Chapter 1

Solving neural field equations on curved surfaces with isogeometric collocation

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Abstract. *The aim of this study is to develop a new numerical algorithm for simulating neural fields on curved geometries, and eventually human brain surfaces, using Computer-Aided Design (CAD) techniques. We present a CAD-integrated analysis approach, referred to as isogeometric collocation, for solving neural field equations on surfaces that closely resemble cortical geometries typically derived from neuroimaging data. Our methodology involves solving partial integro-differential equations directly using isogeometric collocation techniques, combined with efficient numerical procedures such as heat methods for determining geodesic distances between neural units. To demonstrate the effectiveness of our approach, we initially investigate localised activity patterns in a two-dimensional neural field equation posed on a torus, with the eventual goal of extending the analysis to human brain geometries derived directly from neuroimaging point cloud data. We aim to establish a comprehensive methodology that seamlessly integrates realistic geometries with the analysis of partial integro-differential equations in computational neuroscience. This study is particularly significant for two reasons. Firstly, it addresses the highly irregular nature of point clouds derived from modern neuroimaging data while mitigating the limitations of current time-consuming numerical methods. Secondly, by employing efficient geodesic computation schemes, this approach not only models pattern formation on realistic cortical geometries but can also accommodate cortical architectures of greater physiological relevance.*

1.1 Introduction

To understand the emergent dynamics of large neuronal populations in the cortex, neural field theory utilises a continuum description [1]. These models, typically formulated as non-linear partial integro-differential equations, have garnered significant interest due to their their rich mathematical structure and their ability to reproduce experimental neuroscientific findings [2, 3]. Typically, such models take the form

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = -u(\mathbf{x},t) + \int_{\Omega} w(\mathbf{x},\mathbf{x}')f(u(\mathbf{x}',t))d\mathbf{x}'. \tag{1.1}$$

In this context, $u(\mathbf{x},t)$ represents the mean neural activity at position $\mathbf{x} \in \Omega$ and time t. The non-linear function f, typically a sigmoid, models the mean firing rate. The connectivity kernel, w, specifies the interaction strength between neurons at positions $\mathbf{x}$ and $\mathbf{x}'$ [4].

In planar domains and under specific selections of the integral kernel, the neural field model (NFM) in (1.1) reduces to a partial differential equation (PDE), and the theory and tools of PDEs can be deployed to investigate its solutions [5]. However, dealing with physiologically realistic geometries and kernels requires not only the direct solution of the integral form of (1.1) but also the computation of geodesic distances. A notable recent contribution in this area was by Visser *et al.* [6], who employed the full integral formulation of a time-delayed NFM to study wave phenomena on a sphere; however, their approach is constrained by the requirement of analytically available geodesic distances. To address the challenges posed by more complex geometries, Martin *et al.* [7] recently developed a methodology that combines the numerical calculation of geodesic distances with efficient collocation techniques to solve NFMs such as that in (1.1) on general closed surfaces. However, this approach necessitates a triangulated representation of the geometry, a significant limitation as collocation and integration points are constrained to the mesh, potentially leading to inaccuracies in representing the true geometry, especially with low-order interpolation.

To address the limitations identified in [7], in this work, we utilise isogeometric collocation (IGA-C), a method first introduced by Hughes *et al.* [8, 9] in the context of structural mechanics applications. Capitalising on the unified geometry of IGA-C methods for surface representation and numerical simulations, our novel framework incorporates mesh-free geodesic algorithms to solve NFMs, such as (1.1), on closed two-dimensional surfaces, thereby avoiding the restrictive nature of previous mesh-dependent approaches. The mesh-free nature of our method offers greater flexibility in solver selection and, crucially, the freedom to position collocation and integration points independently of any mesh structure. This added flexibility enables efficient upscaling of the method, a key requirement for simulating large-scale cortical-like geometries.

1.2 Mathematical model

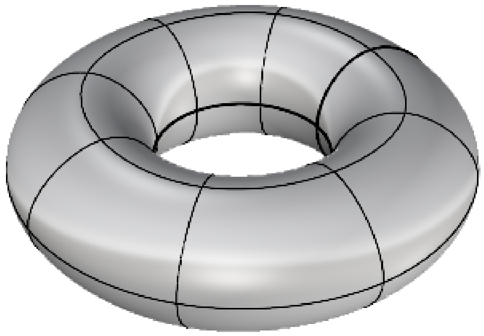
To solve the neural field model (1.1), we make the following choices regarding the functional forms of the connectivity function and firing rate function. The functional form for w is given by

$$w(\mathbf{x},\mathbf{x}') = 2e^{-d(\mathbf{x},\mathbf{x}')^2} - 0.34e^{-0.2d(\mathbf{x},\mathbf{x}')^2}, \tag{1.2}$$

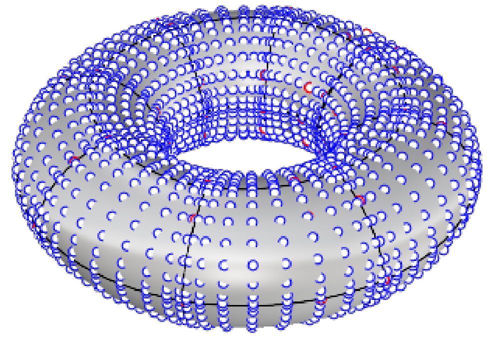
where $d(\mathbf{x},\mathbf{x}')$ signifies the geodesic distance between points $\mathbf{x}$ and $\mathbf{x}'$ on the surface of interest, which we take to be a torus ($\Omega = \mathbb{T}^2$) in this work (see subsection 1.3.1 for details on geodesic distance calculation).

The transformation from population activity u to firing frequency is governed by the sigmoidal firing rate function:

$$f(u) = \frac{1}{1 + e^{-\beta(u-h)}}, \tag{1.3}$$



(a)



(b)

Figure 1.1: (a) NURBS surface representation of a torus with 32 patches. (b) Collocation points (red) and quadrature points (blue) added to patches for IGA-C implementation.

with β determining the rate of the transition. In our experiments, the parameters in (1.3) are chosen as in [10], that is $h = 0.8$ and $\beta = 5$.

1.3 Discretisation using isogeometric collocation

A CAD representation of the torus is expressed using a NURBS geometry as

$$\mathbf{x}(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}^{p,q}(\xi, \eta) \mathbf{P}_{i,j}, \tag{1.4}$$

where $\{P_{i,j}\}$ refer to the control points and $\{R_{i,j}^{p,q}\}$ refer to the bivariate NURBS basis [11] given by

$$R_{i,j}^{p,q}(\xi, \eta) = \frac{N_{i,p}(\xi) N_{j,q}(\eta)}{\sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi) N_{j,q}(\eta) \omega_{ij}}. \tag{1.5}$$

Figure 1.1 shows a NURBS representation of a torus with major radius of curvature $R = 4.5$ and minor radius of curvature $r = 2.5$. Here, we have used cubic B-splines in both the ξ and η directions and a structured grid of 11×7 control points defining the surface. Each basis function is influenced by four control points per segment.

To solve Equation (1.1) using the IGA-C method, we approximate the unknown function $u(\mathbf{x}, t)$ using the same NURBS basis as introduced above:

$$u_h(\mathbf{x}, t) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}^{p,q}(\xi, \eta) u_{i,j}(t), \tag{1.6}$$

where $u_{i,j}(t)$ are unknown control variables, and u_h is our numerical approximation of u.

Rewriting the approximation in terms of physical coordinates using (1.4) and suppressing the degrees yields:

$$u_h(\mathbf{x}, t) = \sum_{i=1}^{n} \sum_{j=1}^{m} u_{i,j}(t) R_{i,j}(\mathbf{x}). \tag{1.7}$$

Substituting this expression into the governing equation (1.1) and enforcing equality at a set of N collocation points $\{\mathbf{x}_j\}_{j=1}^{N}$, we arrive at the following ordinary differential equation approximation:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\mathrm{d}u_{i,j}(t)}{\mathrm{d}t}R_{i,j}(\mathbf{x}_j) = &-\sum_{i=1}^{n}\sum_{j=1}^{m}u_{i,j}(t)R_{i,j}(\mathbf{x}_j)\\ &+\int_{\Omega}w(\mathbf{x}_j,\mathbf{x}')f\left(\sum_{i=1}^{n}\sum_{j=1}^{m}u_{i,j}(t)R_{i,j}(\mathbf{x}')\right)\mathrm{d}\mathbf{x}'\end{aligned}\tag{1.8}$$

for $j = 1,\ldots,N$. The selection of the collocation points $\{\mathbf{x}_j\}$ significantly impacts the precision and stability of the numerical solution. Common and effective choices for these points include Greville abscissae and superconvergent points [12].

To numerically evaluate the integral term in equation (1.8), we employ numerical quadrature. This approximation takes the form:

$$\int_{\Omega}w(\mathbf{x}_j,\mathbf{x}')f\left(\sum_{i=1}^{n}\sum_{j=1}^{m}u_{i,j}(t)R_{i,j}(\mathbf{x}')\right)d\mathbf{x}' \approx \sum_{k=1}^{M}W_k w(\mathbf{x}_j,\mathbf{x}_k)f\left(\sum_{i=1}^{n}\sum_{j=1}^{m}u_{i,j}(t)R_{i,j}(\mathbf{x}_k)\right),$$

where W_k represent the quadrature weights (see [13] for details). The resulting system of ordinary differential equations is then solved using MATLAB's built-in ODE solvers.

For further details, including a recent neuroscience application, see [12, 14, 15].

1.3.1 Heat method for computing geodesics for point cloud data

The key difference between solving the NFM in (1.1) on a curved domain as opposed to a planar domain is the computation of the distance function, $d(\mathbf{x},\mathbf{x}')$, appearing in (1.2), which, except for a small number of cases, must be computed numerically. Geodesic distances in our work are computed using the *heat method* [16]. This technique determines these distances by efficiently solving Varadhan's formula, which provides a fundamental relationship between distance and heat flow on Riemannian manifolds and which is given by

$$d(\mathbf{x},\mathbf{x}') = \lim_{t\longrightarrow 0}\sqrt{-4t\log k_{t,\mathbf{x}}(\mathbf{x}')}.$$

The function $k_{t,\mathbf{x}}(\mathbf{x}')$ is called the *heat kernel*, and it measures the heat transferred from a source $\mathbf{x}$ to a destination $\mathbf{x}'$ in time t. Importantly, the method can be applied to extremely general geometric discretisations, including point-cloud representations. Figure **??** shows the heat method applied directly to a point cloud representation of the torus without any connectivity information.

1.4 Conclusions and future work

In this paper, we have presented a novel computational technique to solve neural field models on curved geometries. An important feature of this approach is its mesh-free nature, which allows us to circumvent many of the limitations encountered by earlier attempts to solve NFMs on curved geometric domains. During the presentation we will provide a detailed explanation of the IGA-C technique's application to solving nonlinear NFMs, such as the one presented in

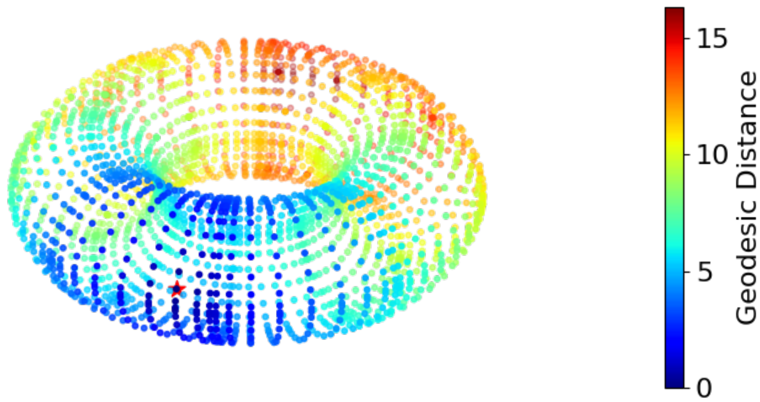


Figure 1.2: The heat method applied to a point cloud representation of a torus, where the point cloud consists of collocation points and Gauss quadrature points allocated on patches, as shown in Fig. 1.1b. The geodesic distance is computed from a collocation point, highlighted with a star.

(1.1), and will demonstrate its potential through the presentation of simulation results obtained on the curved surface of a torus. Future investigations will center on the challenge of scaling the current methodology to more intricate geometries resembling those found in the cortex, motivated by the long-term objective of characterising the effects of curvature on the pattern formation of non-local models such as (1.1).

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