} \boldsymbol{p} \cdot \nabla_{\boldsymbol{x}} u = 0. \tag{1.31}$$

Note that in general, we will consider the slowness $\eta$ to be piecewise constant due to abrupt changes of media or material properties/parameters. Hence Eqn. (1.30) will model our system locally within each region of constant slowness. Fixing the energy as a constant normalised to unity, then the Hamiltonian $H \equiv 1$, and Eqn. (1.27) then gives us:

$$|\boldsymbol{p}| = \eta. \tag{1.32}$$

Taking $p_1$ and $p_2$ as the $x$ and $y$ components of the direction of the ray normalised so that $\boldsymbol{p}$ satisfies (1.32), we obtain

$$\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \eta \cos(\theta) \\ \eta \sin(\theta) \end{pmatrix}, \tag{1.33}$$

where $\theta \in [-\pi, \pi)$ is the angle between the ray vector and the positive $x$-axis. Therefore, Eqn. (1.31) can be written as:

$$\frac{1}{\eta} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = 0. \tag{1.34}$$

Now, setting $\hat{\boldsymbol{s}} = (\cos(\theta), \sin(\theta))^\intercal$, we obtain:

$$\frac{1}{\eta} \hat{\boldsymbol{s}} \cdot \nabla u = 0, \tag{1.35}$$

where we have simplified the notation and used $\nabla$ to represent $\nabla_{\boldsymbol{x}}$. Equation (1.35) describes the transport of energy along ray trajectories. However, it does not account for scenarios where the energy density is reduced as the wave travels through the medium, for example due to material damping. Therefore, to incorporate these energy losses, we introduce a dissipative term with decay rate $\mu$ in agreement with the Beer–Lambert law [13]. This leads to a modified equation

$$\hat{\boldsymbol{s}} \cdot \nabla u + \mu u = 0, \tag{1.36}$$

which is the standard radiative transfer equation without the scattering term [14]. In many contexts, such as tomography or meteorology, the scattering term in the RTE is essential to model the redirection of rays due to turbidity of the medium. However, for our application involving high-frequency waves in complex built-up media, we assume that the waves propagate unimpeded until they encounter a boundary or an abrupt change in material or medium.

## 1.5 Conclusions

In this work, we demonstrated that applying a high-frequency asymptotic ansatz to classical wave equations leads naturally to the eikonal equation, which effectively characterises the evolution of the wave phase. By utilising the method of characteristics, we transformed the eikonal equation into a Hamiltonian system of ray equations, thereby establishing a robust framework for tracking the evolution of ray trajectories. Finally, we introduced a phase-space density that satisfies the stationary Liouville equation, laying the foundation for deriving the radiative transfer equation without scattering. During the conference, I will discuss the numerical solution of the aforementioned RTE by discretising both the spatial and directional variables. We will introduce a computational framework that utilises DG for spatial discretisation, and we will compare the conventional discrete ordinate method with a Fourier-based approach for the directional discretisation. Ultimately, our aim is to apply this integrated framework to simulate high-frequency wave propagation in complex built-up structures.

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