Modelling high-frequency waves using the radiative transfer equation

Robert Lockett[1] and David Chappell
Nottingham Trent University
Clifton Campus
Nottingham, NG11 8NS

ABSTRACT

The effective modelling of high-frequency waves is critical to accurately describing noise generation and control in a broad range of engineering applications. We will explain the connection between high-frequency asymptotic solutions to frequency domain wave equations and the radiative transfer equation (RTE) via a kinetic formulation of the classical Hamiltonian ray dynamics. The numerical solution of the arising RTE requires a discretisation in both the position and direction coordinates. We will introduce a computational framework based on the Discontinuous Galerkin method in space and compare the commonly used discrete ordinates method in direction with a Fourier-based approach. This work builds on previous research into high-frequency vibroacoustic simulation methods, having the same advantages as Dynamical Energy Analysis (DEA), which offers higher precision than the more established Statistical Energy Analysis. The RTE approach adopted here is beneficial since it uses a more familiar finite element based methodology and the degree of the local spatial polynomial approximation can be increased more simply and at lower computational cost than in DEA. We then discuss future work to extend our approach for analysing complex built-up structures, incorporating multi-material modelling and the subsequent wave interactions such as refraction and reflection at the inter-element boundaries.

1. INTRODUCTION

High-frequency wave phenomena arise in a wide range of noise and vibration problems, [1–4], and occur whenever the wavelength is significantly smaller than the characteristic length scales of a vibrating structure. This provides challenges in complex structures where the requirement to adequately resolve the rapid wave oscillations, as well as the complex geometric features can lead to very large and computationally costly numerical schemes as the frequency is increased [5–7]. In this paper, we explore the connection between high-frequency asymptotic solutions of frequency domain wave equations and the radiative transfer equation (RTE) through a kinetic formulation rooted in classical Hamiltonian ray dynamics. This is beneficial as it offers a framework that captures the detailed local wave dynamics accurately and efficiently, which is crucial for modelling high-frequency waves, and thereby enabling the development of simulation techniques that are both robust and computationally efficient in complex structures. We also explain how the RTE is numerically solved using a discretisation framework based on the Discontinuous Galerkin (DG) method. During the presentation, I will discuss simulation results that underscore the potential of this integrated framework in addressing high-frequency wave propagation challenges.

2. MONOCHROMATIC WAVE PROBLEMS AT HIGH-FREQUENCIES

We begin by considering frequency domain wave equations of the form:

$$\Delta^{\alpha}\phi(x,y) - (ik)^{2\alpha}\phi(x,y) = 0, \tag{1}$$

for a given $\alpha = 1, 2$, where $k = \omega/c$ with ω being the angular frequency and $c > 0$ the wave speed in the structure [7]. When $\alpha = 1$, Equation 1 is the Helmholtz equation which models, for example,

[1]rob.lockett2017@my.ntu.ac.uk (corresponding author)

acoustic waves or in-plane waves in thin plates. When $\alpha = 2$ and $c(k) \propto \sqrt{\omega}$, Equation 1 is the biharmonic wave equation describing the flexural motion of thin plates [6]. In the high-frequency regime for the Helmholtz equation, we adopt the usual ansatz [8]:

$$\phi(\boldsymbol{x}) = A(\boldsymbol{x})e^{i\omega S(\boldsymbol{x})} \tag{2}$$

with $\boldsymbol{x} = (x, y)$ representing the two-dimensional spatial coordinates, $A(\boldsymbol{x})$ denoting the amplitude function, and $\mathrm{S}(\boldsymbol{x})$ the phase function [7]. Substituting Equation 2 into the Helmholtz equation, we obtain:

$$\Delta\left(A(\boldsymbol{x})e^{i\omega S(\boldsymbol{x})}\right) + k^2 A(\boldsymbol{x})e^{i\omega S(\boldsymbol{x})} = 0. \tag{3}$$

Equation 3 can be simplified to give:

$$\Delta A + 2i\omega\left(\nabla A \cdot \nabla S\right) + i\omega A\Delta S - \omega^2 A|\nabla S|^2 + k^2 A = 0. \tag{4}$$

In the high-frequency regime $\omega \gg 1$, and thus Equation 4 is predominantly influenced by the highest order ω terms. Therefore, considering only the $O(\omega^2)$ terms and noting that $k = \omega/c$ we obtain:

$$\frac{\omega^2}{c^2}A - \omega^2 |\nabla S|^2 A = 0. \tag{5}$$

Equation 5 can be rearranged to yield the Eikonal equation [6]

$$|\nabla S| = \frac{1}{c} = \eta, \tag{6}$$

where η is referred to as the slowness. Similarly for the biharmonic wave equation, where $\alpha = 2$ in Equation 1, the frequency domain wave equation is:

$$\Delta^2\phi(x, y) - k^4\phi(x, y) = 0. \tag{7}$$

Here the solution ansatz is modified to

$$\phi(\boldsymbol{x}) = A(\boldsymbol{x})e^{i\sqrt{\omega}S(\boldsymbol{x})}, \tag{8}$$

in order to ensure that $|\nabla S|$ is independent of ω. Following a similar procedure to the one outlined for the Helmholtz equation, we again arrive at the Eikonal equation

$$|\nabla S| = \frac{\sqrt{\omega}}{c} = \eta. \tag{9}$$

Note that since $c(k) \propto \sqrt{\omega}$, then $|\nabla S|$ is frequency independent and thus S may be assumed to be slowly varying. In general, the Eikonal equation for the phase S can be written as:

$$|\nabla S| = \frac{\omega^{\frac{\alpha-1}{2}}}{c} = \eta, \tag{10}$$

where α is either 1 or 2 for the Helmholtz or the biharmonic equations, respectively.
We now shall seek to transform the Eikonal Equation 10 into an ordinary differential equation (ODE) system, by using the method of characteristics [9]. To do so we denote the momentum vector by $\boldsymbol{p} = (p_1, p_2)$ and the position by $\boldsymbol{x} = (x, y)$. The method of characteristics gives

$$\frac{d\boldsymbol{p}}{dt} = \frac{1}{\eta}\nabla\eta \quad \text{and} \quad \frac{d\boldsymbol{x}}{dt} = \frac{1}{\eta^2}\boldsymbol{p}, \tag{11}$$

where these equations describe the evolution of both $\boldsymbol{x}$ and $\boldsymbol{p}$ along the ray trajectories. These ODEs are known as the *ray equations* and can be recast in Hamiltonian form as

$$\frac{d\boldsymbol{x}}{dx} = \nabla_{\boldsymbol{p}}H(\boldsymbol{x}, \boldsymbol{p}), \quad \frac{d\boldsymbol{p}}{dx} = -\nabla_{\boldsymbol{x}}H(\boldsymbol{x}, \boldsymbol{p}), \tag{12}$$

with the Hamiltonian defined by

$$H(\boldsymbol{x}, \boldsymbol{p}) = \frac{|\boldsymbol{p}|}{\eta(\boldsymbol{x})}. \tag{13}$$

Therefore, we find that the ray dynamics follow Hamiltonian mechanics [10].

3. FROM THE LIOUVILLE EQUATION TO RADIATIVE TRANSFER

Adapting a kinetic interpretation of the rays as trajectories of particles following the Hamiltonian dynamics of Equation 12 and Equation 13, then we can introduce a phase-space particle density function $\rho(t, \boldsymbol{x}, \boldsymbol{p})$ that will satisfy the Liouville equation [7]:

$$\rho_t + \nabla_{\boldsymbol{p}} H \cdot \nabla_{\boldsymbol{x}} \rho - \nabla_{\boldsymbol{x}} H \cdot \nabla_{\boldsymbol{p}} \rho = 0. \tag{14}$$

Since we are interested in frequency domain wave equations (see Equation 1), which are considered without their harmonic time-dependence until after a solution has been found, we assume that in the high-frequency regime the particle density has reached a steady state. Therefore, we consider the stationary Liouville equation, where $\rho_t = 0$. Now let $u(\boldsymbol{x}, \boldsymbol{p})$ satisfy the stationary Liouville equation as follows

$$\nabla_{\boldsymbol{p}} H \cdot \nabla_{\boldsymbol{x}} u = \nabla_{\boldsymbol{x}} H \cdot \nabla_{\boldsymbol{p}} u. \tag{15}$$

Using Equation 12, Equation 15 can be re-written as

$$\frac{1}{\eta^2} \boldsymbol{p} \cdot \nabla_{\boldsymbol{x}} u = -\frac{1}{\eta} \nabla_{\boldsymbol{x}} \eta \cdot \nabla_{\boldsymbol{p}} u. \tag{16}$$

For a spatially constant slowness function η we have $\nabla_{\boldsymbol{x}} \eta = 0$ and thus the stationary Liouville equation can be simplified again to give:

$$\frac{1}{\eta^2} \boldsymbol{p} \cdot \nabla_{\boldsymbol{x}} u = 0. \tag{17}$$

Note that in general, we will consider the slowness η to be piecewise constant due to changes in the structure or material properties/parameters. Hence Equation 16 will model our system locally within each region of constant slowness. Fixing the energy as a constant normalised to unity, then the Hamiltonian $H \equiv 1$ and Equation 13 gives

$$|\boldsymbol{p}| = \eta. \tag{18}$$

Taking p_1 and p_2 as the x and y components of the direction of the ray normalised so that $\boldsymbol{p}$ satisfies Equation 18, we obtain

$$\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \eta \cos(\theta) \\ \eta \sin(\theta) \end{pmatrix}, \tag{19}$$

where $\theta \in [-\pi, \pi)$ is the angle between the ray vector and the positive x-axis. Therefore, Equation 17 can be written as:

$$\frac{1}{\eta^2} \begin{pmatrix} \eta \cos(\theta) \\ \eta \sin(\theta) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = 0. \tag{20}$$

Now, setting $\hat{\boldsymbol{s}} = (\cos(\theta), \sin(\theta))^\intercal \in S^1$ (the unit circle), we obtain:

$$\frac{1}{\eta} \hat{\boldsymbol{s}} \cdot \nabla u = 0, \tag{21}$$

where we have simplified the notation and used ∇ to represent $\nabla_{\boldsymbol{x}}$. Equation 21 describes the transport of energy along ray trajectories. However, it does not account for realistic scenarios where the energy density is typically reduced as the wave travels through the structure, for example due to material damping. Therefore, to incorporate these energy losses, we introduce a viscous damping term with decay rate μ in agreement with the Beer–Lambert law [11]. This leads to the modified equation

$$\hat{\boldsymbol{s}} \cdot \nabla u + \mu u = 0, \tag{22}$$

where Equation 22 is the homogeneous radiative transfer equation without the scattering term [12]. In many contexts, such as tomography or meteorology, the absent scattering term is essential to model the redirection of rays due to inhomogeneities in the structure. However, for our application involving high-frequency waves in complex built-up media, we assume that the waves propagate unimpeded until they encounter a boundary or an abrupt change in material. To account for internal emitters or energy injections, we also consider an external source term f on the right-hand side, yielding

$$\hat{\boldsymbol{s}} \cdot \nabla u + \mu u = f. \tag{23}$$

4. DISCRETISATION OF THE RADIATIVE TRANSFER EQUATION

In this section, we describe the discretisation process for numerically solving the radiative transfer equation (RTE), addressing both spatial and directional dependence. For the spatial discretisation, we employ the Discontinuous Galerkin (DG) method, while for the directional discretisation, we compare the discrete ordinates method with a Fourier-based approach.

Consider a spatial domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$. We begin by applying the DG method to Equation 23, discretising the spatial domain Ω into N_h non-overlapping elements E_j, where $j = 1, 2, \ldots, N_h$. Each element E_j has boundaries defined by κ edges, denoted $\Gamma_{i,j}$, for $i = 1, 2, \ldots, \kappa$. To capture the essential spatial and angular behaviour within the element E_j we introduce a local approximate solution $u_j(\boldsymbol{x}, \theta) \approx u(\boldsymbol{x}, \theta)$ for $\boldsymbol{x} \in E_j$ and $u_j(\boldsymbol{x}, \theta) = 0$ otherwise, for each $j = 1, 2, \ldots, N_h$. To approximate the spatial dependence of u_j we use orthonormal Legendre polynomials of degree $n = 0, 1, 2, \ldots$, given as

$$\tilde{P}_n(\xi) = \sqrt{\frac{2n+1}{2}} P_n(\xi), \quad -1 \le \xi \le 1, \tag{24}$$

where P_n denotes the standard Legendre polynomials of degree n. The scaled Legendre polynomials satisfy $\int_{-1}^{1} \tilde{P}_m(\xi)\, \tilde{P}_n(\xi)\, \mathrm{d}\xi = \delta_{mn}$, where δ_{mn} is the Kronecker delta. To implement these Legendre polynomials as basis functions, we first enclose each element $E_j \subset \Omega$ in the smallest possible Cartesian bounding box $B_{E_j} = \left(x_c - \frac{h_x}{2}, x_c + \frac{h_x}{2}\right) \times \left(y_c - \frac{h_y}{2}, y_c + \frac{h_y}{2}\right)$ with side-lengths h_x and h_y, and centroid (x_c, y_c). We define the affine map as

$$(\hat{x}, \hat{y}) = \hat{\boldsymbol{x}} = F_j(\boldsymbol{x}) = \left(\frac{2\,(x - x_c)}{h_x}, \frac{2\,(y - y_c)}{h_y} \right), \tag{25}$$

which carries B_{E_j} on to the reference box $\hat{B}_{E_j} = (-1, 1)^2$. The Jacobian of the inverse map F_j^{-1} is given as $|\boldsymbol{J}_E| = \frac{1}{4} h_x h_y$ and as $|\boldsymbol{J}_\Gamma| = \|\boldsymbol{J}_E^\top \hat{\boldsymbol{n}}\|\, |\boldsymbol{J}_E|$ along the boundary Γ [13]. We express the Legendre polynomials in terms of the local co-ordinates $\hat{\boldsymbol{x}}$ as we employ the exact integration method outlined in [13], which is based on integration with respect to $\hat{\boldsymbol{x}}$.

To discretise directionally, we employ two distinct methods, the discrete ordinates method or a Fourier approach. The discrete ordinates method is well-suited for problems where fluxes propagate in discrete directions, whereas the Fourier-based discretisation is more effective when handling unidirectional forces. Firstly, in the Fourier approach we combine a directional Fourier expansion with the spatial Legendre polynomial basis $\tilde{P}_{\alpha_1}(\hat{x})\, \tilde{P}_{\alpha_2}(\hat{y})$, from Equation 24, with values of α_1 and α_2 containing the degrees of the Legendre polynomials in $\hat{x}$ and $\hat{y}$ and satisfying $\alpha_1 + \alpha_2 \le k$. Altogether, in the Fourier approach we expand $u_j(\hat{\boldsymbol{x}}, \theta)$ as

$$u_j(\hat{\boldsymbol{x}}, \theta) = \sum_{\alpha_1=0}^{k} \sum_{\alpha_2=0}^{k-\alpha_1} \sum_{n=-N}^{N} \hat{u}_j \tilde{P}_{\alpha_1}(\hat{x})\, \tilde{P}_{\alpha_2}(\hat{y}) \frac{e^{in\theta}}{\sqrt{2\pi}}. \tag{26}$$

Now, the weak form is found by multiplying by a test function $v(\hat{\boldsymbol{x}}, \theta)$, drawn from the same finite-dimensional space as our spatial and Fourier basis functions in Equation 26, and integrating of E_j and

θ. In the Fourier approach, this test function is set to be

$$v_j(\hat{\boldsymbol{x}}, \theta) = \tilde{P}_{\beta_1}(\hat{x})\, \tilde{P}_{\beta_2}(\hat{y}) \frac{e^{-im\theta}}{\sqrt{2\pi}}, \tag{27}$$

with $m = -N, -N+1, \ldots, N$ and (β_1, β_2) taking the same values as the corresponding indices (α_1, α_2) in Equation 26. Substituting the approximation u_j into Equation 23, multiplying by v_j and integrating over E_j and $\theta \in [-\pi, \pi)$ then yields

$$\int_{-\pi}^{\pi} \int_{E_j} \left(\hat{\boldsymbol{s}}(\theta) \cdot \nabla u_j(\hat{\boldsymbol{x}}, \theta) + \mu u_j(\hat{\boldsymbol{x}}, \theta) \right) v_j(\hat{\boldsymbol{x}}, \theta) \, |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} d\theta = \int_{-\pi}^{\pi} \int_{E_j} f(\hat{\boldsymbol{x}}, \theta)\, v_j(\hat{\boldsymbol{x}}, \theta) \, |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} d\theta. \tag{28}$$

Since $\hat{\boldsymbol{s}}$ is independent of spatial coordinates $\boldsymbol{x}$, we note that $\hat{\boldsymbol{s}} \cdot \nabla u_j = \nabla \cdot (\hat{\boldsymbol{s}} u_j)$. Using this relationship, Equation 28 can be expressed as

$$\begin{aligned} \int_{-\pi}^{\pi} \int_{E_j} (\nabla \cdot (\hat{\boldsymbol{s}}(\theta)\, u_j(\hat{\boldsymbol{x}}, \theta))) v_j(\hat{\boldsymbol{x}}, \theta) \, |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} d\theta + \int_{-\pi}^{\pi} \int_{E_j} \mu u_j(\hat{\boldsymbol{x}}, \theta)\, v_j(\hat{\boldsymbol{x}}, \theta) \, |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} d\theta \\ = \int_{-\pi}^{\pi} \int_{E_j} f(\hat{\boldsymbol{x}}, \theta)\, v_j(\hat{\boldsymbol{x}}, \theta) \, |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} d\theta. \end{aligned} \tag{29}$$

Green's first identity gives

$$\int_{-\pi}^{\pi} \int_{E_j} (\nabla \cdot (\hat{\boldsymbol{s}} u_j)) v_j |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} d\theta = - \int_{-\pi}^{\pi} \int_{E_j} (\hat{\boldsymbol{s}} \cdot \nabla v_j) u_j |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} d\theta + \sum_{i=1}^{\kappa} \int_{-\pi}^{\pi} \int_{\Gamma_{i,j}} \gamma_{i,j} v_j |\boldsymbol{J}_{\Gamma_{i,j}}| d\hat{\boldsymbol{x}} d\theta, \tag{30}$$

where on each face $\Gamma_{i,j} \subset E_j$, we define the numerical flux $\gamma_{i,j}$ by an upwind scheme. Let $\hat{\mathbf{n}}_{i,j}$ be the outward unit normal on $\Gamma_{i,j}$ from element E_j. Then $\gamma_{i,j}(\theta) = (\hat{\mathbf{s}}(\theta) \cdot \hat{\mathbf{n}}_{i,j})\, u^{\mathrm{p}}_{i,j}(\theta)$, where $u^{\mathrm{p}}_{i,j}$ is

$$u^{\mathrm{p}}_{i,j}(\hat{\boldsymbol{x}}, \theta) = \begin{cases} u_j(\hat{\boldsymbol{x}}_{\boldsymbol{i}}, \theta), & \text{if } \hat{\mathbf{s}}(\theta) \cdot \hat{\mathbf{n}}_{i,j} > 0, \\ u_{j_+}(\hat{\boldsymbol{x}}_{i+}, \theta), & \text{otherwise,} \end{cases} \tag{31}$$

and where $\hat{\boldsymbol{x}}_{i+}$ denotes a point on the boundary of the neighbour element E_{j_+} that lies upwind of E_j, connected via $\Gamma_{i,j} = \Gamma_{i+,j+} \subset E_{j+}$ [14]. Applying Equation 30 to the divergence term in Equation 29 yields

$$\begin{aligned} \int_{-\pi}^{\pi} \int_{E_j} \mu u_j(\hat{\boldsymbol{x}}, \theta)\, v_j(\hat{\boldsymbol{x}}, \theta) \, |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} d\theta - \int_{-\pi}^{\pi} \int_{E_j} u_j(\hat{\boldsymbol{x}}, \theta) (\hat{\boldsymbol{s}}(\theta) \cdot \nabla v_j(\hat{\boldsymbol{x}}, \theta)) |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} d\theta \\ + \sum_{i=1}^{\kappa} \int_{-\pi}^{\pi} \int_{\Gamma_{i,j}} \gamma_{i,j}(\hat{\boldsymbol{x}}, \theta)\, v_j(\hat{\boldsymbol{x}}, \theta) \, |\boldsymbol{J}_{\Gamma_{i,j}}| d\hat{\boldsymbol{x}} d\theta = \int_{-\pi}^{\pi} \int_{E_j} f(\hat{\boldsymbol{x}}, \theta)\, v_j(\hat{\boldsymbol{x}}, \theta) \, |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} d\theta. \end{aligned} \tag{32}$$

In the discrete ordinates method we approximate $u(\hat{\boldsymbol{x}}, \theta)$ by its values at a finite set of N_θ directions θ_n for $n = 1, 2, \ldots, N_\theta$ which we take to be uniformly spaced between $[-\pi, \pi)$. For each θ_n and $\boldsymbol{x}$ in element E_j we approximate our solution by

$$u(\hat{\boldsymbol{x}}, \theta_n) \approx u_j(\hat{\boldsymbol{x}}, \theta_n) = u^n_j(\hat{\boldsymbol{x}}) = \sum_{\alpha_1=0}^{k} \sum_{\alpha_2=0}^{k-\alpha_1} \hat{u}^n_j \tilde{P}_{\alpha_1}(\hat{x})\, \tilde{P}_{\alpha_2}(\hat{y}), \quad \alpha_1 + \alpha_2 \leq k, \quad n = 1, 2, \ldots, N_\theta, \tag{33}$$

and choose corresponding test functions to be

$$w_j(\hat{\boldsymbol{x}}) = \tilde{P}_{\beta_1}(\hat{x})\, \tilde{P}_{\beta_2}(\hat{y}), \quad \beta_1 + \beta_2 \leq k. \tag{34}$$

Similarly to the Fourier approach, we substitute Equation 33 into Equation 23 for $\theta = \theta_n$, $n = 1, \ldots, N_\theta$, multiply by w_j from Equation 34 and integrate over E_j to obtain

$$\int_{E_j} \mu u_j^n(\hat{\boldsymbol{x}})\, w_j(\hat{\boldsymbol{x}})\, |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} - \int_{E_j} u_j^n(\hat{\boldsymbol{x}})\, (\hat{\boldsymbol{s}}(\theta_n) \cdot \nabla w_j(\hat{\boldsymbol{x}}))|\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}} + \sum_{i=1}^{\kappa} \int_{\Gamma_{i,j}} \gamma_{i,j}^n(\hat{\boldsymbol{x}})\, w_j(\hat{\boldsymbol{x}})\, |\boldsymbol{J}_{\Gamma_{i,j}}| d\hat{\boldsymbol{x}} = \int_{E_j} f(\hat{\boldsymbol{x}}, \theta_n)\, w_j(\hat{\boldsymbol{x}})\, |\boldsymbol{J}_{E_j}| d\hat{\boldsymbol{x}}, \quad (35)$$

where $\gamma_{i,j}^n = \gamma_{i,j}(\theta_n)$.
Now, Equation 32 for the Fourier-based approach and Equation 35 for the discrete ordinates method, can both be written more succinctly as a matrix system $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{v}$, where $\boldsymbol{A}$ and $\boldsymbol{v}$ are assembled from the left hand and right hand sides of each equation respectively and $\boldsymbol{u}$ is the solution vector. The global solution vector $\boldsymbol{u}$ is assembled by concatenating the element-wise blocks $\boldsymbol{u}_j$ for $j = 1, \ldots, N_h$ via

$$\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{N_h})^\top. \quad (36)$$

Each block $\boldsymbol{u}_j$ gathers all radiance coefficients, $u_{j,\alpha_1,\alpha_2}^n$, corresponding to the spatial and the angular discretisation. Each $\boldsymbol{u}_j$ is of size $\frac{1}{2}(k+1)(k+2)(2N+1)$ for the Fourier approach and $\frac{1}{2}(k+1)(k+2)N_\theta$ for the discrete ordinates method, and is ordered such that for each spatial multi-index, the directional index varies first before advancing to the next spatial multi-index.

5. CONCLUSIONS

In this work, we demonstrated that applying a high-frequency asymptotic ansatz to classical wave equations leads naturally to the eikonal equation, which effectively characterises the evolution of the wave phase. By utilising the method of characteristics, we transformed the eikonal equation into a Hamiltonian system of ray equations, thereby establishing a robust framework for tracking the wave energy densities. We then introduced a phase-space density that satisfies the stationary Liouville equation, laying the foundation for deriving the RTE without scattering. We discussed the numerical solution of the aforementioned RTE by discretising both the spatial and directional variables. We introduced a computational framework that utilises DG for spatial discretisation, and compare the conventional discrete ordinates method for the directional discretisation with a Fourier-based approach. In the oral presentation, I will discuss various simulation results that highlight the method's potential in addressing high-frequency wave problems.

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